Math 51, Spring 2011
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Tangent planes to surfaces in $\mathbb{R}^3$

There are two formulas in section DVC 10 that give the equation for a tangent plane to a surface. The first is Theorem 18, and the second is Proposition 10.1. Theorem 18 is more general than Proposition 10.1. We can only use Proposition 10.1 if we can write the surface explicitly, that is, the surface is the set of all points (x, y, z) such that $z = f(x, y)$ for some function $f\colon \mathcal{D}^2 \to \mathbb{R}$. However, we can use Theorem 18 even if the surface is defined implicitly, that is, the surface is the set of all points (x, y, z) such that $F(x, y, z) = c$ for some function $F\colon \mathcal{D}^3 \to \mathbb{R}$ and constant $c \in \mathbb{R}$.

Example 1. Let S be the surface $2x + 2\ln 2y + z^2 = 9$. Find the tangent plane to S at $(4, \frac{1}{2}, 1)$.

Answer. Since the surface S is implicit, we will use Theorem 18. Define F by $F(x, y, z) = 2x + 2\ln 2y + z^2$. Note that $S = F^{-1}(9)$, that is, surface S is the level set of F at height 9. Hence Theorem 18 applies, and it tells us that the tangent plane to S at $(4, \frac{1}{2}, 1)$ is

$$0 = F_x(4, \frac{1}{2}, 1)(x - 4) + F_y(4, \frac{1}{2}, 1)(y - \frac{1}{2}) + F_z(4, \frac{1}{2}, 1)(z - 1).$$

Finding the partial derivatives, we get

$$0 = 2|_{(4, \frac{1}{2}, 1)}(x - 4) + \frac{2}{y}|_{(4, \frac{1}{2}, 1)}(y - \frac{1}{2}) + 2z|_{(4, \frac{1}{2}, 1)}(z - 1).$$

Plugging the point $(4, \frac{1}{2}, 1)$ into the partial derivatives we get

$$0 = 2(x - 4) + 4(y - \frac{1}{2}) + 2(z - 1),$$

which is our final answer

Remark. The function $F(x, y, z) = 2x + 2\ln 2y + z^2$ is only defined on the set of points $(x, y, z) \in \mathbb{R}^3$ with $y > 0$. You can't take the logarithm of a non-positive number!

Example 2. Let S be the surface $z = \sin(x^2 + y^2)$. Find the tangent plane to S at $(\sqrt{\pi}, 0, 0)$.

Answer. Since the surface S is explicit, we will use Proposition 10.1. Define $f\colon \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = \sin(x^2 + y^2)$. Note that $S = \Gamma_f$, that is, surface S is the graph of f. Hence Proposition 10.1 applies, and it tells us that the tangent plane to S at $(\sqrt{\pi}, 0, 0)$ is

$$z = f(\sqrt{\pi}, 0) + f_x(\sqrt{\pi}, 0)(x - \sqrt{\pi}) + f_y(\sqrt{\pi}, 0)(y - 0).$$

Evaluating $f(\sqrt{\pi}, 0) = 0$ and finding the partial derivatives, we get

$$z = 0 + \cos(x^2 + y^2)(2x)|_{(\sqrt{\pi}, 0)}(x - \sqrt{\pi}) + \cos(x^2 + y^2)(2y)|_{(\sqrt{\pi}, 0)}(y - 0).$$

Plugging the point $(\sqrt{\pi}, 0)$ into the partial derivatives we get

$$z = 0 - 2\sqrt{\pi}(x - \sqrt{\pi}) + 0(y - 0)$$

or

$$z = -2\sqrt{\pi}(x - \sqrt{\pi}),$$

which is our final answer.