Duke Math 431
Spring 2015

Proof of Theorem 2.2.6

Theorem 2.2.6. Let $\{a_n\}$ and $\{b_n\}$ be sequences and suppose that $a_n \to a$ and $b_n \to b$. Suppose that $b \neq 0$ and $b_n \neq 0$ for any n. Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$.

Proof. First we show there is some M with $0 < M \leq |b_n|$ for all n. Since $b_n \to b$, there is some N with $|b_n - b| \leq \frac{|b|}{2}$ for all $n \geq N$. This implies $|b_n| \geq \frac{|b|}{2}$ for all $n \geq N$, for otherwise $|b_n| < \frac{|b|}{2}$ would give the contradiction

$$|b| = |b_n - (b_n - b)| \leq |b_n| + |b_n - b| < \frac{|b|}{2} + \frac{|b|}{2} = |b|.$$

Hence $M = \min\{|b_1|, |b_2|, \ldots, |b_n|, \frac{|b|}{2}\}$ gives $0 < M \leq |b_n|$ for all n.

Now let $\epsilon > 0$ be arbitrary. Since $a_n \to a$ we may pick N_1 so that $n \geq N_1$ gives $|a_n - a| \leq \frac{\epsilon M}{2}$. Since $b_n \to b$ we may pick N_2 so that $n \geq N_2$ gives $|b_n - b| \leq \frac{\epsilon |b| M}{2|a|}$. Let $N = \max\{N_1, n_2\}$. Then $n \geq N$ gives

$$\begin{aligned}
\left|\frac{a_n}{b_n} - \frac{a}{b}\right| &= \left|\frac{a_n b - a b_n}{b b_n}\right| \\
&\leq \frac{|a_n b - a b_n|}{|b| M} \\
&= \frac{|(a_n b - ab) + (ab - a b_n)|}{|b| M} \\
&\leq \frac{|a_n - a||b|}{|b| M} + \frac{|a||b_n - b|}{|b| M} && \text{by the triangle inequality (Proposition 1.1.2(c))} \\
&= \frac{|a_n - a|}{M} + \frac{|a||b_n - b|}{|b| M} \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} && \text{by choice of } N \\
&= \epsilon.
\end{aligned}$$

Hence we have shown $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$.

□