Websites shown for lecture

- Wikipedia "Topology": donut $\longleftrightarrow$ coffee mug
- Klein bottle video & article: https://plus.maths.org/content/os/issue26/features/mathart/Build

Math 472

- Class syllabus and website
- Advice: Come to class, read book, and work with others.

Course overview:

1. Topology is "rubber geometry" - you only care about spaces up to bending or stretching (but not tearing or glueing)

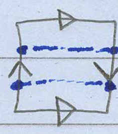
   [Wikipedia "Topology" shows a donut morphing into a coffee cup]

   You forget geometric information such as distances or curvature, but sometimes this allows you to do more (mod 2 integers)

   - Riddle on cover of textbook is optional on Homework 1.

2. Topology studies shapes and surfaces in higher dimensions

   Ex Klein bottle

   Möbius

   (video)

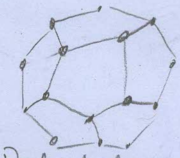
   - non-orientable
   - two Möbius bands glued together

3. Topology is related to
   - geometry (Gauss-Bonnet Theorem)
   - analysis (convergence in a metric space)
   - algebra. Given a topological space X, examples of algebraic invariants of X include
     - fundamental group $\pi_1(X)$
     - homotopy groups $\pi_k(X)$
     - homology groups $H_k(X)$

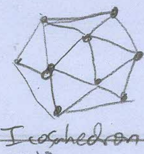
     measure the # of k-dim'l holes in a space X.

     A continuous function $f: X \to Y$ produces $f_*: \pi_k(X) \to \pi_k(Y)$.
   - category theory (functors)

4. Topology is my area of research



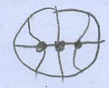
Dodecahedron
$v=20$
$e=30$
$f=12$



Icosahedron
$v=12$
$e=30$
$f=20$

Soccer ball
$v=60$
$e=90$
$f=32$

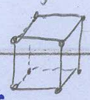
Basketball
$v=6$
$e=12$
$f=8$



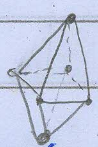
(See other examples in Math 301 notes, 11/2/15.)

§1.1 Euler's theorem

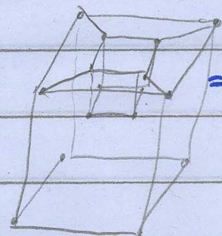
Polyhedra



$v-e+f$
$=8-12+6=2$



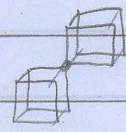
$v-e+f$
$=6-12+8=2$



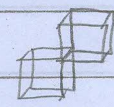
$v-e+f$
$=$

basketball, soccer ball, volleyball

Not a polyhedron:



Fails (ii)



Fails (i)

Def A polyhedron is a finite collection of plane polygons s.t.

(i) If two polygons meet, they meet in an edge shared by no other polygons

(ii) Around each vertex, we can cyclically order the adjacent polygons $Q_1, Q_2, \dots, Q_k$ s.t. Q_i and Q_{i+1} share a common edge $(Q_{k+1}=Q)$

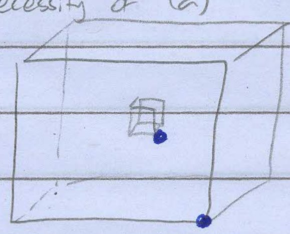
(1.1) **Euler's theorem** Let P be a polyhedron s.t.

(a) Any 2 vertices of P can be connected by a path in P (P connected)

(b) Any loop in P separates P into 2 pieces (P is simply connected, $\pi_1(P)$ is trivial)

Then $v-e+f=2$.

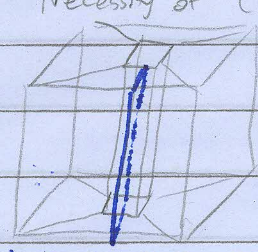
Necessity of (a)



No path

$v-e+f=4$

Necessity of (b)



Loop doesn't separate P

$v-e+f=0$

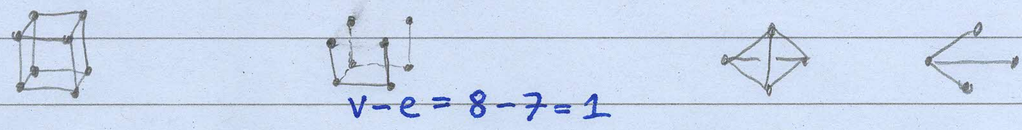
History Letter Euler $\to$ Goldbach, 1750, only convex case

von Staudt, 1847

Book: "Proofs and Refutations" by philosopher Imre Lakatos

Facts needed for proof

Def A connected set of vertices and edges is a graph



$v - e = 8 - 7 = 1$

Def A graph with no loops is a tree.

Fact In a tree T we have $v(T) - e(T) = 1$.

Fact Any graph contains a minimal spanning tree (MST), i.e. a subgraph which

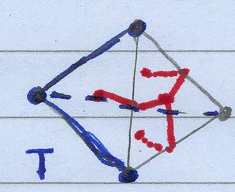
- contains all vertices
- is a tree.

Proof of Theorem 1.1

By (a), the set of all vertices and edges of polyhedron P is a graph. Choose a MST T.

(Keep)

P



Let Γ be a dual graph for T in P;
Γ contains a vertex for each face of P,
and an edge between two adjacent faces
$\Longleftrightarrow$ this edge is not in T.

Note Γ is connected since T contains no loops (this can be proven rigorously).

Note Γ has no loops since otherwise T could not be a spanning tree. Hence Γ is a tree.

It follows that

$$v - e + f = v(T) - (e(T) + e(\Gamma)) + v(\Gamma)$$ since each edge in P is either in T or crossed by a unique edge of Γ

$$= \underbrace{v(T) - e(T)}_{1} + \underbrace{v(\Gamma) - e(\Gamma)}_{1}$$ since T, Γ are trees

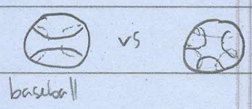
$$= 2.$$

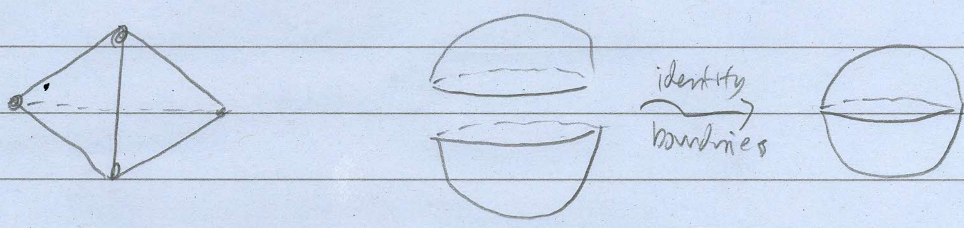
§1.2 Topological equivalence

Our proof of Euler's theorem could be extended to tell us more: P is "homeomorphic" to two discs glued together along their boundaries, i.e. to the 2-sphere $S^2 = \{x \in \mathbb{R}^3 \mid \|x\| = 1\}$.

Aside $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$.

Pic





Def A map $f: X \to Y$ between two "topological spaces" is a homeomorphism if it is bijective (1-to-1 and onto), "continuous" and has continuous inverse.

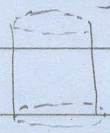
If such an f exists, we say X and Y are homeomorphic (topologically equivalent) spaces, denoted $X \cong Y$ (or $X = Y$)

[There is a weaker notion, homotopy equivalent, denoted $X \simeq Y$].

$[0,1) \not\cong S^1$
Thm 1.2
More general view of χ

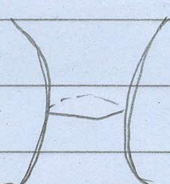
Ex The following spaces are homeomorphic.

(a)



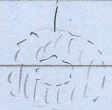
$\{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2=1,\ 1<z<3\}$
$= \{(e^{i\theta}, z) \mid 1<z<3\}$

(b)



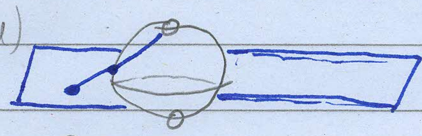
$x^2+y^2-z^2=1$, or $\{(re^{i\theta}, z) \mid r^2-z^2=1, r>0\}$

(c)



$\{(x,y) \mid 1<x^2+y^2<3\}$
$= \{re^{i\theta} \mid 1<r<3\}$

(d)



$S^2 \setminus \{(0,0,1), (0,0,-1)\}$

To see (a) $\cong$ (c), define bijection $f: (a) \to (c)$ via
$$f(e^{i\theta}, z) = ze^{i\theta}$$
Its inverse $f^{-1}: (c) \to (a)$ is defined by $f^{-1}(re^{i\theta}) = (e^{i\theta}, r)$.

To see (b) $\cong$ (c), first note $(-\infty,\infty) \cong (1,3)$ via $f : (-\infty,\infty) \to (1,3)$ defined by $f(z) = \frac{z}{1+|z|} + 2$.

A homeomorphism $g : (b) \to (c)$ is given by

$$g(re^{i\theta}, z) = f(z)e^{i\theta} \qquad (r>0)$$

<u>Ex</u> If $X \cong Y$ and $Y \cong Z$, then $X \cong Z$.

<u>Pf Sketch</u>

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[rr, bend left=50, "g\circ f"] & Y \arrow[l, bend left, "f^{-1}"] \arrow[r, bend left, "g"] & Z \arrow[l, bend left, "g^{-1}"] \arrow[ll, bend left=50, "f^{-1}\circ g^{-1}"] \end{tikzcd}$$

<u>Rmk</u> Our proof of Euler's theorem could be used to give a homeomorphism of P onto S^2 — one disk to each hemisphere.

Ex $[0,1) \not\cong S^1$. Note $f : [0,1) \to S^1$ via $f(t) = e^{2\pi i t}$ is a continuous bijection, but f^{-1} is not continuous.
<u>2 ways to show no homeomorphism exists</u>
- Nbhd about $0 \in [0,1)$
- Fundamental groups: $\pi_1([0,1)) = \langle e \rangle \neq \mathbb{Z} = \pi_1(S^1)$.

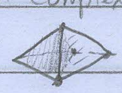
<u>Theorem 1.2</u> Homeomorphic polyhedra have the same Euler characteristic.

Starting point of modern topology!

<u>Aside</u> More generally, homeomorphic (or even homotopy equivalent) topological spaces have the same Euler characteristic χ

$\chi(\text{Polyhedra}) = v - e + f$

$\chi(\text{simplicial complex}) = \sum_i (-1)^i [\#\text{ cells of dim } i]$

$\chi($  $) = 5 - 9 + 7 - 1 = 2$

$$\chi(\text{topological space } Y) = \sum_i (-1)^i \underbrace{\text{rank } H_i(Y)}_{\text{"\# holes of dimension } i\text{"}}$$

$\chi(S^1) = 1 - 1 = 0$

$\chi(S^2) = 1 - 0 + 1 = 2$

$\chi(\text{torus}) = 1 - 2 + 1 = 0$

$\chi(\text{2-holed torus}) = 1 - 4 + 1 = -2$

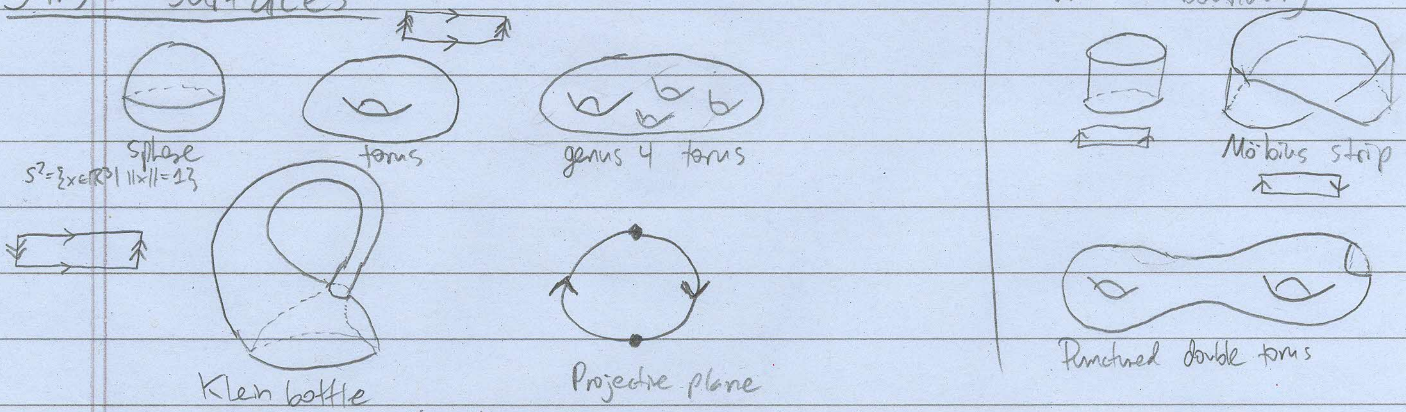
$\chi(\text{g-holed torus}) = 1 - 2g + 1 = 2 - 2g$

<u>Rmk</u> When two spaces are homeomorphic, there are often <u>many</u> homeomorphisms between them.

Problem 1.11 (c)

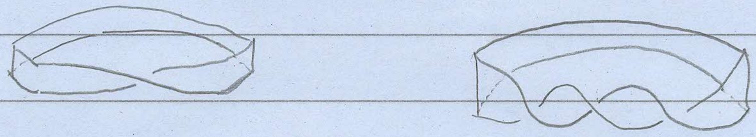
Homeomorphic and ambient "isotopic"

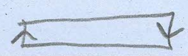
§1.3 Surfaces



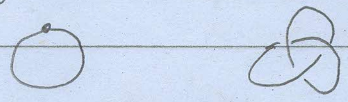
All of these "fit" (can be embedded) in $\mathbb{R}^3$, except for the Klein bottle and projective plane in $\mathbb{R}^4$.
However, we will often not think of them as living inside any larger $\mathbb{R}^n$.

Ex The following are homeomorphic.



Indeed, they're each homeomorphic to  .
They are not "ambient isotopic" – you can't find a homeomorphism between them that extends to a homeomorphism of all of $\mathbb{R}^3$.

Ex Any knot is homeomorphic to a circle, but two knots need not be "ambient isotopic".

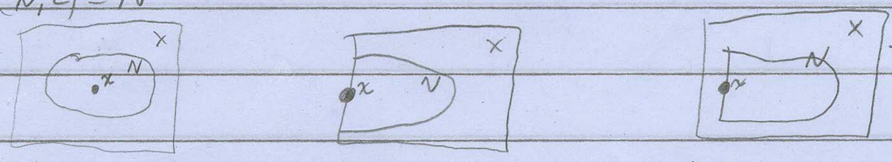


§1.4 Abstract spaces

Recall **Def** If X and Y are metric spaces, then $f: X \to Y$ is <u>continuous</u> if $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $d_X(x, x') < \delta \implies d_Y(f(x), f(x')) < \varepsilon$.

Def A subset $N \subseteq X$ is a <u>neighborhood of $x \in N$</u> if $\exists \ \varepsilon > 0$ s.t. $B(x, \varepsilon) \subseteq N$

Pic



Neighborhood · Neighborhood · <u>Not</u> a nbhd.

Equivalent **Def** If X and Y are metric spaces, then $f: X \to Y$ is <u>continuous</u> if $\forall x \in X$ and nbhd N of $f(x)$ in Y, $f^{-1}(N)$ is a nbhd of x in X.

Rmk We will define topological spaces <u>not</u> as subsets of $\mathbb{R}^n$ and <u>not</u> using distances, but instead using neighborhoods.

Def 1.3 (First def^{n} of topological spaces - the second is Def 2.1)
A <u>topological space</u> is a set X, along with a nonempty collection of nbhds about each $x \in X$, s.t.
(a) x is in each of its nbhds
(b) The intersection of two (really any finite #) nbhds of x is a nbhd of x.
(c) If N is a nbhd of x and $N \subseteq U$, then U is a nbhd of x.
(d) If N is a nbhd of x and
$\mathring{N} = \{ z \in N \mid N \text{ is a nbhd of } z \}$, $\mathring{N}$ is the <u>interior</u> of N.
then $\mathring{N}$ is a nbhd of x

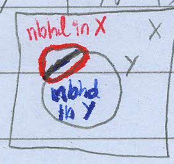
Def If X and Y are topological spaces, then $f: X \to Y$ is <u>continuous</u> if $\forall x \in X$ and nbhd N of $f(x)$ in Y, $f^{-1}(N)$ is a nbhd of x in X.

Examples

- Any metric space is a topological space, with the nbhds defined as above. Equivalent metric spaces induce the same topology! Sketch proof.
- Given a topological space X and a subset $Y \subseteq X$, the <u>subset topology</u> on Y has the following nbhds about $y \in Y$:

  $\{ N \cap Y \mid N \text{ is a nbhd about } y \in X \}$.

<u>Ex</u>



<u>Ex</u> Surfaces in $\mathbb{R}^3$ or $\mathbb{R}^4$.

- The <u>discrete topology</u> on set X is defined by taking the neighborhoods about $x \in X$ to be any set containing x.

  <u>Any</u> function with domain X is continuous w.r.t. this topology
- Three different topologies on $\mathbb{R}$:

  (i) The standard topology ($\mathbb{R}$ is a metric space)

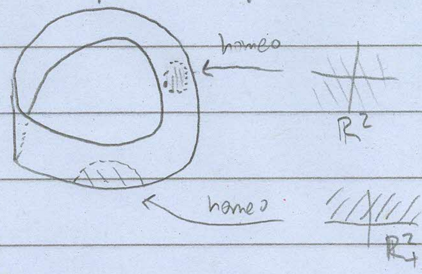
  (ii) The discrete topology] Metrizable via $d(x,y)=1$ for $x \neq y$

  (iii) $N \ni x$ is a nbhd of $x \iff \mathbb{R} \setminus N$ is finite.

  Topology (iii) does not come from a metric (is not "metrizable") since distinct points can't be separated by neighborhoods.

