Duke Math 431
Spring 2015

Proof that zero is less than one

In this note we will prove that $0 < 1$. In order to do so we first need a lemma.

Lemma. For any real number x we have $x^2 \geq 0$.

Proof. We will consider two cases: $x \geq 0$ and $x < 0$. In the first case $x \geq 0$ we have

$$\begin{aligned} x^2 &= x \cdot x \\ &\geq 0 \cdot 0 && \text{by (O5)} \\ &= 0 && \text{by §1.1 \#4.} \end{aligned}$$

In the second case $x < 0$ we have $-x \geq 0$ by Proposition 1.1.1(d), and hence

$$\begin{aligned} x^2 &= x \cdot x \\ &= (-x)(-x) && \text{by Proposition 1.1.1(c)} \\ &\geq 0 \cdot 0 && \text{by (O5) since } -x \geq 0 \\ &= 0 && \text{by §1.1 \#4.} \end{aligned}$$

Hence for any real number x we have $x^2 \geq 0$. □

Claim. We have $0 < 1$.

Proof. First we will show $0 \leq 1$. To see this, note that

$$\begin{aligned} 1 &= 1 \cdot 1 && \text{by (P7)} \\ &\geq 0 && \text{by our Lemma above.} \end{aligned}$$

Now it suffices to show that $0 \neq 1$. Indeed, suppose for a contradiction that $0 = 1$. Choose any real number $x \neq 0$ that is nonzero[1]. Note that we have

$$\begin{aligned} x &= x \cdot 1 && \text{by (P7)} \\ &= x \cdot 0 && \text{since we've assumed for a contradiction that } 0 = 1 \\ &= 0 && \text{by §1.1 Problem \#4.} \end{aligned}$$

This contradicts the fact that we chose $x \neq 0$, and hence it must be the case that $0 \neq 1$.

We have shown $0 \leq 1$ and $0 \neq 1$, and together these imply $0 < 1$. □

[1] Here we've assumed that not all real numbers are zero.