Duke Math 431 **Practice Midterm 1** **February 9, 2015**

Name: ____

- This is the Practice Midterm 1 for Duke Math 431. Partial credit is available. No notes, books, calculators, or other electronic devices are permitted.
- Write proofs that consist of complete sentences, make your logic clear, and justify all conclusions that you make.
- Please sign below to indicate you accept the following statement:

  "I have abided with all aspects of the honor code on this examination."

Signature: ____

Problem	Total Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
Total	60	

$\boxed{1}$ (a) Let $\{a_n\}$ be a sequence of real numbers. Give the precise definition of when $\{a_n\}$ converges to some limit $a \in \mathbb{R}$.

Solution. Sequence $\{a_n\}$ converges to a if, for any $\epsilon > 0$, there is some $N \in \mathbb{N}$ so that $|a_n - a| \leq \epsilon$ for all $n \geq N$.

(b) Let $\{a_n\}$ be a sequence of real numbers. Give the precise definition of when $\{a_n\}$ has $d \in \mathbb{R}$ as a limit point.

Solution. Sequence $\{a_n\}$ has d as a limit point if, for any $\epsilon > 0$ and $N \in \mathbb{N}$, there exists some $n \geq N$ so that $|a_n - d| \leq \epsilon$.

$\boxed{2}$ Show that the sequence $\{a_n\}$ given by $a_n = 5 + \frac{2}{\sqrt[3]{n}}$ satisfies the definition of a Cauchy sequence.

Solution. Given $\epsilon > 0$, choose $N \in \mathbb{N}$ so that $N \geq \frac{4^3}{\epsilon^3}$, which is equivalent to $\frac{2}{\sqrt[3]{N}} \leq \frac{\epsilon}{2}$. Then for any $n, m \geq N$, we have

$$\begin{aligned}
\left|5 + \frac{2}{\sqrt[3]{n}} - \left(5 + \frac{2}{\sqrt[3]{m}}\right)\right| &= \left|\frac{2}{\sqrt[3]{n}} - \frac{2}{\sqrt[3]{m}}\right| \\
&\leq \frac{2}{\sqrt[3]{n}} + \frac{2}{\sqrt[3]{n}} && \text{by the triangle inequality} \\
&\leq \frac{2}{\sqrt[3]{N}} + \frac{2}{\sqrt[3]{N}} \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} && \text{by the choice of } N \\
&= \epsilon.
\end{aligned}$$

Hence $\{a_n\}$ is a Cauchy sequence.

$\boxed{3}$ Let A, B, and C be sets. Prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Solution 1. Note

$$
\begin{aligned}
& x \in A \cup (B \cap C) \\
\iff & x \in A \text{ or } x \in (B \cap C) \\
\iff & x \in A \text{ or } (x \in B \text{ and } x \in C) \\
\iff & (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C) \\
\iff & x \in A \cup B \text{ and } x \in A \cup C) \\
\iff & x \in (A \cup B) \cap (A \cup C)
\end{aligned}
$$

Hence $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

$\boxed{4}$ Let $\{a_n\}$ be a sequence of real numbers and let S be a set of real numbers. Suppose that a_n is an upper bound for S for each $n \in \mathbb{N}$, and that $a_n \to a$. Prove that a is an upper bound for S.

Solution. Suppose for a contradiction that a is not an upper bound for S, which means there exists some $x \in S$ with $a < x$. Choose $\epsilon < x - a$. Since $a_n \to a$, there exists some N so that $|a_n - a| \leq \epsilon < x - a$ for all $n \geq N$. Pick any $n \geq N$. We have

$$\begin{aligned} |a_n - a| &< x - a, && \text{which implies} \\ a_n - a &< x - a, && \text{which implies} \\ a_n &< x. \end{aligned}$$

This contradicts the fact that a_n is an upper bound for S. Hence it must be the case that a is an upper bound for S.

(To get started, it helps to draw a picture).

$\boxed{5}$ Suppose that $a_n \to 0$ and $\{b_n\}$ is bounded. Prove that $a_n b_n \to 0$.

Solution. Let $\epsilon > 0$ be arbitrary. Since $\{b_n\}$ is bounded, there exists some $M > 0$ so that $|b_n| \leq M$ for all $n \in \mathbb{N}$. Since $a_n \to 0$, there exists some $N \in \mathbb{N}$ so that $|a_n| = |a_n - 0| \leq \frac{\epsilon}{M}$ for all $n \geq N$. So for any $n \geq N$, we have

$$\begin{aligned}
|a_n b_n - 0| &= |a_n b_n| \\
&\leq |a_n| \cdot M \\
&\leq \frac{\epsilon}{M} \cdot M && \text{by choice of } N \\
&= \epsilon.
\end{aligned}$$

Hence $a_n b_n \to 0$.

$\boxed{6}$ For the following true and false questions, you do not need to explain your answer at all. Just write "True" or "False".

(a) True or false: There exists a one-to-one function $f\colon \mathbb{Q}\times\mathbb{Q}\to\mathbb{N}$.

Solution. True. $\mathbb{Q}$ is countable by Theorem 1.3.5, hence $\mathbb{Q}\times\mathbb{Q}$ is countable by Proposition 1.3.4, and hence there exists a one-to-one and onto function $f\colon \mathbb{Q}\times\mathbb{Q}\to\mathbb{N}$ by definition.

(b) True or false: If a sequence $\{a_n\}$ is not bounded, then it either diverges to $+\infty$ or diverges to $-\infty$.

Solution. False. Consider the sequence given by

$$a_n = \begin{cases} 0 & \text{if } n \text{ is odd, and} \\ n & \text{if } n \text{ is even.} \end{cases}$$

This sequence is not bounded but does not diverge to either $+\infty$ or $-\infty$.

(c) True or false: If r_1 and r_2 are irrational numbers with $r_1 < r_2$, then there exists a rational number q satisfying $r_1 < q < r_2$.

Solution. True. In homework §1.1 #11 we proved that between any two real numbers $r_1 < r_2$ there exists a rational number q satisfying $r_1 < q < r_2$.

(d) True or false: If a sequence $\{a_n\}$ has exactly one limit point d, then sequence $\{a_n\}$ converges to d.

Solution. False. Consider the sequence given by

$$a_n = \begin{cases} 0 & \text{if } n \text{ is odd, and} \\ n & \text{if } n \text{ is even.} \end{cases}$$

This sequence has 0 as its only limit point, but does not converge to any limit.

(e) True or false: If function $f\colon S\to T$ is one-to-one, then its inverse function $f^{-1}\colon \mathrm{Ran}(f)\to S$ is one-to-one.

Solution. True. Suppose that $f^{-1}(t_1) = f^{-1}(t_2)$ for some $t_1, t_2 \in T$. Let $s \in S$ satisfy $f^{-1}(t_1) = s = f^{-1}(t_2)$. This means that $f(s) = t_1$ and $f(s) = t_2$; hence it must be the case that $t_1 = t_2$. So f^{-1} is one-to-one.