<u>Def 1.4</u> A topological space X is a <u>surface</u> with boundary (2-dim'l manifold) if

- each point has a nbhd homeomorphic to $\mathbb{R}^2$ (or $\mathbb{R}^2_+ = \{(x,y) \mid y \geq 0\}$)
- any two distinct points possess disjoint nbhds.] Technical

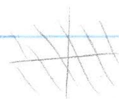


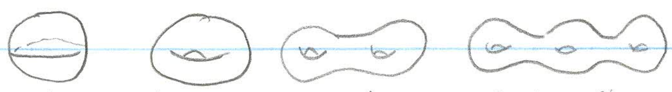
(Definition is often: each point has an open neighborhood homeomorphic to an open subset of $\mathbb{R}^2$)

§1.5 A classification theorem

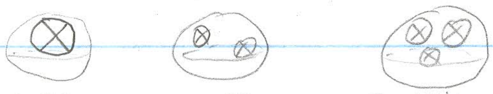
Thm 1.5 Classification theorem for surfaces

Any closed surface (compact w/o boundary) is homeomorphic to either a sphere, a sphere with a finite # of handles added, or a sphere w/ a finite # of discs replaced by Möbius strips. (Assuming connected)

Not closed  $\mathbb{R}^2$ 

Closed 

0 handles 1 handle 2 handles 3 handles — orientable

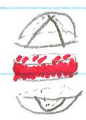


1 Möbius bands (Projective plane) 2 Möbius bands (Klein bottle) 3 Möbius bands — Not orientable (contain Möbius bands!)

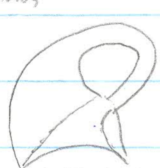
Adding a handle Remove 2 discs & glue in a cylinder

Adding a Möbius band You can remove a disc and glue in a Möbius band since both have boundary S^1

$\partial($  $) = S^1$ $\partial($  $) =$  $= S^1$

Seeing  =  is a Klein bottle

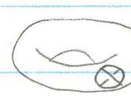
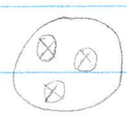
2 Möbius bands



Klein bottle back

=

Klein bottle front

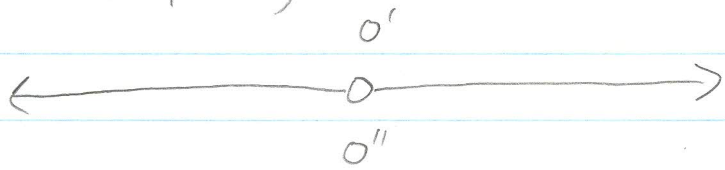
Rmk  = 

More generally, a surface with k handles and $n \geq 1$ Möbius bands is homeomorphic to one with $2k+n$ Möbius bands.

Recall Def 1.4 A topological space is a <u>surface</u> if
(i) each point has a nbhd homeomorphic to $\mathbb{R}^2$. (X is locally Euclidean)
(ii) any two distinct points possess disjoint nbhds. (X is Hausdorff)

What's a space satisfying (i) but not (ii)?

1D version The line with two origins.
Let $X = (\mathbb{R} \setminus \{0\}) \cup \{0', 0''\}$.
- Nbhds about $0'$ are the nbhds about 0 in $\mathbb{R}$ after replacing 0 with $0'$ & possibly adding $0''$
- Analogously for $0''$
- Nbhds about $x \neq 0', 0''$ are all nbhds about x in $\mathbb{R} \setminus \{0\}$, possibly with $0'$ or $0''$ added.



Note X is not Hausdorff since any nbhds about $0'$ and $0''$ intersect.

2D version The plane with two origins
Let $X = (\mathbb{R}^2 \setminus \{\vec{0}\}) \cup \{\vec{0}', \vec{0}''\}$ be equipped with the analogous topology.

Rmk X satisfies (i) but not (ii).

Wikipedia "Non-Hausdorff manifold" or "Locally Euclidean space"

Rmk Most topologists don't study such monsters of point set topology.

§1.6 Topological invariants

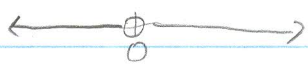
Showing two spaces are homeomorphic is typically constructive: find a homeo $f: X \to Y$.
But to show they're not homeomorphic, you typically can't just check all possible f's!

Def A topological invariant is preserved by homeomorphisms (sometimes homotopy equivalences)

Ex
- The Euler characteristic
- Connectedness
- The fundamental group π_1 & higher homotopy groups π_k.
- Homology groups H_k

Ex Show $\mathbb{R}$ and $\mathbb{R}^2$ aren't homeomorphic

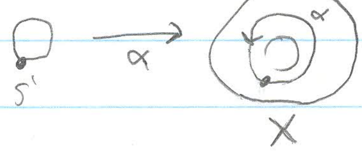
Pf Suppose for a contradiction $h: \mathbb{R} \to \mathbb{R}^2$ were a homeomorphism (non-constructive). This would induce a homeomorphism $h: \mathbb{R} \setminus \{0\} \to \mathbb{R}^2 \setminus \{h(0)\}$.
But $\mathbb{R} \setminus \{0\}$ is not connected while $\mathbb{R}^2 \setminus \{h(0)\}$ is.

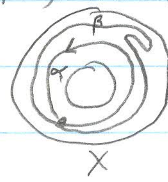


This contradicts the fact that connectedness is a topological invariant, and hence $\mathbb{R}$ and $\mathbb{R}^2$ cannot be homeomorphic.

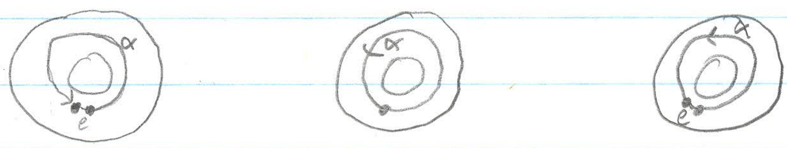
The fundamental group π_1 (Chp 5)

Def Given a topological space X, the fundamental group $\pi_1(X)$ has as its elements all equivalence classes of based maps $\alpha: S^1 \to X$ (really $\alpha: (S^1, 0) \to (X, *)$).





- Two maps $\alpha, \beta: S^1 \to X$ are considered to be the same element if they are "homotopy equivalent" (you can deform one to get the other).
- Group multiplication is concatenation of loops, where $\alpha \cdot \beta$ means doing α then β.
- The identity element e is the constant map $S^1 \to *$.

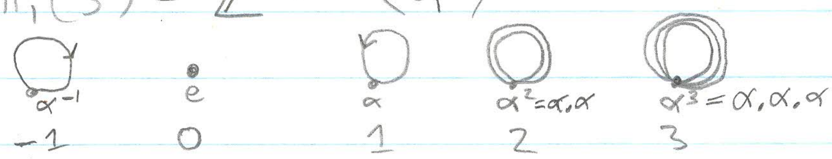


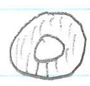
$e \cdot \alpha \simeq \alpha \simeq \alpha \cdot e$

- The inverse α^{-1} follows loop α in the opposite direction.

$\alpha \cdot \alpha^{-1}$ $\qquad\qquad\qquad\qquad$ e

Ex π_1() $= \pi_1(S^1) \cong \mathbb{Z} = \langle \alpha \rangle$



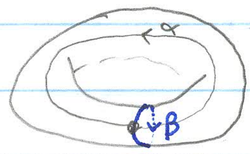
Ex π_1() $= \langle e \rangle$, trivial group. Hence  $\not\cong$ .

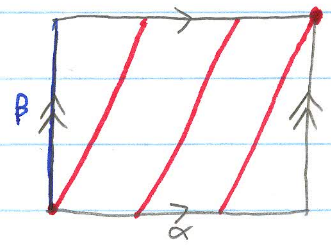
Ex $\pi_1\big($  $\big) = \pi_1(S^1\times S^1) \cong \mathbb{Z}\times\mathbb{Z} = \langle \alpha,\beta \mid \alpha.\beta=\beta.\alpha\rangle$

$(1,0) \leftrightarrow \alpha$

$(0,1) \leftrightarrow \beta$

Free abelian group





$(1,3)\in\mathbb{Z}\times\mathbb{Z}$

Rmk Try to see why $\alpha.\beta \simeq \beta.\alpha$

Ex $\pi_1\big($  $\big) \cong \mathbb{Z} * \mathbb{Z} = \langle \alpha,\beta\rangle$ Free group

Rmk Here $\alpha.\beta \not\simeq \beta.\alpha$

Chp 2 Continuity

§2.1 Open and closed sets

Def 2.1 (Second and primary defⁿ of topological spaces)

A topology on a set X is a nonempty collection of open sets s.t.

- any union of open sets is open
- any finite intersection of open sets is open
- both X and $\emptyset$ are open.

Rmk $\bigcap_{n=1}^{\infty} (-1/n, 1/n) = \{0\}$ is not open in $\mathbb{R}$.

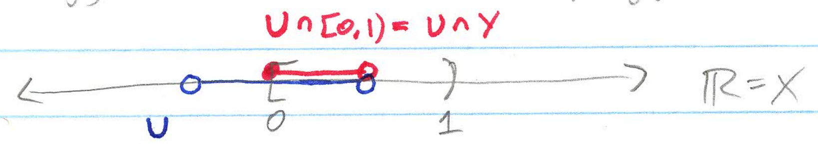
What's the correspondence b/w open sets in Def 2.1 and neighborhoods in §1.4 ?

- A subset $N \subseteq X$ is a neighborhood of $x \in N$ if $\exists$ some open set V with $x \in V \subseteq N$
- A subset $U \subseteq X$ is open if it is a neighborhood of each of its points.

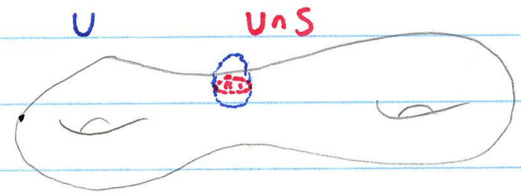
Examples
- If X is a metric space, then we define $U \subseteq X$ to be open if for each $x \in U$, $\exists\, \varepsilon > 0$ s.t. $B(x, \varepsilon) \subseteq U$.
- In the discrete topology on X, every subset of X is declared to be open.

  (Any function $f: X \to Y$ when X has the discrete topology will be continuous.)
- If X is a topological space and $Y \subseteq X$, then the induced or subspace topology on Y has as its open sets all $U \cap Y$, where $U \subseteq X$ is open.

Ex The topology on $\mathbb{R}$ induces a topology on $[0,1)$



Ex The topology on $\mathbb{R}^3$ induces a topology on any surface $S \subseteq \mathbb{R}^3$.



Def A subset $C \subseteq X$ is <u>closed</u> if its complement $X \setminus C$ is open.

- any intersection of closed sets is closed
(Proof sketch: De Morgan formulae)

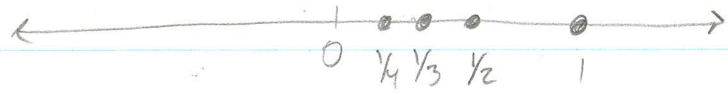
- any finite union of closed sets is closed
(Proof sketch: De Morgan formulae)

Def Let $A \subseteq X$. A point $p \in X$ is a <u>limit point</u> or <u>accumulation point of A</u> if every neighborhood (equivalently open set) about p contains at least one point of $A \setminus \{p\}$.

Rmk Either $p \in A$ or $p \notin A$ are possible.

Ex $X = \mathbb{R}$, $A = [0,1)$. Every point in $[0,1]$ is a limit point of A.

Ex $X=\mathbb{R}$, $A=\{1/1, 1/2, 1/3, 1/4, 1/5, \dots\}$



The only limit point of A is 0.

Ex $X=\mathbb{R}$, $A=\mathbb{Z}$. Then A has no limit points.

Thm 2.2 A set is closed $\Longleftrightarrow$ it contains all its limit points.

Pf ($\Rightarrow$). If A is closed, then $X\setminus A$ is open. Hence no point of $X\setminus A$ can be a limit point of A.

($\Leftarrow$) Suppose A contains all its limit points; let $x\in X\setminus A$. Hence $\exists$ some open $U\ni x$ with $U\cap A=\emptyset$, i.e. $U\subset X\setminus A$. This shows $X\setminus A$ is open, so A is closed.

since $U\cap(A\setminus\{x\})=\emptyset$ and $x\notin A$.

Def The union of A and all its limit points is the <u>closure of A</u>, denoted $\bar{A}$.

Thm 2.3 The closure of A is the smallest closed set containing A, i.e. the intersection of all closed sets containing A.

(**Rmk** Sometimes $\bar{A}$ is defined as this intersection)

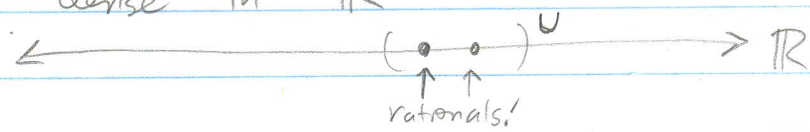
Pf <u>$\bar{A}$ is closed:</u> If $x\in X\setminus\bar{A}$, then we can find open $U\ni x$ with $U\cap A=\emptyset$. Since open U is a nbhd of each of its points, U cannot contain any limit points of A either.

<u>B closed and $A\subseteq B \Rightarrow \bar{A}\subseteq B$:</u> Every limit point of A is also a limit point of B, hence an element of B by Thm 2.2. Hence $\bar{A}\subseteq B$.

Corollary 2.4 A is closed $\iff A = \overline{A}$.

Def $A \subseteq X$ is <u>dense in X</u> if $\overline{A} = X$, or equivalently, if $\emptyset \neq U \subseteq X$ is open $\Rightarrow U \cap A \neq \emptyset$.

Ex $\mathbb{Q}$ is dense in $\mathbb{R}$



Def The <u>interior of a set A</u>, denoted $\mathring{A}$, is the union of all open sets contained in A.

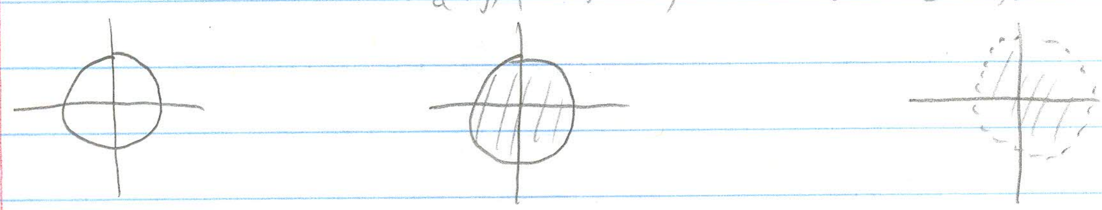
Rmk U is open in $X \iff U = \mathring{U}$.

Def The <u>frontier</u> or <u>boundary</u> of $A \subseteq X$ is $\partial A = \overline{A} \cap \overline{(X \setminus A)}$ [equivalently, $X \setminus (\mathring{A} \cup (X \mathring{\setminus} A))$]

Ex In $\mathbb{R}^2$, we have

$\partial(S^1) = S^1$ $\qquad$ $\partial(D^2) = S^1$ $\qquad$ $\partial(\mathring{D}^2) = S^1$

$D^2 = \{(x,y) \mid x^2+y^2 \leq 1\}$ $\qquad$ $\mathring{D}^2 = \{(x,y) \mid x^2+y^2 < 1\}$



Def Let X be a topological space. A collection β of open sets is a <u>base</u> for X if every open $U \subseteq X$ can be written as a union of sets in β.

Ex A base for $\mathbb{R}$ is $\beta = \{(a,b) \subseteq \mathbb{R} \mid a < b\}$.

Ex Another base for $\mathbb{R}$ is $\beta = \{(a,b) \subseteq \mathbb{R} \mid a < b \text{ and } a, b \in \mathbb{Q}\}$.

<u>Rmk</u> One often specifies a topology on X by giving a base β : a set $U \subseteq X$ is then open if it is a union of sets in β.

<u>Question</u> When does a base β produce a valid topology?

<u>Thm 2.5</u> Let X be a set and β be a nonempty collection of subsets. If a finite intersection of sets in β is in β, and if $\cup\beta = X$,

$\cup\beta := \bigcup_{B\in\beta} B$

then β is a base for a valid topology on X.

<u>Pf sketch</u> Check the axioms for a topology in Def 2.1.

§2.2 Continuous functions

Def / (Theorem 2.6) A function $f: X \to Y$ is continuous $\Leftrightarrow$ $\forall$ open U in Y, $f^{-1}(U)$ is open in X.

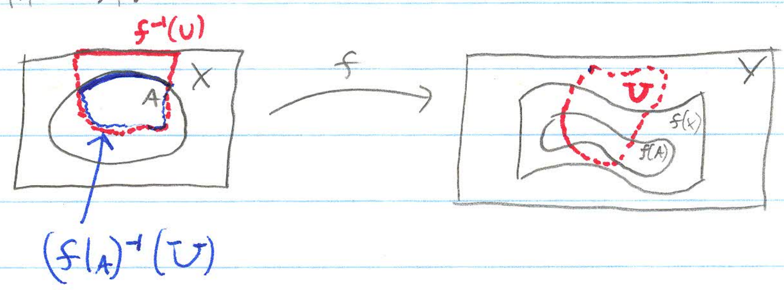
Our book calls a continuous function a map

Thm 2.7 If $f: X \to Y$ and $g: Y \to Z$ are continuous, then so is $g \circ f: X \to Z$.

Pf Let $U \subseteq Z$ be open. Then $g^{-1}(U)$ is open since g cont., and hence $f^{-1}(g^{-1}(U)) = (f \circ g)^{-1}(U)$ is open since f cont.

Thm 2.8 If $f: X \to Y$ is continuous and $A \subseteq X$ has the subspace topology, then $f|_A : A \to Y$ is continuous. (Book: $f|A$ instead of $f|_A$)

Pf Let $U \subseteq Y$ be open. Note $(f|_A)^{-1}(U) = f^{-1}(U) \cap A$.
Since f is continuous, $f^{-1}(A)$ is open in X.
By definition of the subspace topology, $(f|_A)^{-1}(U)$ is open in A.



Thm 2.9 The following are equivalent

(a) $f: X \to Y$ is a map, i.e. $U \subseteq Y$ open $\Rightarrow$ $f^{-1}(U) \subseteq X$ open.

(b) If β is a base for the topology on Y, then the preimage of every set in β is open in Y. (a) $\Rightarrow$ (b) clear

(c) $f(\overline{A}) \subseteq \overline{f(A)}$ for any $A \subseteq X$.

(d) $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ for any $B \subseteq Y$.

(e) $C \subseteq Y$ closed $\Longrightarrow$ $f^{-1}(C) \subseteq X$ closed.

Pf (Sketch)

Book says (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (a) is most efficient

(a) $\Rightarrow$ (b) Every set in β is open in Y.

(b) $\Rightarrow$ (c) See book

(c) $\Rightarrow$ (d) Note $x \in \overline{f^{-1}(B)}$

$\Rightarrow f(x) \in \overline{f(f^{-1}(B))} = \overline{B}$ by (c) with $A = f^{-1}(B)$

$\Rightarrow x \in f^{-1}(\overline{B})$.

(d) $\Rightarrow$ (e) [In book]

Note B is closed in Y

$\Rightarrow \overline{B} = B$

$\Rightarrow \overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B}) = f^{-1}(B)$ by (d)

$\Rightarrow \overline{f^{-1}(B)} = f^{-1}(B)$

$\Rightarrow f^{-1}(B)$ is closed in X.

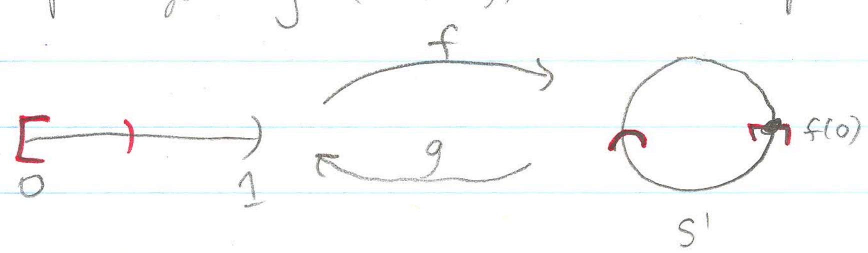
(e) $\Rightarrow$ (a) Let $V \subseteq Y$ be open

$\Rightarrow Y \setminus V$ is closed

$\Rightarrow f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$ is closed in X by (e)

$\Rightarrow f^{-1}(V)$ is open in X.

