Duke Math 431 **Practice Final** **April 19, 2015**

Name: ____

- This is the Practice Final for Duke Math 431. Partial credit is available. No notes, books, calculators, or other electronic devices are permitted.
- Write proofs that consist of complete sentences, make your logic clear, and justify all conclusions that you make.
- Please sign below to indicate you accept the following statement:

  "I have abided with all aspects of the honor code on this examination."

Signature: ____

Problem	Total Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
Total	110	

1 (a) State the precise definition of when a function $\rho\colon \mathcal{M}\times\mathcal{M}\to[0,\infty)$ is a metric.

Solution. Function $\rho\colon \mathcal{M}\times\mathcal{M}\to[0,\infty)$ is a metric if

(a) For all $x,y\in\mathcal{M}$ we have $\rho(x,y)\geq 0$, and $\rho(x,y)=0$ if and only if $x=y$.
(b) For all $x,y\in\mathcal{M}$ we have $\rho(x,y)=\rho(y,x)$.
(c) For all $x,y,z\in\mathcal{M}$ we have $\rho(x,z)\leq\rho(x,y)+\rho(y,z)$.

(b) Let $(\mathcal{M}_1,\rho_1)$ and $(\mathcal{M}_2,\rho_2)$ be metric spaces. Prove that $(\mathcal{M}_1\times\mathcal{M}_2,\rho)$ is a metric space, where $\rho\colon(\mathcal{M}_1\times\mathcal{M}_2)\times(\mathcal{M}_1\times\mathcal{M}_2)\to[0,\infty)$ is defined by the formula

$$\rho\big((x_1,x_2),(y_1,y_2)\big)=\rho_1(x_1,y_1)+\rho_2(x_2,y_2).$$

Solution. For (a), note that

$$\rho\big((x_1,x_2),(y_1,y_2)\big)=\rho_1(x_1,y_1)+\rho_2(x_2,y_2)\geq 0,$$

and that

$$\begin{aligned}&\rho\big((x_1,x_2),(y_1,y_2)\big)=0\\ \Longleftrightarrow\ &\rho_1(x_1,y_1)=0 \text{ and } \rho_2(x_2,y_2)=0\\ \Longleftrightarrow\ &x_1=y_1 \text{ and } x_2=y_2\\ \Longleftrightarrow\ &(x_1,x_2)=(y_1,y_2).\end{aligned}$$

For (b), note that

$$\rho\big((x_1,x_2),(y_1,y_2)\big)=\rho_1(x_1,y_1)+\rho_2(x_2,y_2)=\rho(y_1,x_1)+\rho(y_2,x_2)=\rho\big((y_1,y_2),(x_1,x_2)\big).$$

To see the triangle inequality (c), note

$$\begin{aligned}\rho\big((x_1,x_2),(z_1,z_2)\big)&=\rho_1(x_1,z_1)+\rho_2(x_2,z_2)\\ &\leq\rho_1(x_1,y_1)+\rho_1(y_1,z_1)+\rho_2(x_2,y_2)+\rho_2(y_2,z_2)\\ &=\rho_1(x_1,y_1)+\rho_2(x_2,y_2)+\rho_1(y_1,z_1)+\rho_2(y_2,z_2)\\ &=\rho\big((x_1,x_2),(y_1,y_2)\big)+\rho\big((y_1,y_2),(z_1,z_2)\big)\end{aligned}$$

Hence $(\mathcal{M}_1\times\mathcal{M}_2,\rho)$ is a metric space.

2 Let p and q be integers, $q \neq 0$. Suppose that $f(x) = x^{p/q}$ is differentiable for $x > 0$. Prove that $\frac{d}{dx}x^{p/q} = \frac{p}{q}x^{p/q-1}$.
Hint: differentiate $f(x)^q = x^p$.

Solution. We use the chain rule (Theorem 4.1.3) to differentiate $f(x)^q = x^p$, and we obtain

$$\begin{aligned} px^{p-1} &= \frac{d}{dx}x^p && \text{by §4.1 Example 3} \\ &= \frac{d}{dx}f(x)^q \\ &= q(f(x))^{q-1}f'(x) && \text{by the chain rule (Theorem 4.1.3)} \\ &= qx^{\frac{p(q-1)}{q}}f'(x). \end{aligned}$$

Solving for $f'(x)$ gives

$$f'(x) = \frac{p}{q}x^{p-1-\frac{p(q-1)}{q}} = \frac{p}{q}x^{\frac{pq-q-pq+p}{q}} = \frac{p}{q}x^{\frac{p}{q}-1}.$$

3 Let f_n, f, and g be functions defined on $[a, b]$. Suppose that g is continuous.

(a) Prove that if $f_n \to f$ pointwise, then $gf_n \to gf$ pointwise.

Solution. Note for any $x \in [a, b]$, we have

$$\begin{aligned}
\lim_{n\to\infty} (gf_n)(x) &= \lim_{n\to\infty} g(x)f_n(x) \\
&= g(x) \lim_{n\to\infty} f_n(x) && \text{since } g(x) \text{ is a constant} \\
&= g(x)f(x) && \text{since } f_n \to f \text{ pointwise} \\
&= (gf)(x).
\end{aligned}$$

Hence $gf_n \to gf$ pointwise.

(b) Prove that if $f_n \to f$ uniformly, then $gf_n \to gf$ uniformly.

