CSU Math 510

Homework 3

Due Tuesday, December 8 at 8pm

Instructions. Of the following problems, choose any two (or more) to do. Alternatively, do zero of these problems, and instead do something else that is beneficial to you! That "something else" might be:

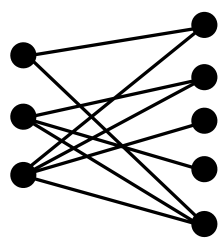
- Do a cooler problem of your choosing.
- Relate the class to your research.
- Learn a side-topic related to linear programming or optimization.
- Code something up.
- Make a blog post.
- Make a YouTube video.
- etc.

Please submit this homework by emailing it to `henry.adams@colostate.edu` with the email subject line "Math 510: Homework 3".

Problems.

The blue text below contains html links.

1. Do any problem or two that you haven't yet done from Homework 1, from Homework 2, or from the optional midterm.
2. Solve a linear programming problem of your choosing in a software package of your choosing (perhaps python, Matlab, etc). There should be built-in commands for doing this.
3. Find and briefly describe a cool example application of linear programming.
4. Is there a topic you'd like to teach the Math 510 class, say taking anywhere from 5 minutes to 30 minutes? If so, email Henry to arrange a day to do this.
5. Solve a linear programming problem to find a maximum matching or a minimum vertex cover in the following bipartite graph.

   If you do both, then check — does König's Theorem hold on this example?
6. In our textbook *Understanding and Using Linear Programming* by Jiří Matoušek and Bernd Gärtner, read the subsection "A logical view" on pages 92–93, and read Section 6.7 on pages 100-104. Briefly explain the connection between the Farkas lemma, logic, and Fourier–Motzkin elimination.
7. Learn the primal-dual simplex algorithm, and explain it on a small example.
8. Learn and explain an interior point method for solving a linear programming problem, such as a central path, a potential reduction, or an affine scaling algorithm.
9. Read parts of Dey, Hirani, and Krishnamoorthy's article *Optimal homologous cycles, total unimodularity, and linear programming* and briefly explain what you learned. I suggest starting with the pictures in Figure 3 :).
10. Find and explain an example showing that if a pivot rule is not chosen carefully, then cycling is possible in the simplex method. One such example is contained on page 52 of the book *Combinatorial Optimization: Algorithms and Complexity* by Christos H Papadimitriou and Kenneth Steiglitz, but I think I prefer the example on page 4 of this paper by Hall and McKinnon. You can find other examples online.

11. Consider the dual linear programming problems

$$\text{Maximize } c^T x \text{ subject to } Ax \leq b \text{ and } x \geq 0$$

and

$$\text{Minimize } b^T y \text{ subject to } A^T y \geq c \text{ and } y \geq 0.$$

Prove the weak duality theorem, namely that for any feasible solutions x and y, we have $c^T x \leq b^T y$.

12. Page 85 of our textbook *Understanding and Using Linear Programming* by Jiří Matoušek and Bernd Gärtner explains the recipe for dualizing any linear programming problem. However, in class we only derived one specific form of this duality, namely that the dual of

$$\text{Maximize } c^T x \text{ subject to } Ax \leq b \text{ and } x \geq 0$$

is

$$\text{Minimize } b^T y \text{ subject to } A^T y \geq c \text{ and } y \geq 0.$$

Indeed, we did this on pages 62–64 of our class notes. Choose any other form of a linear program (say one in which x is not bounded to be nonnegative, or in which $Ax \leq b$ is replaced by $Ax = b$) and derive its dual linear program.

13. I'm part of the Pattern Analysis Lab, an interdisciplinary group of faculty, postdocs, and grad students, that sadly has been taking a break in 2020 on account of the pandemic. Two large collaborative papers from this group that I've been involved on include one paper on topology and machine learning and another on fractal dimensions. Sometimes we also start projects that go unfinished for a long time (or forever). Indeed, in 2019, we were discussing a project on optimal transport matchings, with a view-only Overleaf (untouched for a year) at `https://www.overleaf.com/read/wfhgtvwxhkjf`. Optimal transport matchings are closely connected to linear programming, even though I didn't know linear programming in 2019. What does this have to do with Math 510 homework? This homework problem asks you to check out the above Overleaf link, let me know if you have any thoughts or questions on it, and to think a bit about or work on any interesting questions that these Overleaf notes may inspire for you. I'd be happy to share an editable Overleaf link.

14. Any other problem of your choosing!