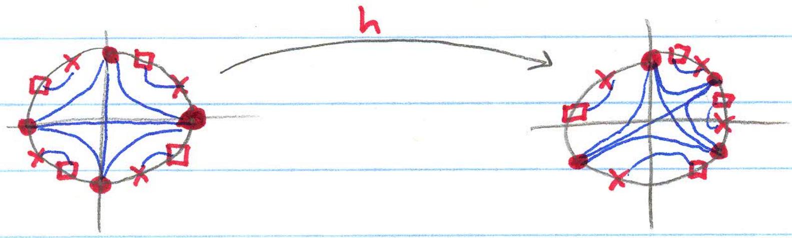
Ex $f : [0,1) \to S^1$ via $f(x) = e^{2\pi i x}$ is continuous.
Its inverse $g : S^1 \to [0,1)$ via $e^{2\pi i x} \longmapsto x$ is not, since $[0, 1/2)$ is open in $[0,1/2)$ but its preimage $g^{-1}([0, 1/2))$ is not open in S^1.



<u>Rmk</u> A homeomorphism $f: X \to Y$ gives a bijection b/w the <u>open sets</u> in X and the open sets in Y.

Not too important:

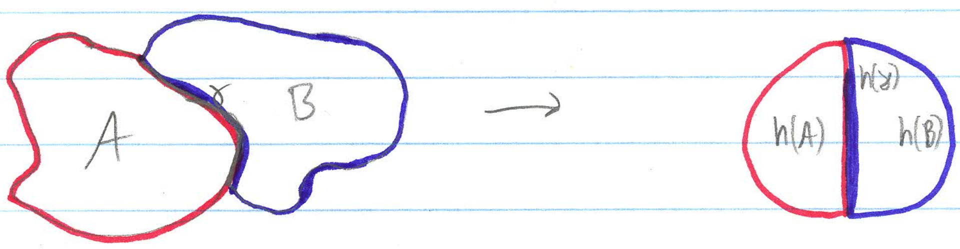
<u>Lemma 2.10</u> Any homeomorphism $h: \partial D^2 \to \partial D^2$ can be extended to a homeomorphism $\tilde{h}: D^2 \to D^2$



$D^2 = \{(x,y) \mid x^2+y^2 \leq 1\}$

$\partial D^2 = S^1 = \{(x,y) \mid x^2+y^2 = 1\}$

<u>Lemma 2.11</u> Let A and B be two discs which intersect along their boundaries in an arc. Then $A \cup B$ is a disc.



§2.3 A space-filling curve : Skipping for now.

§2.4 Tietze extension theorem : Skipping forever.

Def^n on pg 39 A topological space X is Hausdorff if for all $x,y \in X$ with $x \neq y$, $\exists$ open sets $U \ni x$ and $V \ni y$ with $U \cap V = \emptyset$.

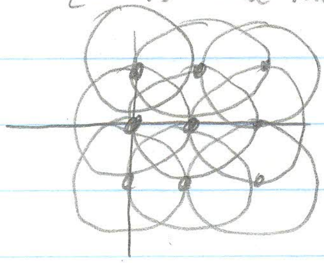
Chapter 3 Compactness and connectedness

§3.1 Closed bounded subsets of $\mathbb{R}^n$

Def 3.2 A topological space X is compact if every open cover of X has a finite subcover.

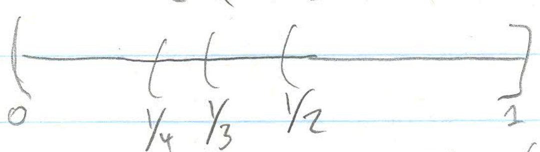
- An open cover of X is a collection $\mathcal{F}$ of open subsets of X s.t. $\bigcup_{U \in \mathcal{F}} U = \bigcup \mathcal{F} = X$.
- A subcover is a subset $\mathcal{F}' \subseteq \mathcal{F}$ with $\bigcup \mathcal{F}' = X$.

Ex $X = \mathbb{R}^2$, $\mathcal{F} = \{$all open balls with integral centers and radius 1$\}$



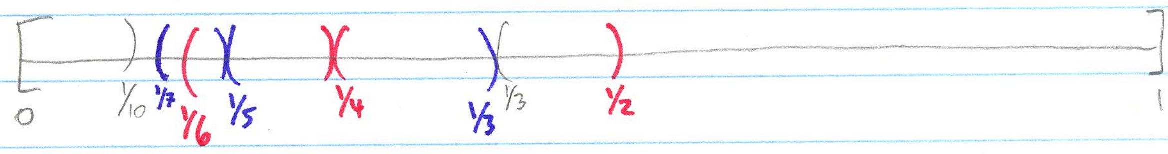
Note $\mathcal{F}$ has no subcovers. Hence $\mathbb{R}^2$ is not compact.

Ex $X = (0,1]$, $\mathcal{F} = \{ (\frac{1}{n}, 1] \mid n = 2, 3, \dots \}$.



Note $\mathcal{F}$ is an open cover of $(0,1]$ with no finite subcover. Hence $(0,1]$ is not compact.

Ex $X=[0,1]$, $\mathcal{F}=\{[0,\frac{1}{10}), (\frac{1}{3},1], \text{ and } (\frac{1}{n+2},\frac{1}{n}) \text{ for } n=2,3,4,\ldots\}$.



Note $\mathcal{F}'=\{[0,\frac{1}{10}), (\frac{1}{3},1], \text{ and } (\frac{1}{n+2},\frac{1}{n}) \text{ for } n=2,3,\ldots,9\}$ is a finite subcover of $\mathcal{F}$. If we could show this for all such open covers $\mathcal{F}$ then we would know $[0,1]$ is compact (it is)

Heine-Borel theorem (2 versions)

Thm 3.3 A closed interal $[a,b]$ of $\mathbb{R}$ is compact

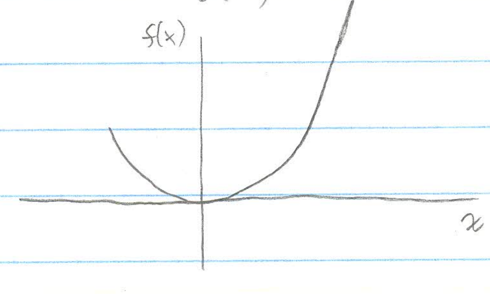
Thm 3.1 $X \subseteq \mathbb{R}^n$ is compact $\Longleftrightarrow$ it is closed and bounded

contained in some ball of finite radius.

Why do we care about compact spaces?

Thm 3.10 If $f: X \to \mathbb{R}$ is continuous and X is compact, then f is bounded and its maximum and minimum values are attained (i.e. $\exists\, x_1, x_2 \in X$ with $f(x_1)=\inf(f(X))$ and $f(x_2)=\sup(f(X))$).

Ex The function $f:(-1,2)\to\mathbb{R}$ via $f(x)=x^2$ achieves its min $[f(0)=0=\inf(f((-1,2)))]$ but not its max $[4=\sup(f((-1,2)))]$.
The function $f:[-1,2]\to\mathbb{R}$ via $f(x)=x^2$ achieves both its min $f(0)$ and its max $f(2)$.



Bolzano-Weierstrass property (3.10) An infinite subset of a compact space must have a limit point.

Ex Fails for $\mathbb{Z} \subseteq \mathbb{R}$.

Rmk A topological space being compact is analogous to a set being finite. [compact spaces need not be finite]

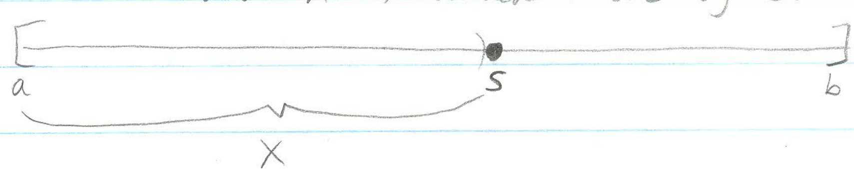
- Any cover of a finite set has a finite subcover.
- Any function on a finite set achieves its maximum and minimum values

- You can prove things about compact spaces by proving them on a finite # of pieces.

§3.2 The Heine-Borel theorem

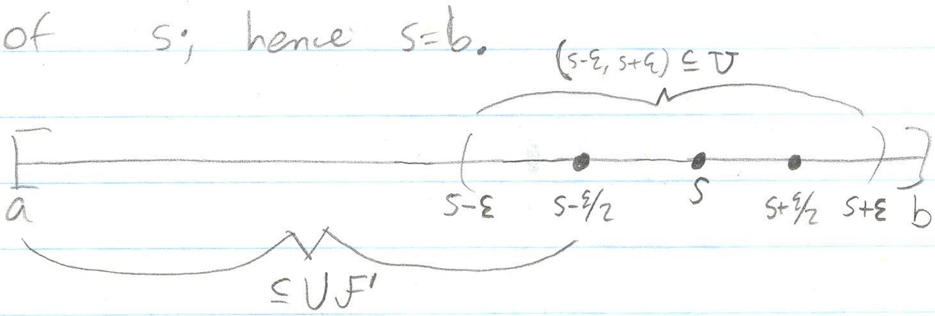
Thm 3.3 A closed interval $[a,b] \subseteq \mathbb{R}$ $(a,b \neq \infty)$ is compact.

Pf 1 Let $\mathcal{F}$ be an open cover of $[a,b]$.
(Our task is to find a finite subcover).
Let $X = \{x \in [a,b] \mid [a,x]$ is contained in the union of a finite subfamily of $\mathcal{F}\}$.
Let $s = \sup(X) = \text{l.u.b}(X)$, which exists since X is nonempty $(a \in X)$ and X is bounded above by b.



We claim $s=b$ and $b \in X$

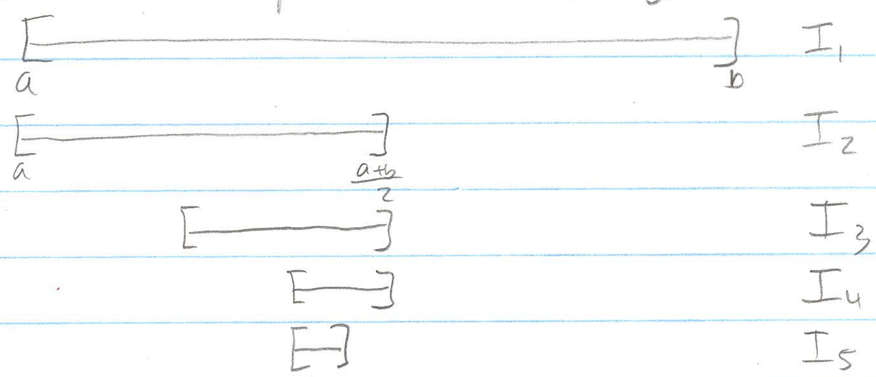
$\underline{s=b}$ Suppose for a contradiction $s<b$. Note $s \in U$ for some $U \in \mathcal{F}$, and $\exists\, \varepsilon > 0$ with $(s-\varepsilon, s+\varepsilon) \subseteq U$ [clearly $s \neq a$].
Let $\mathcal{F}'$ be a finite subfamily with $[a, s-\varepsilon/2] \subseteq \bigcup \mathcal{F}'$;
note $\mathcal{F}' \cup \{U\}$ is a finite subfamily with
$[a, s+\varepsilon/2] \subseteq \bigcup \mathcal{F}' \cup U$. This contradicts the definition of s; hence $s=b$.



$\underline{b \in X}$ Note $s \in U$ for some $U \in \mathcal{F}$, and $\exists\, \varepsilon > 0$ with
$(b-\varepsilon, b] \subseteq U$.
Let $\mathcal{F}'$ be a finite subfamily with $[a, b-\varepsilon/2] \subseteq \bigcup \mathcal{F}'$;
note $\mathcal{F}' \cup \{U\}$ shows $b \in X$.

<u>Pf 2</u> (Sketch) This proof is more general; it works also for squares $[a,b]\times[c,d] \subseteq \mathbb{R}^2$, cubes $\subseteq \mathbb{R}^3$, etc.

Suppose $\mathcal{F}$ is an open cover of $[a,b]$ with no finite subcover.



Bisect $[a,b]$ into $[a, \frac{a+b}{2}]$ and $[\frac{a+b}{2}, b]$; at least one such subinterval must have no finite subcover (else the two finite subcovers would together form a finite subcover of $[a,b]$).

Call this subinterval I_2.

Bisect I_2; at least one such subinterval must have no finite subcover.

Call this subinterval I_3.

We get $I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots$ s.t. each I_n has no finite subcover.

One can show $\bigcap_{n=1}^{\infty} I_n = \{p\}$, a single point.

Note $p \in U$ for some $U \in \mathcal{F}$. Hence $\exists\, \varepsilon > 0$ with $(p-\varepsilon, p+\varepsilon) \cap [a,b] \subseteq U$.

Choose n sufficiently large s.t. length$(I_n) = \frac{b-a}{2^{n-1}} < \varepsilon$.

Hence $I_n \subseteq U$, contradicting the fact that I_n has no finite subcover!

§3.3 Properties of compact spaces

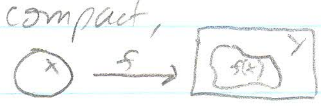
Rmk If $X \cong Y$, then X is compact $\Leftrightarrow$ Y is compact.
homeomorphic

This follows since there is a bijective correspondence between the open sets in X and in Y.

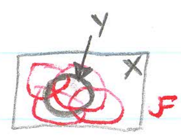
More generally,

Thm 3.4 If $f: X \to Y$ is continuous and surjective, and X is compact, then Y is compact.

Equivalently If $f: X \to Y$ is continuous and X is compact, then $f(X)$ is compact.



Pf Let $\mathcal{F}$ be an open cover of Y.
Note $\{ f^{-1}(U) \mid U \in \mathcal{F} \}$ is an open cover of X since f continuous.
X compact $\Rightarrow \exists$ finite subcover $X = f^{-1}(U_1) \cup \dots \cup f^{-1}(U_n)$.
f surjective $\Rightarrow Y = f(X) = f(f^{-1}(U_1)) \cup \dots \cup f(f^{-1}(U_n))$
$= U_1 \cup \dots \cup U_n$.
So we've found our finite subcover.



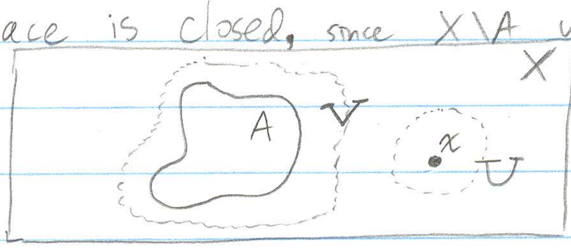
Rmk A subspace $Y \subseteq X$ is compact $\Leftrightarrow \forall$ families $\mathcal{F}$ of open sets in X with $Y \subseteq \bigcup \mathcal{F}$, $\exists$ finite subfamily $\mathcal{F}'$ with $Y \subseteq \bigcup \mathcal{F}'$.

Thm 3.5 A closed subset C of a compact space X is compact.

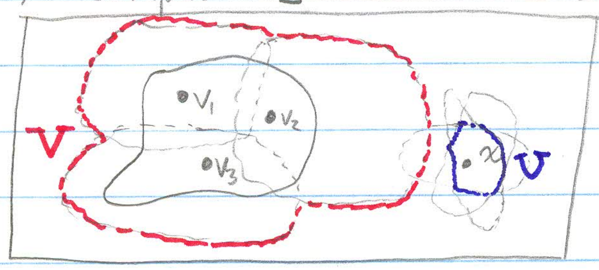
Pf Let $\mathcal{F}$ be a family of open sets in X with $C \subseteq \bigcup \mathcal{F}$.
Add the open set $X \setminus C$ to get an open cover of X.
By compactness of X, $\exists$ finite subcover $X = U_1 \cup \dots \cup U_n \cup \underbrace{(X \setminus C)}_{\text{doesn't hurt to include}}$.
Hence $\mathcal{F}' = \{U_1, \dots, U_n\}$ is a finite subfamily of $\mathcal{F}$ with $C \subseteq \bigcup \mathcal{F}'$.

Thm 3.6 If A is a compact subspace of a Hausdorff space X, and if $x \in X \setminus A$, then $\exists$ disjoint neighborhoods of A and x.
(It follows that a compact subspace of a Hausdorff space is closed, since $X \setminus A$ will be open.)

Pic



Pf $\forall\ a \in A$, $\exists$ disjoint open sets $V_a \ni a$ and $U_a \ni x$.
Note $\{V_a \mid a \in A\}$ is an open family containing A; since A compact $\exists$ finite subfamily $A \subseteq V_{a_1} \cup V_{a_2} \cup \dots \cup V_{a_n}$.



Note $V = V_{a_1} \cup \dots \cup V_{a_n}$ and $U = U_{a_1} \cap \dots \cap U_{a_n}$ are disjoint neighborhoods of A and x.

Thm 3.7 If $f: X \to Y$ is continuous and bijective, X is compact, and Y is Hausdorff, then f is a homeomorphism.

Pf We'll prove $C \subseteq X$ closed $\Rightarrow f(C) = (f^{-1})^{-1}(C)$ closed.
Hence f^{-1} is continuous and f is a homeomorphism.

Indeed, $C \subseteq X$ closed $\Rightarrow C$ compact by Thm 3.5
$\Rightarrow f(C)$ compact by Thm 3.4
$\Rightarrow f(C)$ closed by Thm 3.6.

What types of spaces can be compact? The following theorem says an infinite set of points in a compact space must "crowd together".

Thm 3.8 Bolzano–Weierstrass property

An infinite subset of a compact space must have a limit point.

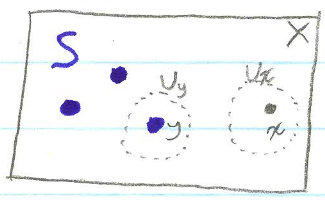
Non-example $\mathbb{Z} \subseteq \mathbb{R}$ has no limit points.

Pf Let X be compact and $S \subseteq X$ have no limit points; we will show S is finite.

Given $x \in X$, $\exists$ open $U_x \ni x$ s.t.

$$U_x \cap S = \begin{cases} \phi & \text{if } x \notin S \\ \{x\} & \text{if } x \in S \end{cases}$$

since S has no limit points.



By compactness of X, the open cover $\{U_x \mid x \in X\}$ has a finite subcover.

Since each U_x contains at most one point of S, this finite subcover shows S is finite.

Thm 3.9 A compact subspace C of $\mathbb{R}^n$ is closed and bounded.

Rmk Thm 3.1 states "$\Longleftrightarrow$", which we prove in §3.4.

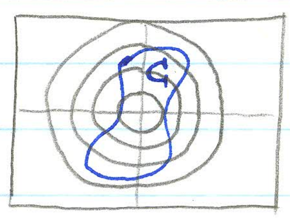
Pf C is closed by Thm 3.6 since $\mathbb{R}^n$ is Hausdorff.

