CSU Math 510

Midterm

Due Friday, October 16 at 8pm

Instructions. Please submit this optional midterm by Friday, Oct 16 at 8pm by emailing it to `henry.adams@colostate.edu` with the email subject line “Math 510: Midterm”.

Each problem is worth 5 points, and so the exam is out of 20 points.

This an optional midterm, in the sense that it can only improve your grade if you decide to take it.

This midterm was written to be taken in 75 minutes with no books, notes, or calculators (so that you can use it as qualifying exam practice if you want to). But I am not proctoring the exam, and the point of my grading is only to give you feedback on how I will grade the qual. So please feel free to use extra time or outside resources if it will help you learn the material better (in preparation for the qualifying exam).

1. Put the following linear program into equational form (maximize $c^T x$ subject to $Ax = b$ and $x \geq 0$).

Minimize x_3
subject to

$$\begin{array}{rrrrl} -2x_1 & +x_2 & +2x_3 & & \leq 1 \\ x_1 & & -x_3 & +2x_4 & = 2 \\ x_1 & +x_2 & & +x_4 & \geq 3 \\ & & & x_1, x_2, x_4 & \geq 0 \end{array}$$

Change min to max and add slack variables

Max $-x_3$

subject to

$$\begin{array}{rrrrrrl} -2x_1 & +x_2 & +2x_3 & & +x_5 & & = 1 \\ x_1 & & -x_3 & +2x_4 & & & = 2 \\ -x_1 & -x_2 & & -x_4 & & +x_6 & = -3 \end{array}$$

$$x_1, x_2, x_4, x_5, x_6 \geq 0$$

Handle the fact that x_3 is not necessarily nonnegative

Max $-x_3' + x_3''$

Subject to

$$\begin{array}{rrrrrrrl} -2x_1 & +x_2 & +2x_3' & -2x_3'' & & +x_5 & & = 1 \\ x_1 & & -x_3' & +x_3'' & +2x_4 & & & = 2 \\ -x_1 & -x_2 & & & -x_4 & & +x_6 & = -3 \end{array}$$

$$x_1, x_2, x_3', x_3'', x_4, x_5, x_6 \geq 0$$

After changing to $x_1, \dots, x_7$, we have

$$c = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad A = \begin{bmatrix} -2 & 1 & 2 & -2 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 & 2 & 0 & 0 \\ -1 & -1 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}.$$

2. Use the simplex method to find an optimal solution to the following linear program. You can use any pivot rule you like.

Maximize $x_1 + x_2 + x_3 + x_4$
subject to

$$\begin{array}{lllll} -4x_1 & +x_2 & +x_3 & & = 10 \\ x_1 & & & +x_4 & = 3 \\ x_1 & +x_2 & & & \geq -1 \\ & & & x_1, x_2, x_3, x_4 & \geq 0 \end{array}$$

Equational form: Maximize $x_1 + x_2 + x_3 + x_4$

subject to

$$\begin{array}{lllll} -4x_1 & +x_2 & +x_3 & & = 10 \\ x_1 & & & +x_4 & = 3 \\ -x_1 & -x_2 & & +x_5 & = 1 \\ & & & x_1, x_2, x_3, x_4, x_5 & \geq 0 \end{array}$$

Simplex tableau

$$\begin{aligned} x_3 &= 10 + 4x_1 - x_2 \\ x_4 &= 3 - x_1 \\ x_5 &= 1 + x_1 + x_2 \\ z &= 13 + 4x_1 \end{aligned}
\quad \xrightarrow{\text{pivot}} \quad
\begin{aligned} x_1 &= 3 - x_4 \\ x_3 &= 22 - x_2 - 4x_4 \\ x_5 &= 4 + x_2 - x_4 \\ z &= 25 - 4x_4 \end{aligned}$$

The max is 25, obtained when $(x_1, x_2, x_3, x_4) = (3, 0, 22, 0)$.

Don't return x_5 since it's not part of the original problem!

3. Use the simplex method on an auxiliary problem to find *any* basic feasible solution to the following linear program. This basic feasible solution does not need to be optimal. You may chose pivot variables however you like.

Remark: Don't just find a basic feasible solution by eye — I am asking you to use the simplex method.

Maximize $x_1 + x_2$
subject to

$$\begin{aligned} -2x_1 \quad\quad\;\; &+3x_3 &= 12 \\ x_1 \quad -x_2 \;\; &+x_3 &= 3 \\ &x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Auxiliary problem: Maximize $-x_4 - x_5$

Subject to

$$\begin{aligned} -2x_1 \quad\quad\;\; + 3x_3 + x_4 \quad\;\; &= 12 \\ x_1 \quad -x_2 + x_3 \quad\quad + x_5 &= 3 \\ x_1, \dots, x_5 &\geq 0 \end{aligned}$$

Simplex tableau

$$\begin{aligned} x_4 &= 12 + 2x_1 \quad\quad - 3x_3 \\ x_5 &= 3 - x_1 + x_2 - x_3 \\ z &= -15 - x_1 - x_2 + 4x_3 \end{aligned}$$

$$\begin{aligned} x_3 &= 3 - x_1 + x_2 - x_5 \\ x_4 &= 3 + 5x_1 - 3x_2 + 3x_5 \\ z &= -3 - 5x_1 + 3x_2 - 4x_5 \end{aligned}$$

$$\begin{aligned} x_2 &= 1 + \tfrac{5}{3}x_1 - \tfrac{1}{3}x_4 + x_5 \\ x_3 &= 4 + \tfrac{2}{3}x_1 - \tfrac{1}{3}x_4 \\ z &= \quad\quad\quad\quad - x_4 - x_5 \end{aligned}$$

$$(x_1, x_2, x_3) = (0, 1, 4)$$

4. The dual of the linear program

Maximize $x_1 + x_2$
subject to

$$\begin{array}{cccc} x_1 & & +x_3 & \leq 5 \\ x_1 & -x_2 & +x_3 & \leq 4 \\ x_1 & +x_2 & & \leq 6 \\ & & x_1, x_2, x_3 & \geq 0 \end{array}$$

is:

Minimize $5y_1 + 4y_2 + 6y_3$
subject to

$$\begin{array}{cccc} y_1 & +y_2 & +y_3 & \geq 1 \\ & -y_2 & +y_3 & \geq 1 \\ y_1 & +y_2 & & \geq 0 \\ & & y_1, y_2, y_3 & \geq 0 \end{array}$$

Show how this dual linear program is derived, briefly explaining your logic.

We want to find upper bounds on the primal problem

$X_1 + X_2 \leq d_1 X_1 + d_2 X_2 \leq h$ for $d_1 \geq 1$, $d_2 \geq 1$.

Obtain such bounds via y_1, y_2, y_3 times the three equations, with $y_i \geq 0$ to not flip the inequality:

Note $y_1(x_1 + x_3) + y_2(x_1 - x_2 + x_3) + y_3(x_1 + x_2) \leq 5y_1 + 4y_2 + 6y_3$

$\quad\quad\quad\quad ||$

$(y_1 + y_2 + y_3)x_1 + (-y_2 + y_3)x_2 + (y_1 + y_2)x_3$

So $y_1 + y_2 + y_3 = d_1 \geq 1$ $\quad -y_2 + y_3 = d_2 \geq 1$ $\quad y_1 + y_2 = d_3 \geq 0$

We want to minimize $h = 5y_1 + 4y_2 + 6y_3$,

and combined all together this gives the dual linear program.

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