CSU Math 472
Fall 2022

Homework B

Due Wednesday, November 2, in class (else by email)

Problems.

1. For each of the following topological spaces X, find an open cover of X which does not contain a finite subcover. You do not need to prove that your answer is correct. It follows that each of the below spaces X is not compact.
   (a) $X = \mathbb{R}$.
   (b) $X = \mathbb{Z}$ with the discrete topology (meaning every subset of $\mathbb{Z}$ is an open set).
   (c) $X = (-1, 1)$.
   (d) $X = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$.
2. For each of the four non-compact spaces X in #1 (a)–(d) above, write down a continuous function $f\colon X \to \mathbb{R}$ that has no upper bound, i.e., such that $\sup_{x\in X} f(x) = \infty$. Also, write down a continuous function $g\colon X \to \mathbb{R}$ that has a finite supremum that is not attained. You do not need to prove that your answers are correct. Note this would not be possible if X were compact.
3. (Intermediate value theorem). Let $a, b \in \mathbb{R}$ with $a < b$, and let $f\colon [a, b] \to \mathbb{R}$ be a continuous map with $f(a) < 0$ and $f(b) > 0$. Use the connectedness of $[a, b]$ to prove that there exists some point $c \in [a, b]$ with $f(c) = 0$.
4. Suppose $X = A \cup B$ where A is path-connected, B is path-connected, and $A \cap B \neq \emptyset$. Prove that X is path-connected.
5. Define $id\colon S^1 \to S^1$ by $id(p) = p$, and define $g\colon S^1 \to S^1$ by $g(p) = -p$. Find a homotopy $H\colon S^1 \times I \to S^1$ from id to g.

   Notational aside: It's perhaps easiest to represent S^1 using complex coordinates, $S^1 = \{e^{i\theta}\} \subseteq \mathbb{C}$, where θ varies between 0 and 2π. Then $id(e^{i\theta}) = e^{i\theta}$ and $g(e^{i\theta}) = -e^{i\theta} = e^{i(\theta+\pi)}$.