Note $C \subseteq \mathbb{R}^n = \bigcup_{k \geq 1} B(\vec{O}, k)$

open ball of radius k about $\vec{O} \in \mathbb{R}^n$

C compact $\Rightarrow$ C is contained in a finite # of such balls (in fact one since they're nested)

$\Rightarrow$ C bounded.



<u>Thm 3.10</u> If $f: X \to \mathbb{R}$ is continuous and X is compact, then f is bounded and attains its bounds.

<u>Pf</u> $f(X)$ is compact by Thm 3.4, hence closed and bounded by Thm 3.9.

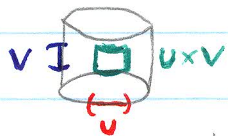
Since $f(X)$ is closed, $\sup(f) \in f(X)$ and $\inf(f) \in f(X)$, i.e. we can find $x_1, x_2 \in X$ with $f(x_1) = \sup(f)$, $f(x_2) = \inf(f)$.

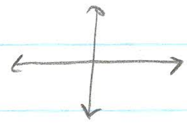
§3.4 Product spaces

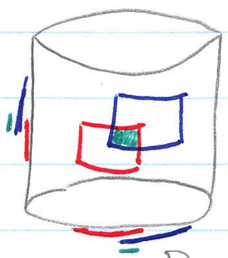
Let X and Y be topological spaces.

$X \times Y = \{(x,y) \mid x \in X,\ y \in Y\}$

Ex $S^1 \times [0,1]$ $\qquad$ $S^1 \times S^1$ $\qquad$ $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$



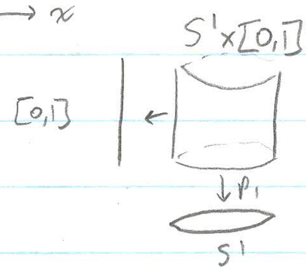




To put a topology on $X \times Y$, let
$\beta = \{ U \times V \mid U \subseteq X \text{ and } V \subseteq Y \text{ are open}\}$.
Note $\bigcup \beta = X \times Y$ (take $U = X$ and $V = Y$) and
the intersection of any finite # of sets in β is in β
$[(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \cap (V_1 \cap V_2)]$. Hence by
Thm 2.5,

Def The topology on $X \times Y$ with base β
(an open is a union of sets in β)
is the <u>product topology</u>.

Rmk Note we have projections
$p_1 : X \times Y \longrightarrow X$ and $p_2 : X \times Y \longrightarrow Y$
$\quad (x,y) \longmapsto x \qquad\qquad (x,y) \longmapsto y$



Thm 3.12 (i) p_1 and p_2 are continuous ($A \subseteq X$ open $\Rightarrow p_1^{-1}(A) \subseteq X \times Y$ open)
(ii) p_1 and p_2 map open sets to open sets
($A \subseteq X \times Y$ open $\Rightarrow p_1(A) \subseteq X$ open)
(iii) The product topology is the smallest topology on $X \times Y$
(fewest open sets) for which p_1 and p_2 are both continuous.

Pf (i) To see p_1 is continuous, note if $A \subseteq X$ is open then so is $p_1^{-1}(A) = A \times Y$. Similar for p_2.

(ii) Let $U \times V \in \beta$.

Then $p_1(U \times V) = U$ is open in X.

Any open set A in $X \times Y$ is a union of sets in β, hence $p_1(A)$ will be a union of open sets in X, which is open.

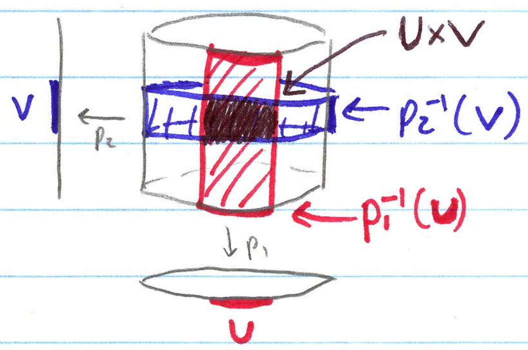
(iii) Fix an arbitrary topology on $X \times Y$.

If p_1, p_2 are continuous, then for any opens $U \subseteq X$ and $V \subseteq Y$ we have

$p_1^{-1}(U) = U \times Y$ is open in $X \times Y$

$p_2^{-1}(V) = X \times V$ is open in $X \times Y$

Hence $U \times V = p_1^{-1}(U) \cap p_2^{-1}(V)$ is open in $X \times Y$.



This topology contains as open sets all sets in β, and hence all open sets in the product topology.

Thm 3.13 A function $f: Z \to X \times Y$ is continuous $\iff$ $p_1 f: Z \to X$ and $p_2 f: Z \to Y$ are both continuous

Pic

$$\begin{tikzcd} & & X \\ Z \arrow[r, "f"] \arrow[urr, bend left, "p_1 f"] \arrow[drr, bend right, "p_2 f"'] & X \times Y \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & \\ & & Y \end{tikzcd}$$

Rmk f is often denoted $f = (f_1, f_2)$, where $f_1 = p_1 f$ and $f_2 = p_2 f$ are the coordinate functions.

Pf ($\Rightarrow$) f continuous $\Rightarrow$ $p_1 f$, $p_2 f$ continuous since p_1 and p_2 are continuous, and composition preserves continuity

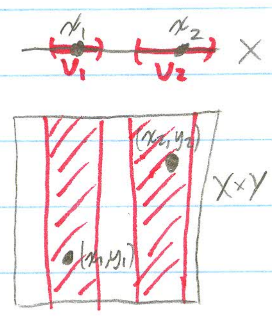
($\Leftarrow$) Let $U \times V \subseteq X \times Y$ be in β. Note

$$f^{-1}(U \times V) = (p_1 f)^{-1}(U) \cap (p_2 f)^{-1}(V)$$

is open in Z (as the intersection of two opens).
Hence the preimage of <u>any</u> open set in $X \times Y$ will be the union of open sets in Z, hence open in Z.

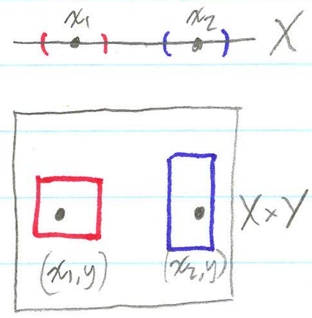
Thm 3.14 Product space $X \times Y$ is Hausdorff $\iff$ X and Y are both Hausdorff.

Pf ($\Leftarrow$) Let $(x_1, y_1) \neq (x_2, y_2)$.



Then $x_1 \neq x_2$ or $y_1 \neq y_2$ (or both).
WLOG, assume the former.
Since X is Hausdorff, we can find disjoint open sets $U_1 \ni x_1$, $U_2 \ni x_2$.
Note $U_1 \times Y$ and $U_2 \times Y$ are disjoint open sets about (x_1, y_1) and (x_2, y_2) in $X \times Y$.

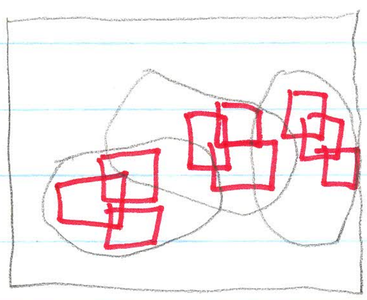
($\Longrightarrow$) Given $x_1, x_2 \in X$ with $x_1 \neq x_2$, pick any $y \in Y$. Since $X \times Y$ Hausdorff, we can find basic disjoint open sets $U_1 \times V_1 \ni (x_1, y)$, $U_2 \times V_2 \ni (x_2, y)$. Then U_1, U_2 are disjoint open sets about x_1, x_2.



Lemma 3.16 Let β be a base for topological space X. Then X compact $\Longleftrightarrow$ every open cover of X by sets in β has a finite subcover.

Pf ($\Rightarrow$) Clear.

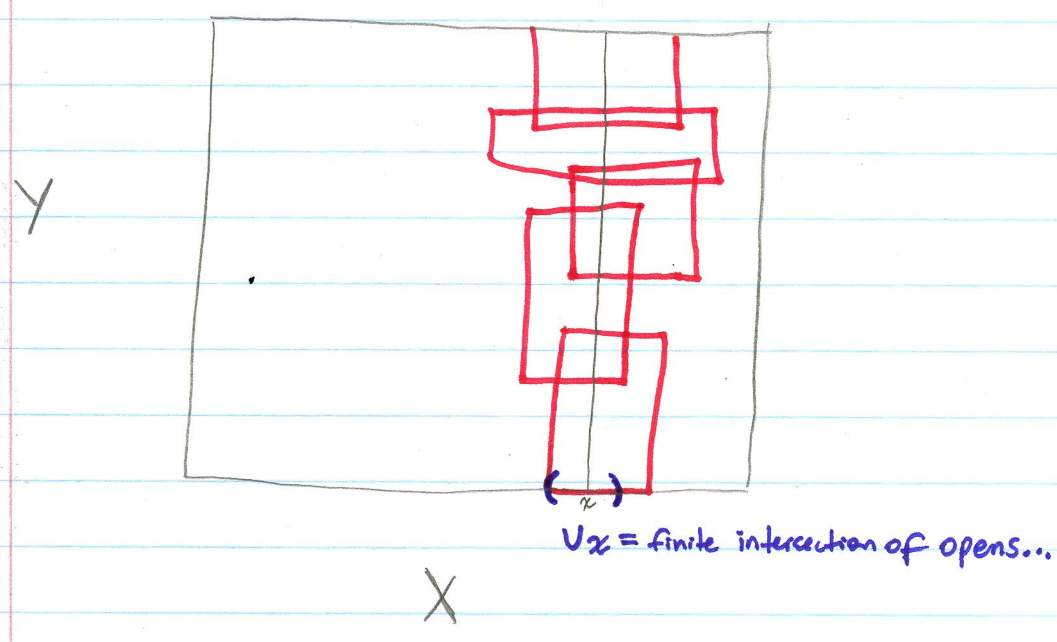
($\Leftarrow$) Let $\mathcal{F}$ be an arbitrary open cover of X. Write each set in $\mathcal{F}$ as a union of sets in β; let β' be the collection of basic open sets used. We have $\bigcup \beta' = \bigcup \mathcal{F} = X$. By assumption β' has a finite subcover β''. Choose one set in $\mathcal{F}$ containing each set in β''; this is a finite subcover of $\mathcal{F}$.



Thm 3.15 $X \times Y$ is compact $\Longleftrightarrow$ both X and Y are compact.

Pf of ($\Rightarrow$) Note $p_1 : X \times Y \to X$, $p_2 : X_2 \times Y \to Y$ are continuous and surjective; hence X and Y are compact by Thm 3.4.

Picture of ($\Leftarrow$) Let $\mathcal{F}$ be an open cover of $X \times Y$ by basic open sets (we will use Lemma 3.16).



Thm 3.1 A subset X of $\mathbb{R}^n$ is compact $\Leftrightarrow$ X is closed & bounded.

Pf ($\Rightarrow$) was Thm 3.9

($\Leftarrow$) X bounded implies $\exists s > 0$ s.t.

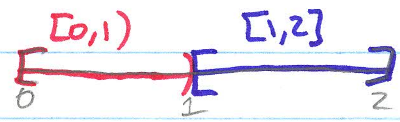
$$X \subseteq \underbrace{[-s,s] \times \dots \times [-s,s]}_{n \text{ copies}}.$$

The Heine-Borel theorem (3.3) says $[-s,s]$ is compact, hence so is $[-s,s] \times \dots \times [-s,s]$ by Thm 3.15. So X is a closed subset of compact space $[-s,s] \times \dots \times [-s,s]$ and therefore compact by Thm 3.5.

§3.5 Connectedness

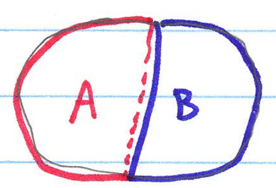
Intuitively, a topological space is connected if it is "all in one piece".

Def 3.17 A space X is <u>connected</u> if whenever it is decomposed as $X = A \cup B$ with $A, B \neq \emptyset$, then either $\overline{A} \cap B \neq \emptyset$ or $A \cap \overline{B} \neq \emptyset$.

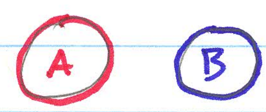


Ex We'll see $[0,2]$ is connected.
Note $[0,2] = [0,1) \cup [1,2]$ and $\overline{[0,1)} \cap [1,2] = \{1\} \neq \emptyset$.

<u>Typical definition (Thm 3.20(c))</u> A space X is <u>connected</u> if it cannot be expressed as the union of two disjoint open sets.



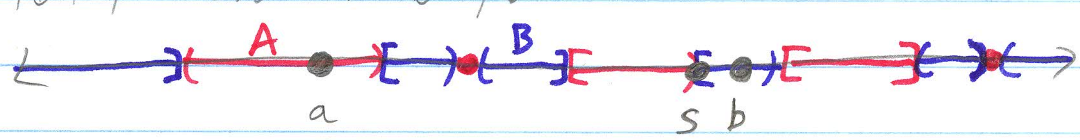
X connected



X not connected

Thm 3.18 $\mathbb{R}$ is connected.

Pf We'll use Def 3.17. Suppose $\mathbb{R} = A \cup B$ with $A, B \neq \emptyset$ and $A \cap B = \emptyset$.



WLOG, choose $a \in A$ and $b \in B$ with $a < b$.
Let $X = \{a' \in A \mid a' < b\}$ and $s = \sup X$. [Do you see why s exists?]
<u>Case I: $s \in A$.</u> Note $(s, b] \subseteq B$, hence s is a limit point of B and $A \cap \overline{B} \neq \emptyset$.
<u>Case II: $s \notin A$.</u> Then $s \in B$, and also $s = \sup X$ is a limit point of A. So $\overline{A} \cap B \neq \emptyset$.

Thm 3.19 A nonempty subset of $\mathbb{R}$ is connected $\iff$ it is an interval $\left(\begin{array}{l} (a,b), [a,b), (a,b], [a,b] \\ (-\infty,b), (-\infty,b], (a,\infty), \text{or } [a,\infty) \end{array} \right)$

Pf Omitted

Thm 3.20 The following conditions on space X are equivalent:

(a) X is connected

(b) The only subsets of X which are both open and closed are X and ϕ.

(c) X cannot be expressed as the union of two disjoint nonempty open sets.

(d) There is no surjective continuous $f: X \to Y$ where Y is a discrete space with more than one point.

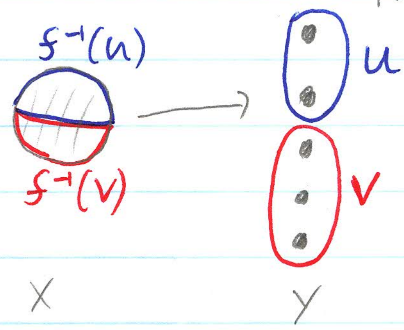
Pf We'll show (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (a).

(a) $\Rightarrow$ (b) Let A be both open and closed in X. Let $B = X \setminus A$; note B is both open and closed. Hence $\bar{A} = A$ and $\bar{B} = B$, so $\bar{A} \cap B = A \cap \bar{B} = A \cap B = \phi$. Then (a) gives $A = \phi$ or $B = \phi$, so $A = \phi$ or $A = X$.

(b) $\Rightarrow$ (c) Clear.

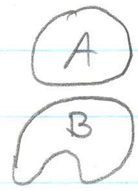
(c) $\Rightarrow$ (d) Assume (c). Suppose for a contradiction $\exists$ surjective continuous $f: X \to Y$ with Y discrete and $|Y| \geq 2$.

$$X \xrightarrow{f} Y$$



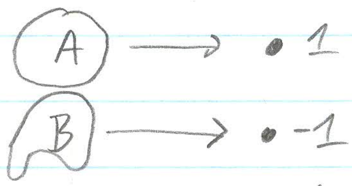
By assumption, we can write $Y = U \cup V$ with $U \cap V = \emptyset$ and both U, V open in Y.
Then $X = f^{-1}(U) \cup f^{-1}(V)$ is the disjoint union of two disjoint nonempty open sets in X, contradicting (c).

(d) $\Rightarrow$ (a) Assume (d). Suppose for a contradiction X is not connected. Then $X = A \cup B$ with $A, B \neq \emptyset$ and $\bar{A} \cap B = A \cap \bar{B} = \emptyset$.

$X =$ 

Note $A = X \setminus \bar{B}$ is open in X, as is $B = X \setminus \bar{A}$.
Define $f : X \to \{-1, 1\}$ via

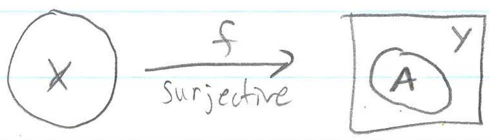
$$f(x) = \begin{cases} -1 & \text{if } x \in A \\ 1 & \text{if } x \in B. \end{cases}$$



Then f is continuous and onto, contradicting (d).

Thm 3.21 The continuous image of a connected space is connected.

Pf Let $f : X \to Y$ be continuous and surjective with X connected.
Let A be both open and closed in Y.
Then f continuous $\Rightarrow$ $f^{-1}(A)$ is both open and closed in X.
X connected $\Rightarrow$ $f^{-1}(A) = \emptyset$ or X (Thm 3.20 (b)).
Hence $A = \emptyset$ or $A = Y$.
So Y is connected by Thm 3.20(b).



Corollary 3.22 If X and Y are homeomorphic, then X is connected $\iff$ Y is connected.

Def A subset $Z \subseteq X$ is <u>dense</u> in X when $\overline{Z} = X$

Rmk Z is dense in $X \iff Z$ intersects every nonempty open set in X.

Ex $\mathbb{Q}$ is dense in $\mathbb{R}$ 

$(0,1)$ is dense in $[0,1]$

$\{(x,y) \mid x^2+y^2<1\}$ is dense in $\{(x,y) \mid x^2+y^2 \le 1\}$

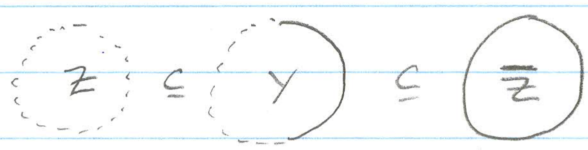
Thm 3.23 Let $Z \subseteq X$. If Z is connected and dense in X, then X is connected. <u>Ex</u> Z = open ball, X = closed ball.

Pf Let $A \neq \phi$ be an open and closed subset of X.
Z dense $\Rightarrow Z \cap A \neq \phi$.
Since $Z \cap A$ is both open and closed in Z, and Z is connected, this shows $Z \cap A = Z$, i.e. $Z \subseteq A$.
Hence $X = \overline{Z} \subseteq \overline{A} = A$, giving $X = A$ as required.

Corollary 3.24 Let X be a topological space and $Z \subseteq Y \subseteq \overline{Z}$.
If Z is connected, then so is Y.
(In particular, if Z is connected then so is $\overline{Z}$).

Pf Note the closure of Z in Y is all of Y.
Apply Thm 3.23 to $Z \subseteq Y$.

