Duke Math 431 **Practice Midterm 2** **March 21, 2015**

Name: ____

- This is the Practice Midterm 2 for Duke Math 431. Partial credit is available. No notes, books, calculators, or other electronic devices are permitted.
- Write proofs that consist of complete sentences, make your logic clear, and justify all conclusions that you make.
- Please sign below to indicate you accept the following statement:

  "I have abided with all aspects of the honor code on this examination."

Signature: ____

Problem	Total Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
Total	60	

1 (a) Give an example of a function f and a domain D so that $f\colon D \to \mathbb{R}$ is continuous but not uniformly continuous. No proofs are necessary.

Solution. The function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is continuous but not uniformly continuous.

(b) Give an example of a function $f\colon [0,1] \to \mathbb{R}$ that is not Riemann integrable.

Solution. Recall from §3.3 #1 that the function $f\colon [0,1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1 & \text{if x is rational} \end{cases}$$

is not Riemann integrable.

(c) Suppose f is n times continuously differentiable on $[a,b]$ and that $f^{(n+1)}$ exists. Let $T^{(n)}(x, x_0)$ be the n-th Taylor polynomial of f at x_0. State the conclusion of Taylor's Theorem.

Solution. Under these hypotheses, for all $x \in [a,b]$ with $x \neq x_0$ there exists a ξ between x and x_0 such that

$$f(x) = T^{(n)}(x, x_0) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}.$$

2 Suppose $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ are continuously differentiable, that $f(0) = g(0)$, and that $f'(x) \le g'(x)$ for all $x \ge 0$. Prove that $f(x) \le g(x)$ for all $x \ge 0$.

Solution. Let $x \ge 0$. We have

$$\begin{aligned} f(x) - f(0) &= \int_0^x f'(t)dt && \text{by the Part I of the Fundamental Theorem (Theorem 4.2.4)} \\ &\le \int_0^x g'(t)dt && \text{by Theorem 3.3.4} \\ &= g(x) - g(0) && \text{by the Part I of the Fundamental Theorem (Theorem 4.2.4).} \end{aligned}$$

Since $f(0) = g(0)$, this implies that $f(x) \le g(x)$ for all $x \ge 0$.

3 Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is continuously differentiable and that its derivative f' is bounded. Prove that f is uniformly continuous on $\mathbb{R}$.

Solution. Since f' is bounded, there exists some $M \in \mathbb{R}$ so that $|f'(x)| \leq M$ for all $x \in \mathbb{R}$. Note that if $x, c \in \mathbb{R}$, then by the Mean Value Theorem there exists some d between x and c satisfying $f'(d) = \frac{f(x)-f(c)}{x-c}$. Hence we have

$$|f(x) - f(c)| = |f'(d)||x - c| \leq M|x - c|.$$

Hence given any $\epsilon > 0$, choose $\delta = \frac{\epsilon}{M}$. Then note that$|x - c| \leq \delta = \frac{\epsilon}{M}$ implies

$$|f(x) - f(c)| \leq M|x - c| \leq M\frac{\epsilon}{M} = \epsilon.$$

We have shown that f is uniformly continuous.

4 Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable at x. Prove that $\lim_{h\to 0} \frac{f(x+h)-f(x-h)}{2h} = f'(x)$.

Solution. Suppose $\epsilon > 0$. Since $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(x)$ exists, there is a $\delta > 0$ such that $|f'(x) - \frac{f(x+h)-f(x)}{h}| \leq \epsilon$ if $0 < |h| \leq \delta$ (see the top of page 122). Hence if $0 < |h| \leq \delta$ then we have

