Proof of Theorem 2.2.6

**Theorem 2.2.6.** Let \( \{a_n\} \) and \( \{b_n\} \) be sequences and suppose that \( a_n \to a \) and \( b_n \to b \). Suppose that \( b \neq 0 \) and \( b_n \neq 0 \) for any \( n \). Then \( \lim_{n \to \infty} \frac{a_n}{b_n} = \frac{a}{b} \).

**Proof.** First we show there is some \( M \) with \( 0 < M \leq |b_n| \) for all \( n \). Since \( b_n \to b \), there is some \( N \) with \( |b_n - b| \leq \frac{|b|}{2} \) for all \( n \geq N \). This implies \( |b_n| \geq \frac{|b|}{2} \) for all \( n \geq N \), for otherwise \( |b_n| < \frac{|b|}{2} \) would give the contradiction

\[
|b| = |b_n - (b_n - b)| \leq |b_n| + |b_n - b| < \frac{|b|}{2} + \frac{|b|}{2} = |b|.
\]

Hence \( M = \min\{|b_1|, |b_2|, \ldots, |b_n|, \frac{|b|}{2}\} \) gives \( 0 < M \leq |b_n| \) for all \( n \).

Now let \( \epsilon > 0 \) be arbitrary. Since \( a_n \to a \) we may pick \( N_1 \) so that \( n \geq N_1 \) gives \( |a_n - a| \leq \frac{\epsilon M}{2} \).

Since \( b_n \to b \) we may pick \( N_2 \) so that \( n \geq N_2 \) gives \( |b_n - b| \leq \frac{|b|M}{2|a|} \). Let \( N = \max\{N_1, n_2\} \).

Then \( n \geq N \) gives

\[
\left| \frac{a_n}{b_n} - \frac{a}{b} \right| = \left| \frac{a_n b - ab_n}{b_n bb} \right| \\
\leq \frac{|a_n b - ab_n|}{|b|M} \\
= \frac{|(a_n b - ab) + (ab - ab_n)|}{|b|M} \\
\leq \frac{|a_n - a||b|}{|b|M} + \frac{|a||b_n - b|}{|b|M} \\
= \frac{|a_n - a|}{M} + \frac{|a||b_n - b|}{|b|M} \\
\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \quad \text{by the triangle inequality (Proposition 1.1.2(c))} \\
= \epsilon.
\]

Hence we have shown \( \lim_{n \to \infty} \frac{a_n}{b_n} = \frac{a}{b} \).