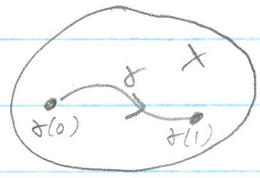
<u>Ex</u>



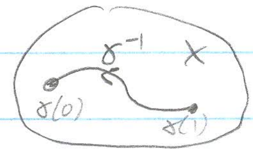
Skipping Theorems 3.25, 3.26, 3.27 for now.

§3.6 Joining points by paths

Def A path in a topological space X is a continuous function $\gamma:[0,1]\to X$.



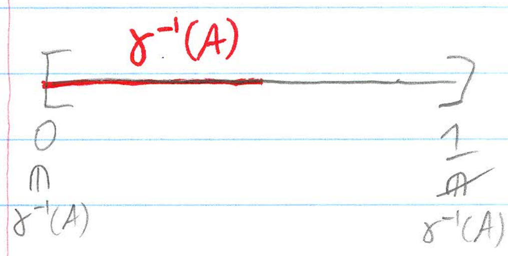
Rmk Note $\gamma^{-1}:[0,1]\to X$ defined by $\gamma^{-1}(t)=\gamma(1-t)$ is a path from $\gamma(1)$ to $\gamma(0)$

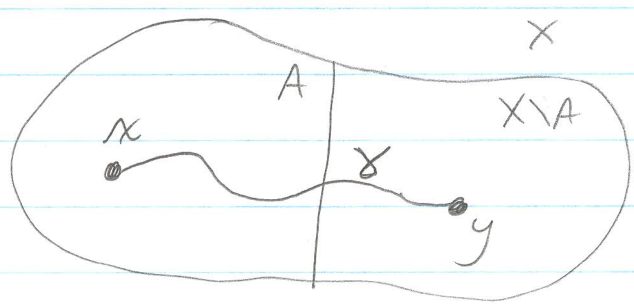


Def A space is path-connected if any two of its points can be joined by a path.

Thm 3.29 If X is path-connected, then X is connected.

Pf Let $A\subseteq X$ with $A\neq\emptyset$ be both open and closed.
We must show $A=X$.
Suppose for a contradiction $A\neq X$.
Then we can choose $x\in A$ and $y\in X\setminus A$.
By assumption $\exists$ path $\gamma:[0,1]\to X$ with $\gamma(0)=x$, $\gamma(1)=y$.
Then $\gamma^{-1}(A)$ is a nonempty proper subset of $[0,1]$ which (by continuity of γ) is both open and closed.
This contradicts the fact that $[0,1]$ is connected.
Hence $A=X$ as required, and X is connected.



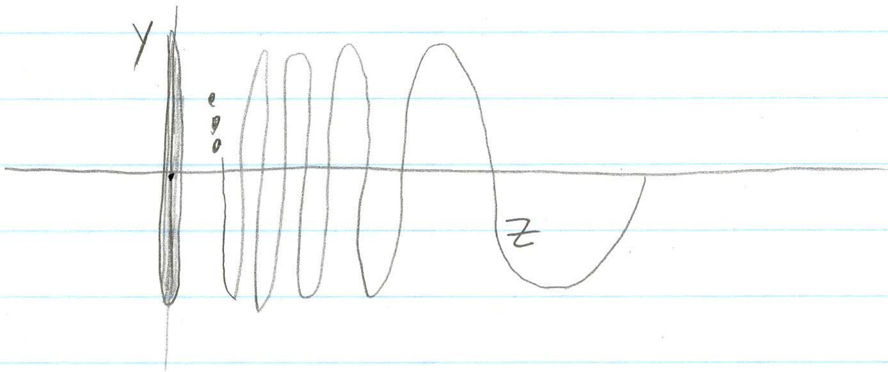


Rmk Not every connected space is path-connected

Ex Let $Y = \{(0,y) \in \mathbb{R}^2 \mid -1 \le y \le 1\}$

$Z = \{(x, \sin\frac{\pi}{x}) \in \mathbb{R}^2 \mid 0 < x \le 1\}$

Then $X = Y \cup Z$ is connected but not path-connected



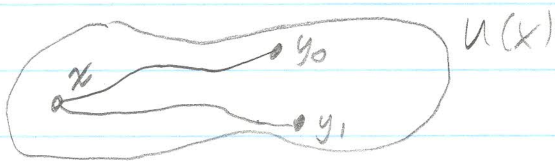
To see X is connected, note Z is connected and $\overline{Z} = X$; hence X is connected by Corollary 3.24. [Z connected $\Rightarrow \overline{Z}$ connected]

To see X is not path-connected, our book verifies that one cannot join any point $z \in Z$ with any point $y \in Y$ via a path in X.

Thm 3.30 A connected open subset X of Euclidean space $\mathbb{R}^n$ is path-connected.

Pf Given $x \in X$, let $U(x) = \{y \in X \mid \exists \text{ path in } X \text{ joining } x \text{ to } y\}$

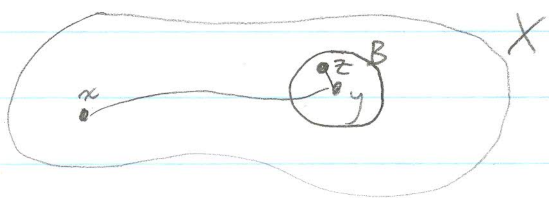
Note $U(x)$ is path-connected, since if $y_0, y_1 \in U(x)$, then we can join y_0, y_1 by a path through x:



Our strategy is to show $U(x)=X$ [for any fixed $x \in U(x)$]. Then X will be path-connected since $U(x)$ is. To do this, we will show $U(x)$ is nonempty, open, and closed in X, which implies $U(x)=X$ since X is connected.

To see $U(x)$ is nonempty, note $x \in U(x)$.

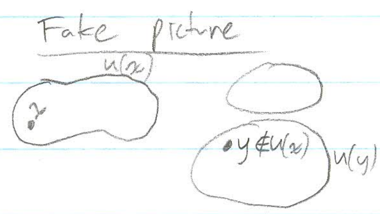
To see $U(x)$ is open in X, let $y \in U(x) \subseteq X$.
X open $\Rightarrow \exists$ open ball B with $y \in B \subseteq X$.



If $z \in B$, then $\exists$ straight-line path in B from y to B.
This shows $y \in B \subseteq U(x)$.
Hence $U(x)$ is open in X.

To see $U(x)$ is closed in X, note

$$X \setminus U(x) = \bigcup_{y \in X \setminus U(x)} U(y).$$



Each $U(y)$ is open by the argument above, so $X \setminus U(x)$ is open and $U(x)$ is closed. $\square$

Chapter 5 The fundamental group

For X a topological space, fundamental group $\pi_1(X)$ roughly speaking measures the "# of 1-dim'l holes in X".

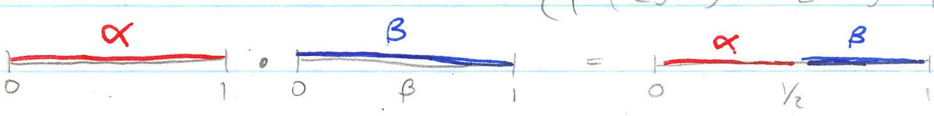
[Homotopy group $\pi_k(X)$ roughly measures the "k-dim'l holes".]

§ 5.1 Homotopic maps

Let $* \in X$ be a designated point.

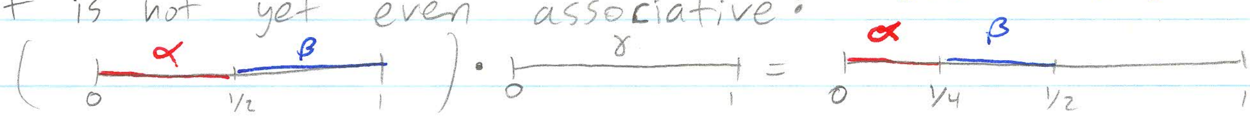
A loop in X based at $*$ is a map $\alpha : [0,1] \to X$ with $\alpha(0) = * = \alpha(1)$.

Given two such loops α, β, their product is the loop $\alpha . \beta : [0,1] \to X$ defined by
$$\alpha . \beta(s) = \begin{cases} \alpha(2s) & 0 \le s \le \frac{1}{2} \\ \beta(2s-1) & \frac{1}{2} \le s \le 1 \end{cases}$$

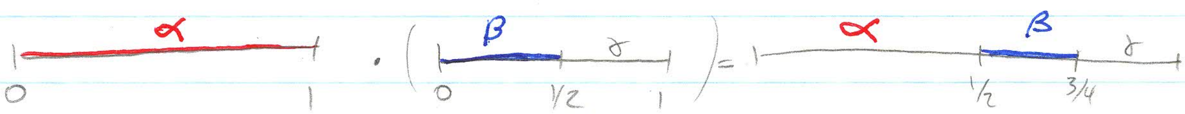


This product does not yet give a group structure: [$(ij)k \overset{?}{=} i(jk)$]
it is not yet even associative:

$(\alpha . \beta) . \gamma$



$\alpha . (\beta . \gamma)$



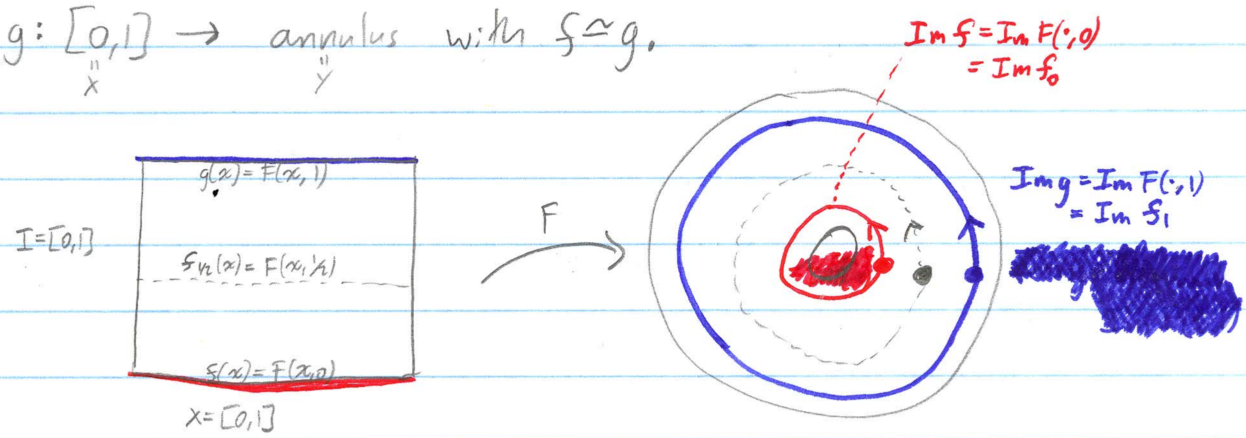
We will define $\pi_1(X)$ to be the "homotopy classes" of based loops [equiv. based maps $S^1 \to X$] under this will-be product structure.

[$\pi_k(X)$ will be the "homotopy classes" of based maps $S^k \to X$.]

Def $I = [0,1]$

Def 5.1 Let $f, g : X \to Y$. Then f and g are homotopic (denoted $f \simeq g$) if $\exists$ continuous $F : X \times I \to Y$ with
$F(x,0) = f(x)$ and $F(x,1) = g(x)$ $\forall x \in X$.

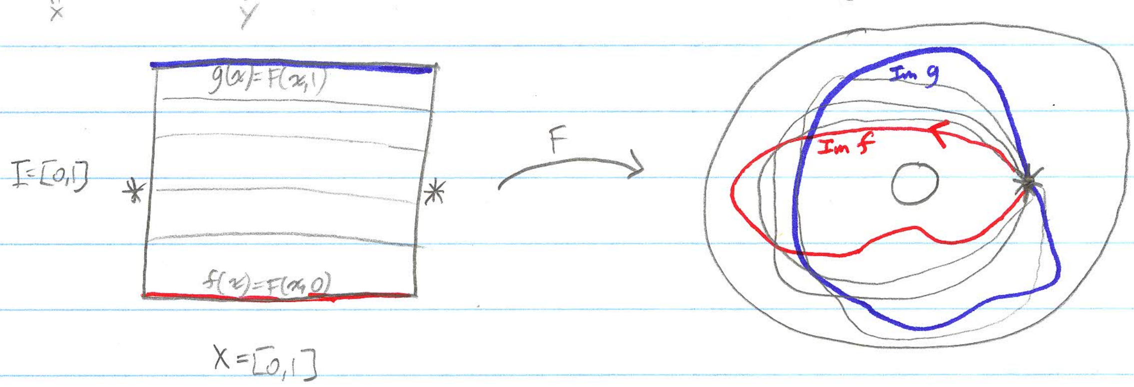
Ex $f, g : \underset{X}{\underset{\shortparallel}{[0,1]}} \to \underset{Y}{\underset{\shortparallel}{\text{annulus}}}$ with $f \simeq g$.



Notation We often denote $F(\cdot, t) : X \to Y$ by $f_t : X \to Y$ [i.e. $F(x,t) = f_t(x)$]. We think of F as a continuous family of maps $f_t : X \to Y$ from $f_0 = f$ to $f_1 = g$.

Def If there is some $A \subseteq X$ with $f|_A = g|_A$, and if $\exists$ some homotopy $F : X \times I \to Y$ with
$F(a,t) = f(a) = g(a)$ $\forall a \in A$, then we say f and g are homotopic relative A (denoted $f \simeq g$ rel A).

Ex $f, g : \underset{X}{\underset{\shortparallel}{[0,1]}} \to \underset{Y}{\underset{\shortparallel}{\text{annulus}}}$ with $A = \{0,1\}$ and $f \simeq g$ rel A.

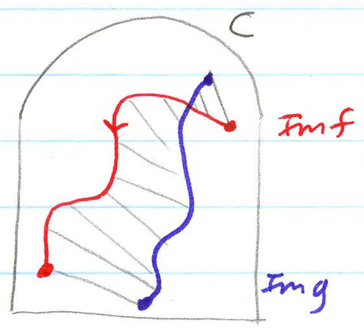


Examples of homotopies

- Let $f, g : X \to C$ where C is a convex subset of $\mathbb{R}^n$ (if two points are in C, then so is the line segment between them).

Pic

$X = [0,1]$

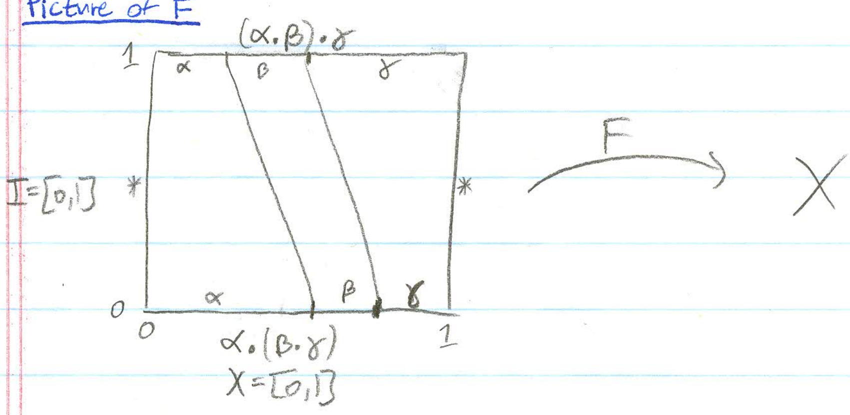


The straight-line homotopy between f and g is defined by $F : X \times I \to C$ via $F(x,t) = (1-t)f(x) + t\,g(x)$.

- $\alpha, \beta, \gamma : [0,1] \to X$ are loops based at $* \in X$.

Then $\alpha\cdot(\beta\cdot\gamma) \simeq (\alpha\cdot\beta)\cdot\gamma$ rel $\{0,1\}$ via some homotopy F.

Picture of F

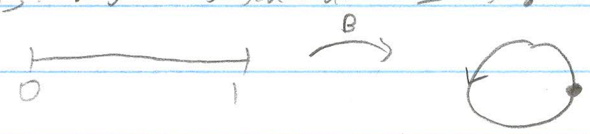


We'll prove this later.

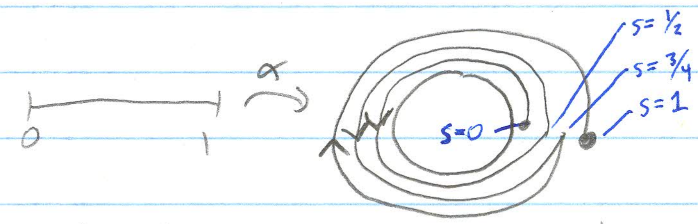
- Let S^1 be the unit circle in the complex plane.

Consider loops $\alpha, \beta : [0,1] \to S^1$ based at $1 \in S^1$.

$$\beta(s) = e^{2\pi i s}$$



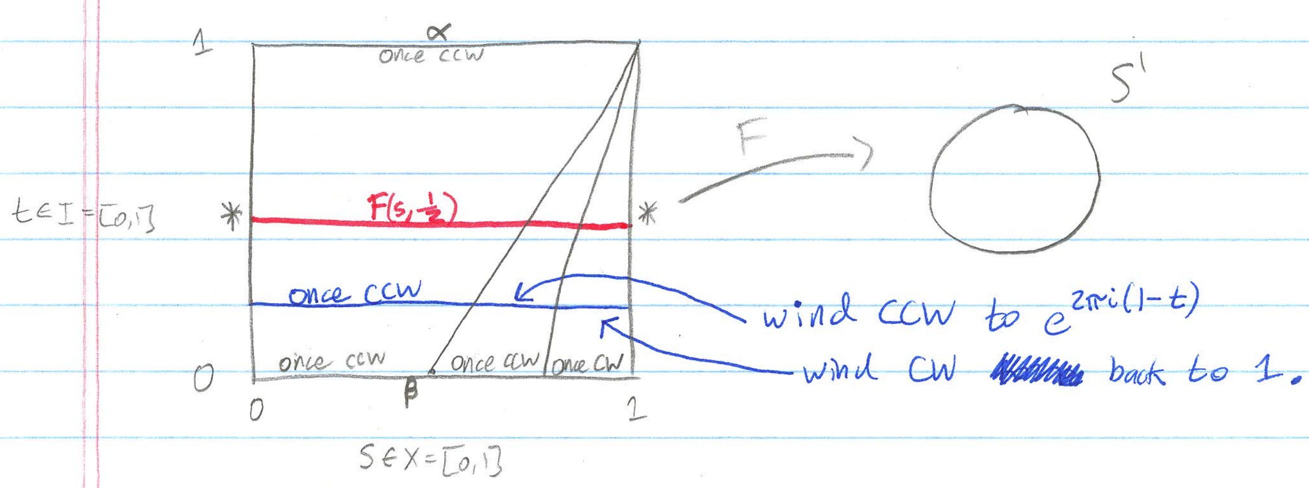
$$\alpha(s) = \begin{cases} e^{4\pi i s} & 0 \le s \le \frac{1}{2} \\ e^{4\pi i (2s-1)} & \frac{1}{2} \le s \le \frac{3}{4} \\ e^{8\pi i (1-s)} & \frac{3}{4} \le s \le 1 \end{cases}$$

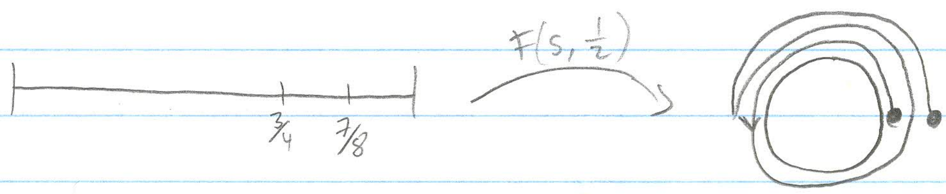


Note α wraps CCW around S^1, then CCW around S^1, then CW around S^1.