$$\begin{aligned}
\Big|\frac{f(x+h)-f(x-h)}{2h} - f'(x)\Big| &= \Big|\Big(\frac{f(x+h)-f(x)}{2h} - \frac{f'(x)}{2}\Big) + \Big(\frac{f(x)-f(x-h)}{2h} - \frac{f'(x)}{2}\Big)\Big| \\
&\leq \Big|\frac{f(x+h)-f(x)}{2h} - \frac{f'(x)}{2}\Big| + \Big|\frac{f(x)-f(x-h)}{2h} - \frac{f'(x)}{2}\Big| \\
&= \frac{1}{2}\Big|\frac{f(x+h)-f(x)}{h} - f'(x)\Big| + \frac{1}{2}\Big|\frac{f(x)-f(x-h)}{h} - f'(x)\Big| \\
&= \frac{1}{2}\Big|\frac{f(x+h)-f(x)}{h} - f'(x)\Big| + \frac{1}{2}\Big|\frac{f(x+(-h))-f(x)}{(-h)} - f'(x)\Big| \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

We have shown $\lim_{h\to 0} \frac{f(x+h)-f(x-h)}{2h} = f'(x)$.

Remark: You could also prove this limit by considering an arbitrary sequence $h_n \to 0$ with $h_n \neq 0$.

5 Let $f\colon [0,3] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & 1 < x < 2 \\ 2 & x = 2 \\ 1 & 2 < x \leq 3 \end{cases}$$

Prove that f is Rieman integrable and compute $\int_0^3 f(x)dx$.

Solution. Consider the partition P given by $0 < \frac{1}{N} < \frac{2}{N} < \ldots < \frac{3N-1}{N} < 3$ for some $N \in \mathbb{N}$. Note

$$\begin{aligned} U_P(f) - L_P(f) &= \sum_{i=1}^{3N} (M_i - m_i) \frac{1}{N} \\ &= \frac{1}{N} \sum_{i=1}^{3N} (M_i - m_i) \\ &= \frac{1}{N} \big[(M_{N+1} - m_{N+1}) + (M_{2N} - m_{2N}) + (M_{2N+1} - m_{2N+1}) \big] \\ & \qquad \qquad \text{since all other terms are zero} \\ &= \frac{1}{N} \big[(1-0) + (2-0) + (2-1) \big] \\ &= \frac{4}{N}. \end{aligned}$$

Hence given any $\epsilon > 0$, choose $N \in \mathbb{N}$ so that $N \geq \frac{4}{\epsilon}$. This gives $U_P(f) - L_P(f) \leq \epsilon$, and hence by Lemma 3.3 we have shown that f is Riemann integrable.

To compute $\int_0^3 f(x)dx$, note

$$\begin{aligned} U_P(f) &= \sum_{i=1}^{3N} M_i \frac{1}{N} \\ &= \frac{1}{N} \sum_{i=1}^{3N} M_i \\ &= \frac{1}{N} \big[(1)(N+1) + (0)(N-2) + (2)(2) + (1)(N-1) \big] \\ &= \frac{2N+4}{N}. \end{aligned}$$

Hence Corollary 3.3.2 gives

$$\int_0^3 f(x)dx = \lim_{N \to \infty} U_P(f) = \lim_{N \to \infty} \frac{2N+4}{N} = 2.$$

6 Let f be a continuous function on the interval $[a, b]$. Suppose that for every $c \in [a, b]$ and $d \in [a, b]$ we know that $\int_c^d f(x)dx = 0$. Prove that $f(x) = 0$ for all x.

Solution. Suppose for a contradiction that $f(y) \neq 0$ for some $y \in [a, b]$. We'll assume $f(y) > 0$; the case $f(y) < 0$ is analogous. Since f is continuous, there exists some $\delta > 0$ such that $|x - y| \leq \delta$ implies $|f(x) - f(y)| \leq \frac{f(y)}{2}$, and hence $f(x) \geq \frac{f(y)}{2}$. Therefore we have

$$0 = \int_{y-\delta}^{y+\delta} f(x)dx \geq \int_{y-\delta}^{y+\delta} \frac{f(y)}{2} dx = \frac{f(y)}{2} \int_{y-\delta}^{y+\delta} 1 dx = \frac{f(y)}{2} 2\delta = f(y)\delta > 0.$$

This is a contradiction, and hence it must be the case that $f(x) = 0$ for all x.