Duke Math 431
Spring 2015

Why is the definition of two sets having the same cardinality symmetric?

Two sets S and T have the same cardinality if there is a one-to-one and onto function $f\colon S \to T$. At first this doesn't feel symmetric. Is it okay to instead give a one-to-one and onto function $g\colon T \to S$? Does the existence of one such function imply the existence of the other?

The answer to both questions is yes. To see why, note that if you give a one-to-one and onto function $f\colon S \to T$, then the inverse function f^{-1} is a one-to-one and onto function in the opposite direction, $f^{-1}\colon T \to S$. Alternatively, if you give a one-to-one and onto function $g\colon T \to S$, then the inverse function g^{-1} is a one-to-one and onto function in the opposite direction, $g^{-1}\colon S \to T$.

In particular, recall that a set is countable if it has the cardinality of $\mathbb{N}$, the set of natural numbers. To show that a set S is countable, it is fine to give either a one-to-one and onto function $f\colon \mathbb{N} \to S$, or alternatively to give a one-to-one and onto function $g\colon S \to \mathbb{N}$.