Then $\beta \simeq \alpha$ rel $\{0,1\}$ via some homotopy F.

<u>Picture of F</u>





See our book for an (ugly) explicit formula for F.

We will gradually get used to such formulas.

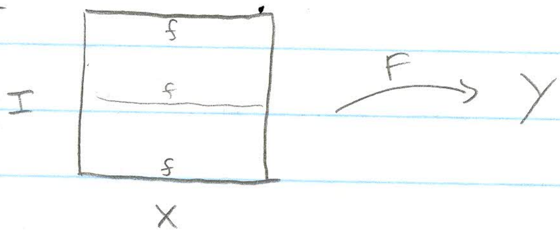
Lemma 5.2 The relation of homotopy ($f \simeq g$) is an <u>equivalence relation</u> on the set of all maps from X to Y.

Pf We must show homotopy equivalence is reflexive, symmetric, and transitive.

Let $f, g, h: X \to Y$

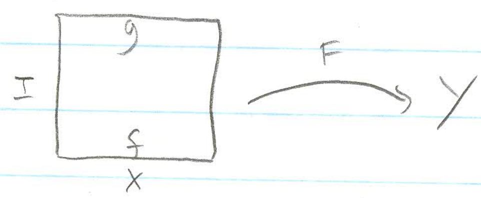
<u>Reflexive</u> Note $f \simeq f$ via $F: X \times I \to Y$ defined by $F(x,t) = f(x)$

<u>Pic</u>

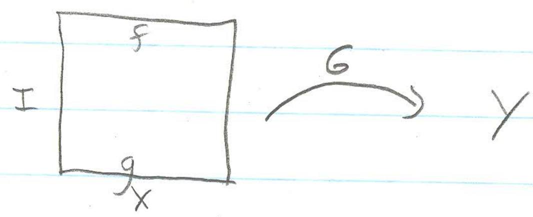


<u>Symmetric</u> If $f \simeq g$, we must show $g \simeq f$.

Suppose $f \simeq g$ via a homotopy $F: X \times I \to Y$ with $F(x,0) = f(x)$ and $F(x,1) = g(x)$

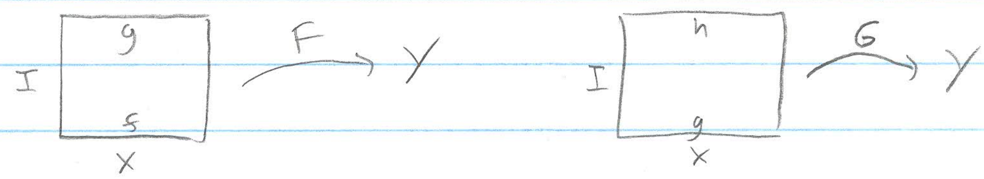


Then $g \simeq f$ via a homotopy $G: X \times I \to Y$ defined by $G(x,t) = F(x, 1-t)$



<u>Transitive</u> If $f \simeq g$ and $g \simeq h$, we must show $f \simeq h$.
Suppose $f \simeq g$ via a homotopy F
and $g \simeq h$ via a homotopy G

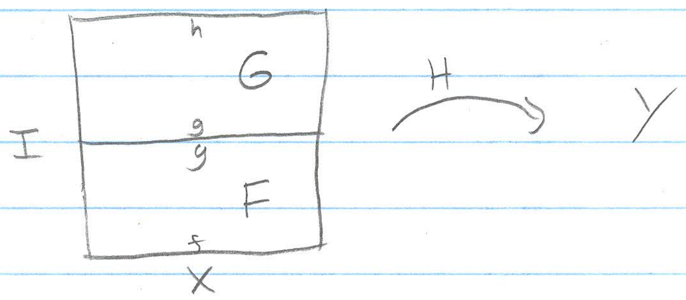
<u>Pic</u>



Then $f \simeq h$ via a homotopy H defined by

$$H(x,t) = \begin{cases} F(x,2t) & 0 \le t \le 1/2 \\ G(x,2t-1) & 1/2 \le t \le 1 \end{cases}$$

<u>Pic</u>



<u>Lemma 5.3</u> The relation $f \simeq g$ rel A is also an equivalence relation

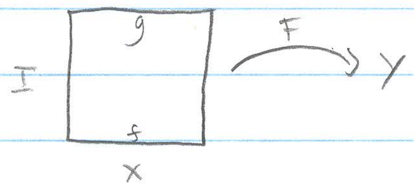
<u>Pf sketch</u> If all given maps f, g, h fix A and if all given homotopies fix A, then so do all the new homotopies we constructed above.

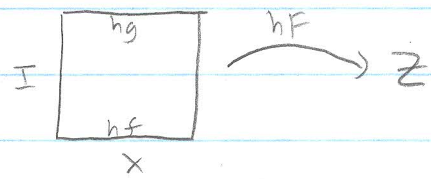
<u>Lemma 5.4</u> (i) If $f, g : X \to Y$ with $f \simeq g$ and $h : Y \to Z$, then $hf \simeq hg$.
(ii) If $f : X \to Y$ and $g, h : Y \to Z$ with $g \simeq h$, then $gf \simeq hf$.
IE, "homotopy behaves well w.r.t. composition."

(i) $\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\simeq"] & Y \arrow[r, "h"] & Z \end{tikzcd} \Rightarrow hf \simeq hg$

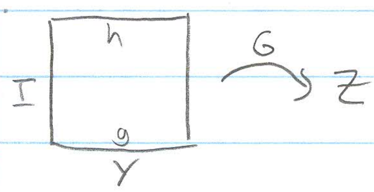
(ii) $\begin{tikzcd} X \arrow[r, "f"'] & Y \arrow[r, bend left, "g"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\simeq"] & Z \end{tikzcd} \Rightarrow gf \simeq hf$

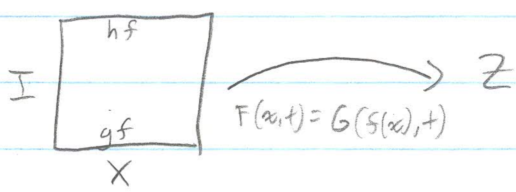
Pf (i) $f \simeq g$ gives $F : X \times I \to Y$ with $F(\cdot, 0) = f$, $F(\cdot, 1) = g$.
Note $hF : X \times I \to Z$ is a homotopy from $hF(\cdot, 0) = hf$ to $hF(\cdot, 1) = hg$.





(ii) $g \simeq h$ gives $G : Y \times I \to Z$ with $G(y, 0) = g(y)$, $G(y, 1) = h(y)$.
Note $F : X \times I \to Z$ via $F(x, t) = G(f(x), t)$ is a homotopy from $F(x, 0) = G(f(x), 0) = gf(x)$ to $F(x, 1) = G(f(x), 1) = hf(x)$.

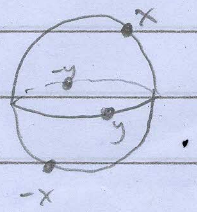


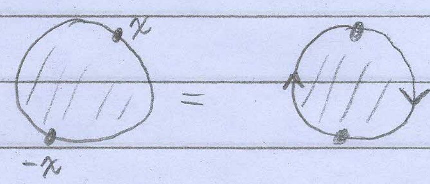


The projective plane

The projective plane $\mathbb{R}P^2$ (or $\mathbb{P}^2$) parametrizes the set of all lines (unoriented) in $\mathbb{R}^3$.

$$\mathbb{R}P^2 = S^2 / x \sim -x \qquad \mathbb{R}P^2 = D^2 / \substack{x \sim -x \\ \text{for } x \in \partial D^2}$$





$\mathbb{R}P^2$ is an unorientable surface.

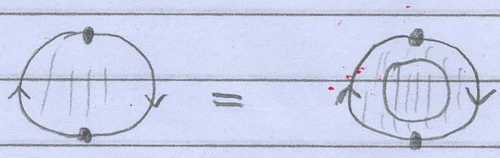
More generally, projective space $\mathbb{R}P^n$ parametrizes the set of all lines in $\mathbb{R}^{n+1}$.

$$\mathbb{R}P^n = S^n / x \sim -x \qquad \mathbb{R}P^n = D^n / \substack{x \sim -x \\ \text{for } x \in \partial D^n}$$

$\mathbb{R}P^n$ is an n-dimensional manifold.

<u>Ex</u> $\mathbb{R}P^1 = S^1 / x \sim -x \cong S^1$

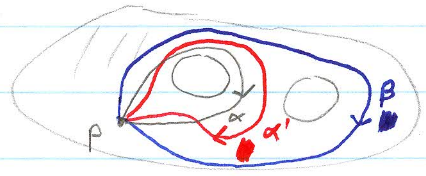
<u>Recall</u> $\mathbb{R}P^2 =$ , a sphere with one disk removed and a Möbius band glued in.



Note outer ring is a cylinder with antipodal points identified on one boundary circle, i.e. a Möbius band!

§5.2 Construction of the fundamental group

Let X be a topological space and $p \in X$.
Given a loop α in X based at p
(meaning $\alpha : [0,1] \to X$ with $\alpha(0) = p = \alpha(1)$),
let $\langle \alpha \rangle$ denote the equivalence class of
homotopic maps, rel $\{0,1\}$ (fixing endpoints).



Pic $\langle \alpha \rangle$ represents both α and α' but not β

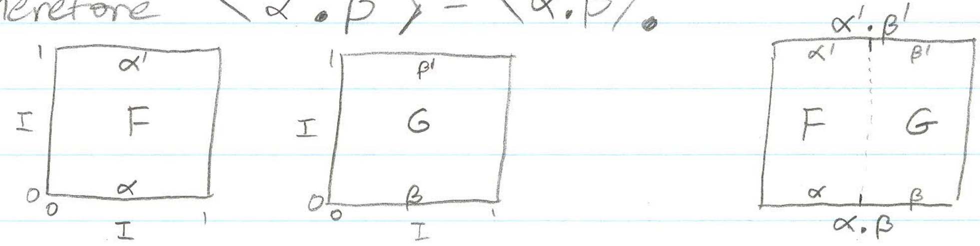
Concatenation of loops defines a product on homotopy classes:
$\langle \alpha \rangle \circ \langle \beta \rangle = \langle \alpha . \beta \rangle$.
To see this multiplication is well-defined, note
if $\alpha \underset{F}{\simeq} \alpha'$ rel $\{0,1\}$ and $\beta \underset{G}{\simeq} \beta'$ rel $\{0,1\}$,
then $\alpha . \beta \underset{H}{\simeq} \alpha' . \beta'$ rel $\{0,1\}$, where

$$H(s,t) = \begin{cases} F(2s,t) & 0 \le s \le \frac{1}{2} \\ G(2s-1,t) & \frac{1}{2} \le s \le 1. \end{cases}$$

[Recall $\alpha.\beta(s) = \begin{cases} \alpha(2s) & 0 \le s \le \frac{1}{2} \\ \beta(2s-1) & \frac{1}{2} \le s \le 1 \end{cases}$]

Therefore $\langle \alpha' . \beta' \rangle = \langle \alpha . \beta \rangle$.

Pic



Def Let fundamental group (or 1st homotopy group) $\pi_1(X,p)$ be the set of homotopy classes of loops in X based at p, with multiplication $\langle\alpha\rangle\cdot\langle\beta\rangle = \langle\alpha\cdot\beta\rangle$.

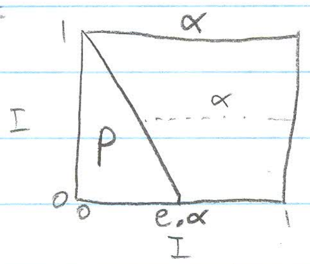
Thm 5.5 $\pi_1(X,p)$ is indeed a group.

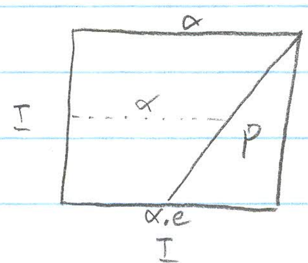
Fact (☆) Since I is convex, if $f,g: I\to I$ with $f(0)=g(0)$ and $f(1)=g(1)$, then there is a straight-line homotopy between f and g rel $\{0,1\}$.

$$[F(x,t)=(1-t)f(x)+tg(x)]$$

Pf of Thm Identity element Let $e:[0,1]\to X$ be the constant loop $e(s)=p$ $\forall s\in[0,1]$. We need $\langle e\rangle\cdot\langle\alpha\rangle=\langle\alpha\rangle=\langle\alpha\rangle\cdot\langle e\rangle$.

Pic





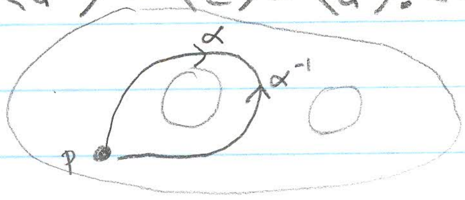
One could write down formulas for these homotopies. Or, to see $e\cdot\alpha\simeq\alpha$, note $e\cdot\alpha=\alpha\circ f$, where $f: I\to I$ via
$$f(s)=\begin{cases}0 & 0\le s\le\frac{1}{2}\\ 2s-1 & \frac{1}{2}\le s\le 1.\end{cases}$$

Hence
$$\begin{aligned} e\cdot\alpha &= \alpha\circ f \\ &\simeq \alpha\circ id_I \quad \text{rel } \{0,1\} \text{ by (☆) and Lemma 5.4} \\ &= \alpha. \end{aligned}$$

Similarly for $\alpha\cdot e\simeq\alpha$.

Inverses

The inverse of $\langle \alpha \rangle$ is $\langle \alpha^{-1} \rangle$, where $\alpha^{-1}(s) = \alpha(1-s)$.
We need $\langle \alpha \rangle . \langle \alpha^{-1} \rangle = \langle e \rangle = \langle \alpha^{-1} \rangle . \langle \alpha \rangle$.



Here Wed →

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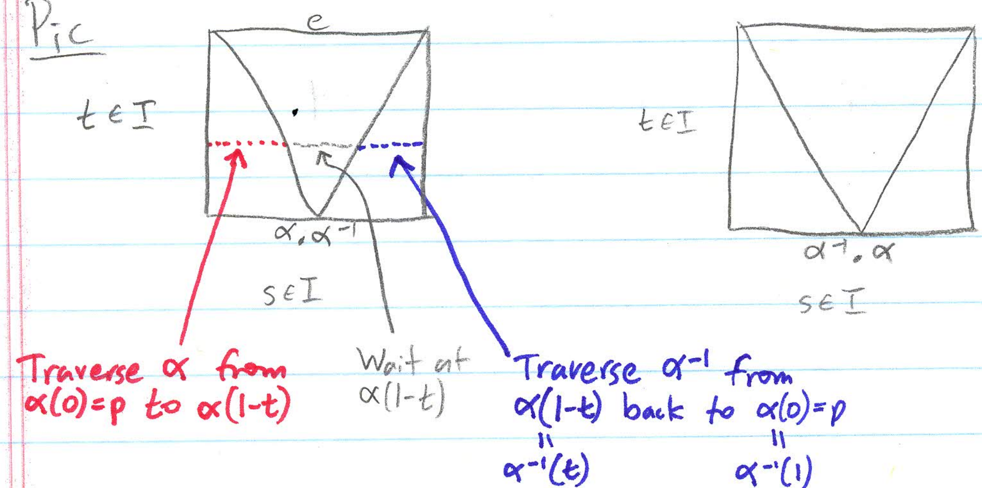


One could write down formulas for these homotopies.
Or, to see $\alpha . \alpha^{-1} \simeq e$, note $\alpha . \alpha^{-1} = \alpha \circ f$, where $f : I \to I$ via
$$f(s) = \begin{cases} 2s & 0 \le s \le \frac{1}{2} \\ 2-2s & \frac{1}{2} \le s \le 1. \end{cases}$$

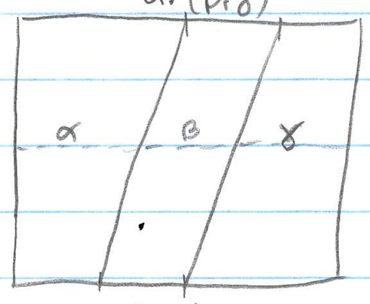
If $g : I \to I$ via $g(s) = 0$ is a constant map, note $f \simeq g$ rel $\{0,1\}$ by ($\star$).
Hence
$$\begin{aligned} \alpha . \alpha^{-1} &= \alpha \circ f \\ &\simeq \alpha \circ g \text{ rel } \{0,1\} \text{ by lemma 5.4} \\ &= e. \end{aligned}$$

Similarly for $\alpha^{-1} . \alpha \simeq e$.

Associativity For any three loops α, β, γ based at p, we must show $\langle \alpha.\beta \rangle . \langle \gamma \rangle = \langle \alpha \rangle . \langle \beta.\gamma \rangle$, ie

$$(\alpha.\beta).\gamma \simeq \alpha.(\beta.\gamma) \text{ rel } \{0,1\}.$$

Pic



Note $(\alpha.\beta).\gamma = (\alpha.(\beta.\gamma)) \circ f$, where $f: I \to I$ via

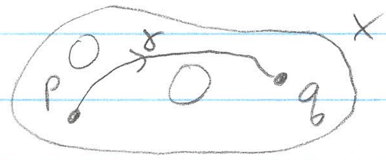
$$f(s) = \begin{cases} 2s & 0 \le s \le \frac{1}{4} \\ s + \frac{1}{4} & \frac{1}{4} \le s \le \frac{1}{2} \\ \frac{s+1}{2} & \frac{1}{2} \le s \le 1 \end{cases}$$

Hence
$$\begin{aligned} (\alpha.\beta).\gamma &= (\alpha.(\beta.\gamma)) \circ f \\ &\simeq (\alpha.(\beta.\gamma)) \circ id_I \text{ rel } \{0,1\} \text{ by } (\star) \text{ and Lemma 5.4} \\ &= \alpha.(\beta.\gamma) \end{aligned}$$

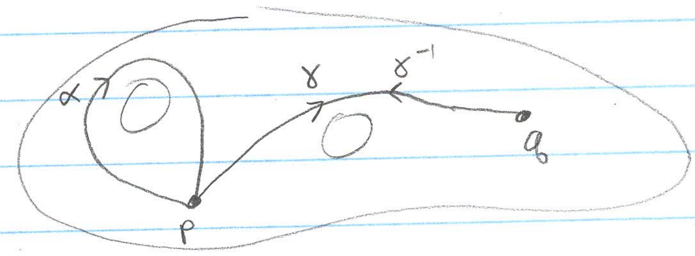
This completes the proof of Theorem 5.5

Thm 5.6 If X is path-connected then $\pi_1(X,p)$ and $\pi_1(X,q)$ are isomorphic groups for any $p, q \in X$.
[Hence we often just write $\pi_1(X)$].

Pf sketch Since X is path-connected $\exists$ a path $\gamma: [0,1] \to X$ with $\gamma(0) = p$ and $\gamma(1) = q$



Define $\gamma_* : \pi_1(X,p) \longrightarrow \pi_1(X,q)$ via
$$\langle \alpha \rangle \longmapsto \langle \gamma^{-1} \cdot \alpha \cdot \gamma \rangle$$



<u>To check</u>:

<u>γ_* well-defined</u>: $\alpha \simeq \alpha' \Rightarrow \gamma^{-1} \cdot \alpha \cdot \gamma \simeq \gamma^{-1} \cdot \alpha' \cdot \gamma$

<u>γ_* is a homomorphism</u>: (<u>not</u> a homeomorphism)

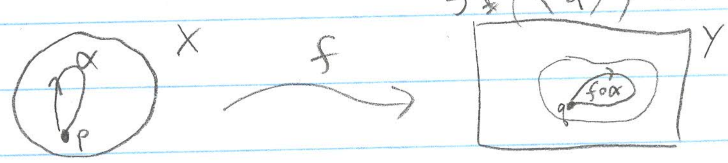
$$\begin{aligned}
\gamma_*(\langle \alpha \rangle \cdot \langle \beta \rangle) &= \gamma_*(\langle \alpha \cdot \beta \rangle) \\
&= \langle \gamma^{-1} \cdot (\alpha \cdot \beta) \cdot \gamma \rangle \\
&= \langle (\gamma^{-1} \cdot \alpha \cdot \gamma) \cdot (\gamma^{-1} \cdot \beta \cdot \gamma) \rangle \\
&\qquad \text{since } \gamma^{-1} \cdot (\alpha \cdot \beta) \cdot \gamma \simeq (\gamma^{-1} \cdot \alpha \cdot \gamma) \cdot (\gamma^{-1} \cdot \beta \cdot \gamma) \\
&= \langle \gamma^{-1} \cdot \alpha \cdot \gamma \rangle \cdot \langle \gamma^{-1} \cdot \beta \cdot \gamma \rangle \\
&= \gamma_*(\langle \alpha \rangle) \cdot \gamma_*(\langle \beta \rangle)
\end{aligned}$$

<u>γ_* has an inverse homeomorphism, namely $(\gamma^{-1})_*$</u>

<u>"Functoriality"</u>

So far we've assigned groups $\pi_1(X)$ to spaces. But we can also assign group homomorphisms $f_* : \pi_1(X) \to \pi_1(Y)$ to maps $f : X \to Y$.

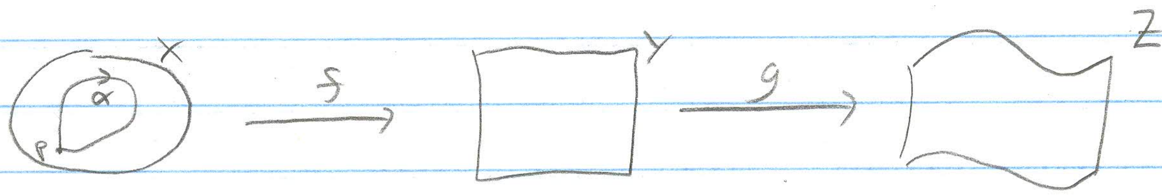
Indeed, let $f : X \to Y$. Fix $p \in X$ and let $q = f(p) \in Y$. We define $f_* : \pi_1(X,p) \longrightarrow \pi_1(Y,q)$ via
$$f_*(\langle \alpha \rangle) = \langle f \circ \alpha \rangle$$



To see that f_* is a group homomorphism, we need $f_*(\langle\alpha\rangle.\langle\beta\rangle) = f_*(\langle\alpha\rangle).f_*(\langle\beta\rangle)$, which is true since

$$f\circ(\alpha.\beta) = (f\circ\alpha).(f\circ\beta)$$
$$\Rightarrow f\circ(\alpha.\beta) \simeq (f\circ\alpha).(f\circ\beta)$$
$$\Rightarrow f_*(\langle\alpha\rangle.\langle\beta\rangle) = \langle f\circ(\alpha.\beta)\rangle$$
$$= \langle(f\circ\alpha).(f\circ\beta)\rangle$$
$$\doteq \langle f\circ\alpha\rangle.\langle f\circ\beta\rangle$$
$$= f_*(\langle\alpha\rangle).f_*(\langle\beta\rangle).$$

Thm 5.7 If $X \xrightarrow{f} Y \xrightarrow{g} Z$, then $(g\circ f)_* = g_*\circ f_*$.



Pf $(g\circ f)_*(\langle\alpha\rangle) = \langle g\circ f\circ\alpha\rangle$

$g_*(f_*(\langle\alpha\rangle)) = g_*(\langle f\circ\alpha\rangle) = \langle g\circ f\circ\alpha\rangle$.

Corollary Homeomorphic (path-connected) spaces have isomorphic fundamental groups.

Pf Let $h: X\to Y$ be a homeomorphism.

Then
$$\begin{tikzcd} X \arrow[r, "h"] \arrow[rr, bend right, "id_X"'] & Y \arrow[r, "h^{-1}"] & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Y \arrow[r, "h^{-1}"] \arrow[rr, bend right, "id_Y"'] & X \arrow[r, "h"] & Y \end{tikzcd}$$

give $h^{-1}_*\circ h_* = (id_X)_* : \pi_1(X)\to\pi_1(X)$

and $h_*\circ h^{-1}_* = (id_Y)_* : \pi_1(Y)\to\pi_1(Y)$.

Since $(id_X)_*$ and $(id_Y)_*$ are identity maps, this shows $h_*:\pi_1(X)\to\pi_1(Y)$ is an isomorphism with inverse h^{-1}_*.

Fact More is true: homotopy equivalent (path-connected) spaces have isomorphic fundamental groups.

This gives us one way to try to test if two spaces are homeomorphic or homotopy equivalent:
If their fundamental groups are not isomorphic, then the spaces are not homeomorphic or homotopy equivalent.
Otherwise, we must look for a finer, more sophisticated invariant to try to distinguish the spaces (or show they're the same).

§5.3 Calculations of $\pi_1(X)$

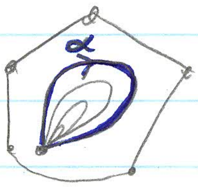
We will skip some of this section: orbit spaces, Thm 5.13

Space X	$\pi_1(X)$
Convex subset of $\mathbb{R}^n$	trivial group $\{e\}$
Contractible space, homotopy equivalent to $*$	$\{e\}$
Circle S^1	$\mathbb{Z}$ ← requires work to show $= \langle a \rangle$
Sphere S^n for $n \geq 2$	$\{e\}$
Torus $S^1 \times S^1$	$\mathbb{Z} \times \mathbb{Z}$ ← Thm 5.14 $= \langle a,b \mid aba^{-1}b^{-1} \rangle$
Projective plane $\mathbb{P}^2$	$\mathbb{Z}/2\mathbb{Z} = \langle a \mid a^2 \rangle$
Klein bottle	$\langle a,b \mid abab^{-1} \rangle$
Figure eight  $S^1 \vee S^1$	$\langle a,b \rangle$
$S^1 \vee S^1 \vee S^1$ 	$\langle a,b,c \rangle$

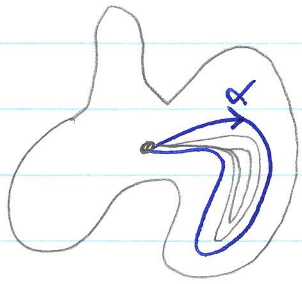
Group notation. Not "homotopy class of"

Def Given two spaces X and Y, the wedge sum $X \vee Y$ is formed by joining the spaces at a single point. Ex $S^1 \vee S^2 =$ 

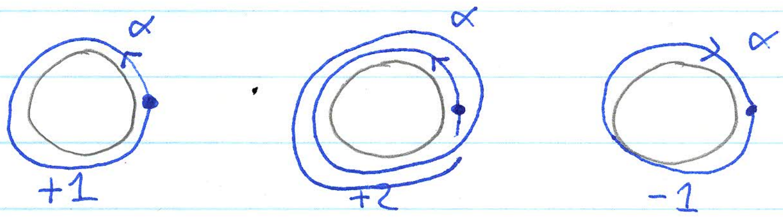
Pictures



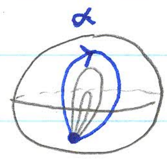
Convex



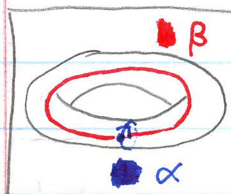
Contractible

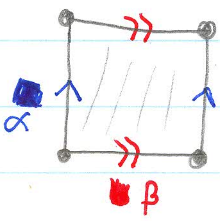


Circle



Sphere





Torus

You can see $\alpha \cdot \beta \simeq \beta \cdot \alpha$, which gives the commutativity relation in

homotopy class of α

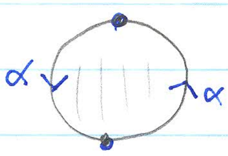
homotopy class of β

$$\pi_1(S^1 \times S^1) = \mathbb{Z} \times \mathbb{Z} = \langle a, b \mid aba^{-1}b^{-1} \rangle.$$

$$aba^{-1}b^{-1} = e \iff ab = ba$$

Group multiplication in $\mathbb{Z} \times \mathbb{Z}$: $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

$$(1,0) + (2,1) = (3,1)$$



Projective plane $\mathbb{P}^2$

$$\pi_1(\mathbb{P}^2) = \mathbb{Z}/2\mathbb{Z} = \langle a \mid a^2 \rangle$$

homotopy class of α

Group multiplication in $\mathbb{Z}/2\mathbb{Z}$:

+	0	1
0	0	1
1	1	0

identity

You can see $\alpha \cdot \alpha \simeq$ constant path e, which gives the relation $a^2 = e$ in $\pi_1(\mathbb{P}^2) = \langle a \mid a^2 \rangle$.

Free group $\langle a_1, \ldots, a_n \rangle$

(This is different from the free abelian group $\mathbb{Z}^n = \underbrace{\mathbb{Z} \times \ldots \times \mathbb{Z}}_{n \text{ times}}$)

A <u>reduced word</u> in the symbols $a_1, \ldots, a_n$ is of the form $a_{i_1}^{e_1} a_{i_2}^{e_2} \ldots a_{i_k}^{e_k}$ such that $i_j \neq i_{j+1}$.

<u>Ex</u> $a_1^2 a_3 a_2^3 a_1^{-2} a_2$ reduced

$a_1^2 a_3 a_3^3 a_1^{-2} a_2$ not reduced, reduces to $a_1^2 a_3^4 a_1^{-2} a_2$.

$a_1^2 a_3^{-3} a_3^3 a_1^{-2} a_2$ not reduced, reduces to $a_1^2 a_1^{-2} a_2$ and then to a_2.

<u>Def</u> The free group $\langle a_1, \ldots, a_n \rangle$ on n generators is the set of all reduced words, with group multiplication given by concatenation and then reducing.

→ <u>Ex</u> In $\langle a_1, a_2, a_3 \rangle$ we have

$$(a_1^2 a_3 a_2)(a_2^{-1} a_3^2 a_2) = a_1^2 a_3 a_2 a_2^{-1} a_3^2 a_2$$
$$= a_1^2 a_3 a_3^2 a_2$$
$$= a_1^2 a_3^3 a_2$$

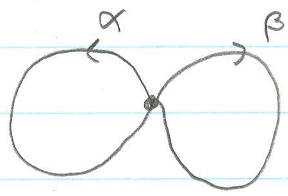
The identity element is the empty word

Note $(a_1^3 a_3^{-2} a_2)^{-1} = a_2^{-1} a_3^2 a_1^{-3}$ since

$(a_1^3 a_3^{-2} a_2)(a_2^{-1} a_3^2 a_1^{-3}) = a_1^3 a_3^{-2} a_3^2 a_1^{-3} = a_1^3 a_1^{-3} =$ empty word

and $(a_2^{-1} a_3^2 a_1^{-3})(a_1^3 a_3^{-2} a_2) =$ empty word.

$\pi_1(\text{figure } 8) = \pi_1(S^1 \vee S^1) = \langle a, b \rangle$



a is the homotopy class represented by α
b is the homotopy class represented by β

Note $(aba)(a^{-1}ba) = ab^2a$, which corresponds to $(\alpha . \beta . \alpha) . (\alpha^{-1} . \beta . \alpha) \simeq \alpha . \beta . \beta . \alpha$.

Note $ab \neq ba$, which corresponds to $\alpha . \beta \not\simeq \beta . \alpha$.

Presentation of a group via generators and relations

Def If $r_1, \dots, r_k$ are reduced words in $a_1, \dots, a_n$, then group $\langle a_1, \dots, a_n \mid r_1, \dots, r_k \rangle$ is the quotient group $\langle a_1, \dots, a_n \rangle / N$ where N is the smallest normal subgroup of $\langle a_1, \dots, a_n \rangle$ containing $r_1, \dots, r_k$.

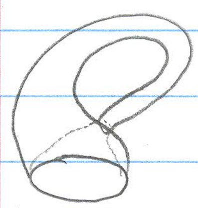
Consequence In $\langle a_1, \dots, a_n \mid r_1, \dots, r_k \rangle$, each r_i has been identified with the identity (empty word)

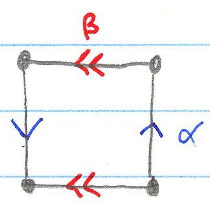
Ex $\langle a \mid a^2 \rangle$ has two elements, a and the empty word.
Note $(a)(a) = a^2 =$ empty word.
$\langle a \mid a^2 \rangle = \mathbb{Z}/2\mathbb{Z}$

Ex In $\langle a, b \mid abab^{-1} \rangle$, we have

$$
\begin{aligned}
&(abab)(ab^{-1}a^{-1}b^{-1}a^{-1}b^2) \\
&= ab(abab^{-1})a^{-1}b^{-1}a^{-1}b^2 \\
&= aba^{-1}b^{-1}a^{-1}b^2 \\
&= a(abab^{-1})^{-1}b^2 \\
&= ab^2.
\end{aligned}
$$

NOT easy to see





Klein bottle $\quad \pi_1(\text{Klein}) = \langle a, b \mid abab^{-1} \rangle$

homotopy class of α

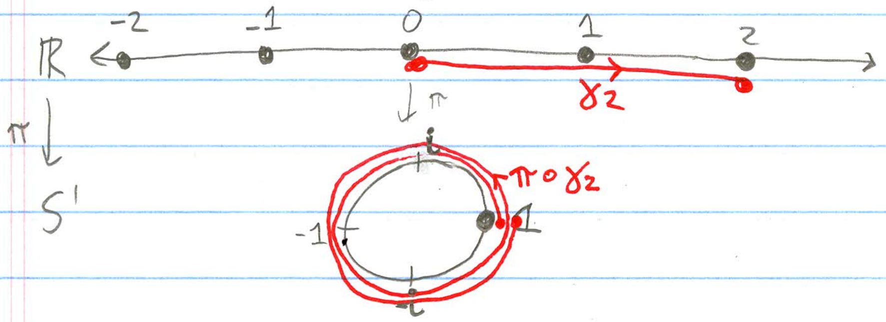
You can see $\alpha.\beta.\alpha.\beta^{-1} \simeq e$, which gives the relation $abab^{-1}$.

$\pi_1(S^1) = \mathbb{Z}$

Let S^1 be the unit circle in $\mathbb{C}$.

Define $\pi : \mathbb{R} \to S^1$ by

$$x \longmapsto e^{2\pi i x}$$



Note $\pi^{-1}(1) = \mathbb{Z} \subseteq \mathbb{R}$.

Given $n \in \mathbb{Z}$, let $\gamma_n : [0,1] \to \mathbb{R}$ be the path

$$\gamma_n(s) = ns$$

joining 0 to n in $\mathbb{R}$.

Note $\pi \circ \gamma_n$ is a loop in S^1 based at 1, which wraps n times around S^1

(counter clockwise for n positive
clockwise for n negative.)

Thm 5.8 The function $\phi : \mathbb{Z} \to \pi_1(S^1, 1)$ defined by $\phi(n) = \langle \pi \circ \gamma_n \rangle$ is a group isomorphism.

(So $\pi_1(S^1) = \mathbb{Z}$)

Pf Omitted, along with Lemmas 5.9, 5.10, 5.11

We're also omitting Theorem 5.13 (about fundamental groups of orbit spaces).

Thm 5.14 If X and Y are path-connected spaces, then $\pi_1(X \times Y)$ is isomorphic to $\pi_1(X) \times \pi_1(Y)$.

Consequence $\pi_1(\text{torus}) = \pi_1(S^1 \times S^1) = \pi_1(S^1) \times \pi_1(S^1) = \mathbb{Z} \times \mathbb{Z}$.

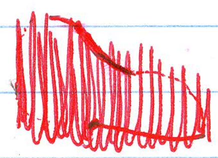
Pf The projections $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ induce homomorphisms

$\langle\alpha\rangle \longmapsto \langle p_1 \circ \alpha\rangle$

$p_{1*} : \pi_1(X \times Y) \to \pi_1(X)$

and $p_{2*} : \pi_1(X \times Y) \to \pi_1(Y)$ (pg 94)

$\langle\alpha\rangle \longmapsto \langle p_2 \circ \alpha\rangle$

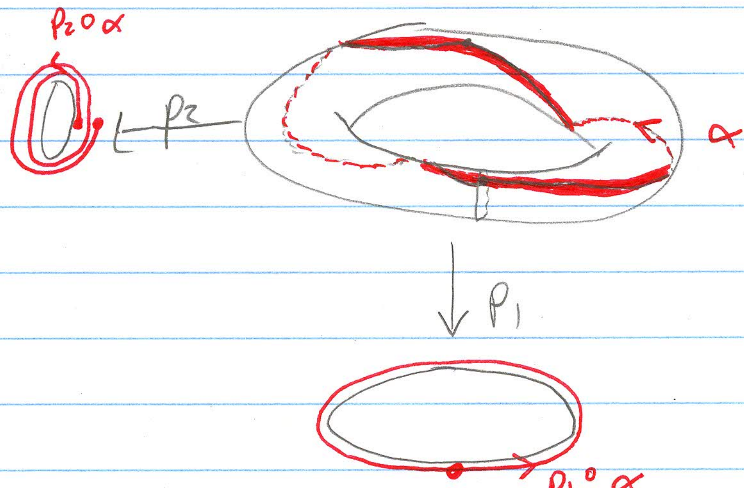


Define homomorphism $\psi : \pi_1(X \times Y) \to \pi_1(X) \times \pi_1(Y)$

by $\langle\alpha\rangle \longmapsto (p_{1*}(\langle\alpha\rangle), p_{2*}(\langle\alpha\rangle))$

$\qquad\qquad\qquad\qquad\qquad\qquad \| $

$(\langle p_1 \circ \alpha\rangle, \langle p_2 \circ \alpha\rangle)$.



$\underline{\psi \text{ is a homomorphism}}$ since p_{1*} and p_{2*} are:

$$\begin{aligned}
\psi(\langle\alpha\rangle . \langle\beta\rangle) &= (p_{1*}(\langle\alpha\rangle . \langle\beta\rangle), p_{2*}(\langle\alpha\rangle . \langle\beta\rangle)) && \text{def of } \psi \\
&= (p_{1*}(\langle\alpha\rangle) . p_{1*}(\langle\beta\rangle), p_{2*}(\langle\alpha\rangle) . p_{2*}(\langle\beta\rangle)) && p_{1*}, p_{2*} \text{ homomorphisms} \\
&= (p_{1*}(\langle\alpha\rangle), p_{2*}(\langle\alpha\rangle))(p_{1*}(\langle\beta\rangle), p_{2*}(\langle\beta\rangle)) && \text{multiplication in } \pi_1(X) \times \pi_1(Y) \\
&= \psi(\langle\alpha\rangle)\, \psi(\langle\beta\rangle). && \text{def of } \psi
\end{aligned}$$

$\alpha : [0,1] \to X \times Y$

ψ is injective (one-to-one) since if $\psi(\langle\alpha\rangle)$ is the identity in $\pi_1(X) \times \pi_1(Y)$, then
$\langle p_1 \circ \alpha \rangle$ is the identity in $\pi_1(X)$ and $\langle p_2 \circ \alpha \rangle$ the identity in $\pi_1(Y)$

$\Longrightarrow \quad p_1 \circ \alpha \underset{F}{\simeq} e \quad$ and $\quad p_2 \circ \alpha \underset{G}{\simeq} e$

$\Longrightarrow \quad \alpha \underset{H}{\simeq} e \quad$ where $H(s,t) = \big(F(s,t), G(s,t) \big)$.

ψ is surjective (onto) since given an arbitrary element $(\langle\beta\rangle, \langle\gamma\rangle) \in \pi_1(X) \times \pi_1(Y)$, define loop $\alpha : [0,1] \to X \times Y$ by $\alpha(s) = (\beta(s), \gamma(s))$.
By construction

$$\begin{aligned} \psi(\langle\alpha\rangle) &= (\langle p_1 \circ \alpha \rangle, \langle p_2 \circ \alpha \rangle) \\ &= (\langle\beta\rangle, \langle\gamma\rangle) \quad \text{as required.} \end{aligned}$$

Hence $\pi_1(X \times Y)$ is isomorphic to $\pi_1(X) \times \pi_1(Y)$.

§5.4 Homotopy type (of a space)

Rmk Already defined when maps are homotopy equivalent; we now define when spaces are homotopy equivalent.

Rmk Homeomorphic spaces are homotopy equivalent, but not vice versa. Ex $\mathbb{R}^2 \not\cong \mathbb{R}^3$ but $\mathbb{R}^2 \simeq \mathbb{R}^3$.

Rmk Homotopy equivalent spaces have isomorphic fundamental groups, homotopy groups, homology groups, and equal Euler characteristics.

Rmk Homotopy equivalent spaces need not have the same "dimensions".

Def 5.15 Two spaces X and Y are homotopy equivalent (denoted $X \simeq Y$) if $\exists$ maps $f: X \to Y$, $g: Y \to X$ with $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$.

Pic

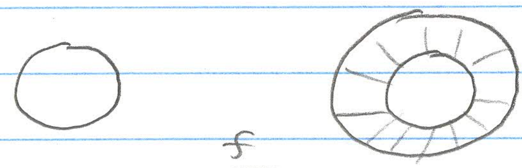
$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$$

Notation g is a homotopy inverse for f, and f is a homotopy inverse for g.

Examples

- Homeomorphic spaces have the same homotopy type. Indeed, $X \cong Y$ if $\exists$ maps $f: X \to Y$, $g: Y \to X$ with $g \circ f = id_X$ and $f \circ g = id_Y$, immediately giving $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$.

- The circle $S^1 = \{e^{i\theta} \mid 0 \le \theta < 2\pi\}$ is homotopy equivalent to the annulus $A = \{re^{i\theta} \mid 0 \le \theta < 2\pi, 1 \le r \le 2\}$.



$$\begin{tikzcd} S^1 \arrow[r, bend left, "f"] & A \arrow[l, bend left, "g"] \end{tikzcd}$$

Define $f : S^1 \to A$ by
$$e^{i\theta} \mapsto e^{i\theta} \quad \text{or} \quad x \mapsto x$$

Define $g : A \to S^1$ by
$$re^{i\theta} \mapsto e^{i\theta} \quad \text{or} \quad x \mapsto x/\|x\|.$$

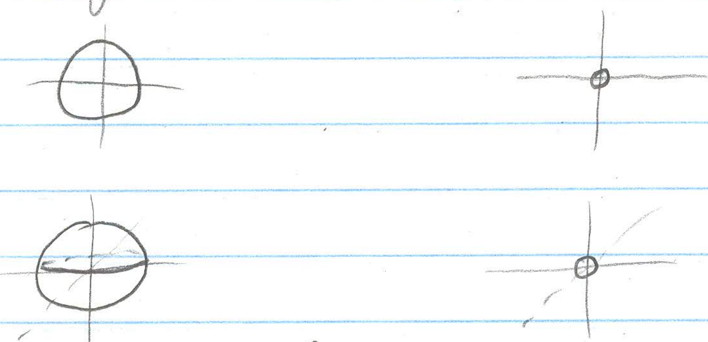
Note $g \circ f = id_{S^1}$ since
$$\begin{aligned}(g\circ f)(e^{i\theta}) &= g(f(e^{i\theta}))\\ &= g(e^{i\theta})\\ &= e^{i\theta}.\end{aligned}$$

Note $f \circ g \neq id_A$ since
$$\begin{aligned}(f\circ g)(re^{i\theta}) &= f(g(re^{i\theta}))\\ &= f(e^{i\theta})\\ &= e^{i\theta}.\end{aligned}$$

However, $f \circ g \simeq id_A$ via the homotopy $H : A \times I \to A$ given by
$$\begin{aligned}H(re^{i\theta}, t) &= (1-t)e^{i\theta} + tre^{i\theta}\\ &= (1-t+tr)e^{i\theta}\end{aligned}$$
or equivalently
$$H(x,t) = (1-t)\frac{x}{\|x\|} + tx.$$

Rmk Even though A is not convex, the straight-line homotopy lives in A in this case.

- $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ and $\mathbb{R}^{n+1} \setminus \{0\}$ are homotopy equivalent.



$$\begin{tikzcd} S^n \arrow[r, bend left, "f"] & \mathbb{R}^{n+1} \setminus \{0\} \arrow[l, bend left, "g"] \end{tikzcd}$$

Define $f: S^n \to \mathbb{R}^{n+1} \setminus \{0\}$ by $f(x) = x$.
Define $g: \mathbb{R}^{n+1} \setminus \{0\} \to S^n$ by $g(x) = x/\|x\|$.

Note $g \circ f = id_{S^n}$ since
$(g \circ f)(x) = g(f(x)) = g(x) = x$
$\uparrow$ since $\|x\| = 1$ since $x \in S^n$.

Note $f \circ g \neq id_{\mathbb{R}^{n+1} \setminus \{0\}}$ since $(f \circ g)(x) = x/\|x\|$.

We see $f \circ g \simeq id_{\mathbb{R}^{n+1} \setminus \{0\}}$ via the homotopy
$H: \mathbb{R}^{n+1} \setminus \{0\} \times \underline{I} \to \mathbb{R}^{n+1} \setminus \{0\}$ given by

$$H(x,t) = (1-t)x + t\frac{x}{\|x\|}$$

- Any convex subset C of $\mathbb{R}^n$ is homotopy equivalent to a point.

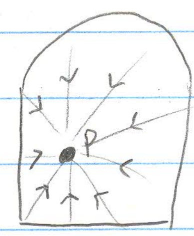
Indeed, pick any point $p \in C$.

$$\{p\} \underset{g}{\overset{f}{\rightleftarrows}} C$$

Define $f : \{p\} \to C$ by $f(p) = p$
Define $g : C \to \{p\}$ by $g(x) = p \quad \forall x \in C$.

Note $g \circ f = id_{\{p\}}$.
Note $f \circ g \simeq id_C$ via the straight-line homotopy
$H : C \times I \to C$ defined by

$$H(x,t) = (1-t)\, id_C(x) + t\,(f \circ g)(x)$$
$$= (1-t)x + tp.$$


C

Lemma 5.16 The relation $X \simeq Y$ is an equivalence relation on topological spaces.

Pf Reflexivity ($X \simeq X$) is clear: consider $X \underset{id}{\overset{id}{\rightleftarrows}} X$.

Symmetry ($X \simeq Y \Rightarrow Y \simeq X$) is clear:
Given $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ consider $Y \underset{f}{\overset{g}{\rightleftarrows}} X$.

Transitivity ($X \simeq Y$ and $Y \simeq Z \Rightarrow X \simeq Z$):

$X \simeq Y$ gives $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ with $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$.

$Y \simeq Z$ gives $Y \underset{v}{\overset{u}{\rightleftarrows}} Z$ with $v \circ u \simeq id_Y$ and $u \circ v \simeq id_Z$.

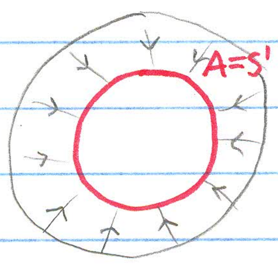
Consider
$$\begin{tikzcd} X \arrow[r, bend left, "u\circ f"] & Z \arrow[l, bend left, "g\circ v"] \end{tikzcd}$$

Note
$$\begin{aligned}(g\circ v)\circ(u\circ f) &= g\circ(v\circ u)\circ f \\ &\simeq g\circ id_Y\circ f \quad \text{by Lemma 5.4} \\ &= g\circ f \\ &\simeq id_X\end{aligned}$$
and
$$\begin{aligned}(u\circ f)\circ(g\circ v) &= u\circ(f\circ g)\circ v \\ &= u\circ v \quad \text{by Lemma 5.4} \\ &\simeq id_Z\end{aligned}$$

__Rmk__ A deformation retraction is a specific type of homotopy "collapsing" space X onto a subspace A.

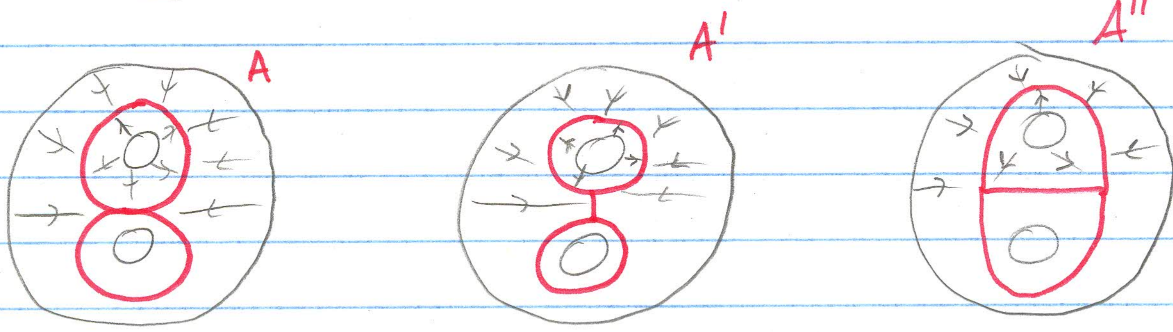
__Def__ Let $A\subseteq X$. A homotopy $G: X\times I\to X$ rel A (meaning $G(a,t)=a \ \forall a\in A$) for which
$$G(x,0)=x \quad \text{and} \quad G(x,1)\in A \quad \forall x\in X$$
is a __deformation retraction from X onto A__.

__Pic 1__



X = annulus

__Pic 2__



X = disk w/ 2 holes

<u>Prop</u> If $\exists$ a deformation retraction G from X onto $A \subseteq X$, then $X \simeq A$.

$$A \underset{G(\cdot,1)}{\overset{\underset{a \mapsto a}{f}}{\rightleftarrows}} X$$

$\boxed{G(\cdot,1) \text{ maps into } A}$

<u>Pf</u> Note $G(\cdot,1) \circ f = id_A$ since $\forall a \in A$,

$$\begin{aligned}(G(\cdot,1)\circ f)(a) &= G(\cdot,1)(f(a)) \\ &= G(\cdot,1)(a) \\ &= G(a,1) \\ &= a \quad \text{since } G \text{ is a homotopy rel } A\end{aligned}$$

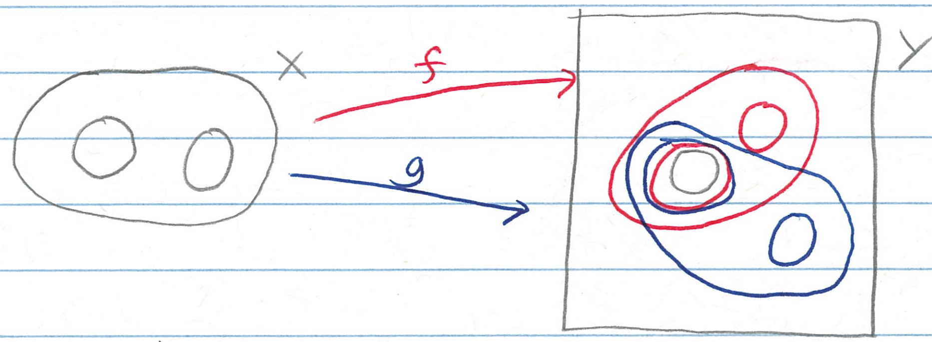
Note
$$\begin{aligned} f \circ G(\cdot,1) &= G(\cdot,1) \quad \text{since } f \text{ is an inclusion} \\ &\simeq G(\cdot,0) \quad \text{since } G \text{ is a homotopy} \\ &= id_X \end{aligned}$$

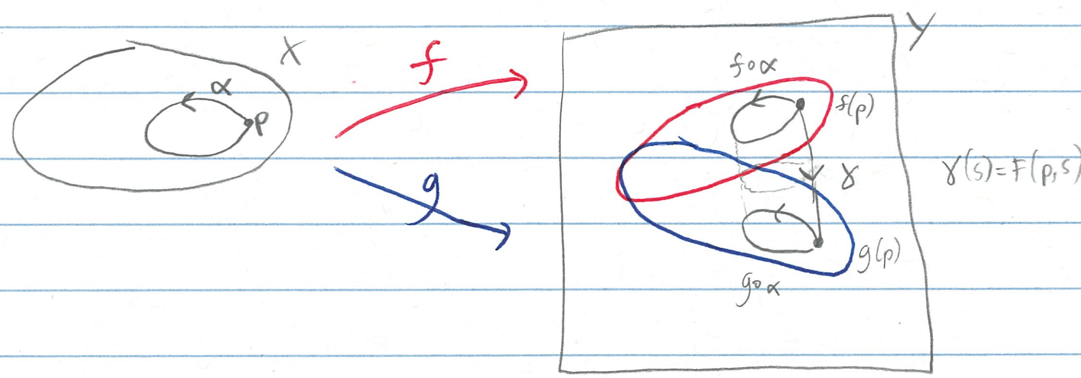
<u>Rmk</u> In Pic 2, we have $A \simeq X$, $A' \simeq X$, $A'' \simeq X$. Since $\simeq$ is an equivalence relation, the spaces A, A', A'' are all pairwise homotopy equivalent.

<u>Rmk</u> I think all examples of $X \simeq Y$ we've seen so far have been examples of deformation retractions, except $A \simeq A'$, $A \simeq A''$, $A' \simeq A''$.

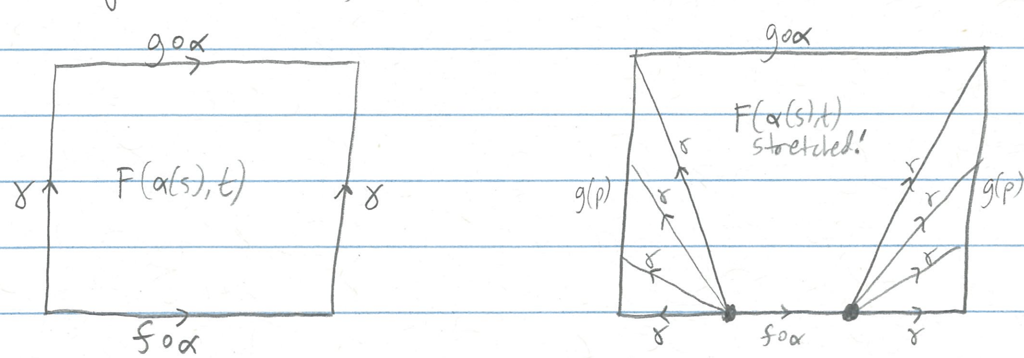
Thm 5.17 If $f \underset{F}{\simeq} g : X \to Y$, then
$g_* = \gamma_* \circ f_* : \pi_1(X,p) \longrightarrow \pi_1(Y, g(p))$
where $\gamma(t) = F(p,t)$ is a path in Y joining $f(p)$ to $g(p)$

Pic





Pf Recall $g_*(\langle \alpha \rangle) = \langle g \circ \alpha \rangle$ and
$\gamma_* f_*(\langle \alpha \rangle) = \langle \gamma^{-1} \cdot (f \circ \alpha) \cdot \gamma \rangle$.



The square on the right shows
$g \circ \alpha \simeq \gamma^{-1} \cdot (f \circ \alpha) \cdot \gamma$ rel $\{0,1\}$.

Thm 5.18 If X, Y path-connected with $X \simeq Y$, then $\pi_1(X) \cong \pi_1(Y)$

isomorphic groups

Pf Let $X \underset{k}{\overset{h}{\rightleftarrows}} Y$ with

$k \circ h \underset{F}{\simeq} id_X$ and $h \circ k \underset{G}{\simeq} id_Y$

Consider

$$\begin{tikzcd} \pi_1(X,p) \arrow[r, "h_*"] \arrow[rr, bend right, "k_* \circ h_*"'] & \pi_1(Y, h(p)) \arrow[r, "k_*"] \arrow[rr, bend right, "h_* \circ k_*"'] & \pi_1(X, k(h(p))) \arrow[r, "h_*"] & \pi_1(Y, h(k(h(p)))) \end{tikzcd}$$

We'll see h_* is an isomorphism.

$k \circ h \underset{F}{\simeq} id_X \Rightarrow k_* \circ h_* = \gamma_*$ where $\gamma(s) = F(s,p)$ by Theorem 5.17

$\Rightarrow k_* \circ h_*$ is an isomorphism by Theorem 5.6

$\Rightarrow h_*$ is injective.

$h \circ k \underset{G}{\simeq} id_Y \Rightarrow h_* \circ k_* = \gamma_*$ where $\gamma(s) = G(s, h(p))$ by Theorem 5.17

$\Rightarrow h_* \circ k_*$ is an isomorphism by Theorem 5.6

$\Rightarrow h_*$ is surjective

Hence $h_* : \pi_1(X,p) \to \pi_1(Y, h(p))$ is an isomorphism showing $\pi_1(X) \cong \pi_1(Y)$.