Solution. Since $g\colon [a, b] \to \mathbb{R}$ is continuous, Theorem 3.2.1 says there is some $M > 0$ with $|g(x)| \le M$ for all $x \in [a, b]$. Let $\epsilon > 0$ be arbitrary. Since $f_n \to f$ uniformly, there is some $N \in \mathbb{N}$ so that $n \ge N$ implies $|f_n(x) - f(x)| \le \frac{\epsilon}{M}$ for all $x \in [a, b]$. Then note $n \ge N$ implies

$$\begin{aligned}
|(gf_n)(x) - (gf)(x)| &= |g(x)f_n(x) - g(x)f(x)| \\
&= |g(x)| \cdot |f_n(x) - f(x)| \\
&\le M \cdot \frac{\epsilon}{M} \\
&= \epsilon && \text{for all } x \in [a, b].
\end{aligned}$$

Hence $gf_n \to gf$ uniformly on $[a, b]$.

4 Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers such that $\{a_n\}$ converges to a limit $a \in \mathbb{R}$ and $\{b_n\}$ is bounded. Prove that

$$\limsup_{n\to\infty}(a_n + b_n) = \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

Solution. Let $a = \lim_{n\to\infty} a_n = \limsup a_n$ and let $\overline{b} = \limsup b_n$. We will show that $a + \overline{b}$ satisfies the properties of Theorem 6.1.1(a) for sequence $a_n + b_n$, and hence the uniqueness part of this theorem will guarantee that $\limsup(a_n + b_n) = a + \overline{b}$.

Let $\epsilon > 0$. Since $a_n \to a$ we know there is some $N_1 \in \mathbb{N}$ so that for all $n \geq N_1$ we have $a_n \leq a + \frac{\epsilon}{2}$, and since $\overline{b} = \limsup b_n$ we know there is some $N_2 \in \mathbb{N}$ so that for all $n \geq N_2$ we have $b_n \leq \overline{b} + \frac{\epsilon}{2}$. Hence for all $n \geq \max\{N_1, N_2\}$ we have

$$a_n + b_n \leq \left(a + \frac{\epsilon}{2}\right) + \left(\overline{b} + \frac{\epsilon}{2}\right) = a + \overline{b} + \epsilon.$$

Let $\epsilon > 0$ and $N \in N$. Since $a_n \to a$ we know there is some $N_1 \in \mathbb{N}$ so that for all $n \geq N_1$ we have $a_n \geq a - \frac{\epsilon}{2}$, and since $\overline{b} = \limsup b_n$ we know that there is some $n \geq \max\{N, N_1\}$ so that $b_n \geq \overline{b} - \frac{\epsilon}{2}$. Hence we have found some $n \geq N$ so that

$$a_n + b_n \geq \left(a - \frac{\epsilon}{2}\right) + \left(\overline{b} - \frac{\epsilon}{2}\right) = a + \overline{b} - \epsilon.$$

By Theorem 6.1.1(a) we have

$$\limsup(a_n + b_n) = a + \overline{b} = \limsup a_n + \limsup b_n.$$

5 Let $f\colon (0,\infty) \to \mathbb{R}$ be defined by $f(x) = \frac{1}{x^2}$.

(a) Prove that f is continuous on $(0,\infty)$.

Solution. Since x^2 is a polynomial, we know (by page 74 of the book) that x^2 is continuous on $(0,\infty)$. Theorem 3.1.1(d) says the ratio of two continuous functions is continuous wherever the denominator is not zero, and hence $f(x) = \frac{1}{x^2}$ is continuous on $(0,\infty)$.

(b) Prove that f is not uniformly continuous on $(0,\infty)$.

Solution. Let $\epsilon = 1$. Given any $\delta > 0$, choose $n \in \mathbb{N}$ so that $n \geq \frac{1}{\delta}$, which gives $\frac{1}{n} \leq \delta$. Let $x = \frac{1}{n}$ and let $c = \frac{1}{n+1}$. Note that we have

$$|x - c| = \left|\frac{1}{n} - \frac{1}{n+1}\right| = \frac{1}{n} - \frac{1}{n+1} < \frac{1}{n} \leq \delta$$

but

$$|f(x) - f(c)| = \left|\frac{1}{1/n^2} - \frac{1}{1/(n+1)^2}\right| = |n^2 - (n+1)^2| = 2n + 1 > \epsilon.$$

Hence we've shown that f is not uniformly continuous on $(0,\infty)$.

6 Consider the series $\sum_{j=1}^{\infty} 2^{(-1)^j} (\frac{1}{2})^j$.

(a) Use the comparison test to determine whether this series converges or diverges.

Solution. Note

$$0 \leq 2^{(-1)^j} (\tfrac{1}{2})^j \leq 2(\tfrac{1}{2})^j.$$

The series $\sum 2(\frac{1}{2})^j$ converges because it is a constant multiple of a geometric series with ratio $\frac{1}{2} < 1$. Hence the Comparison Test (Theorem 6.2.2) implies that $\sum 2^{(-1)^j} (\frac{1}{2})^j$ converges.

(b) Does the ratio test determine whether this series converges or diverges, or is it inconclusive?

Solution. Note

$$\frac{|a_{j+1}|}{|a_j|} = \begin{cases} \frac{2\cdot(1/2)}{1/2} = 2 & \text{if } j \text{ is odd} \\ \frac{(1/2)\cdot(1/2)}{2} = \frac{1}{8} & \text{if } j \text{ is even.} \end{cases}$$

Since $\limsup \frac{|a_{j+1}|}{|a_j|} = 2 \geq 1$ and $\liminf \frac{|a_{j+1}|}{|a_j|} = \frac{1}{8} \leq 1$, the Ratio Test (Theorem 6.2.4) is inconclusive.

(c) Does the root test determine whether this series converges or diverges, or is it inconclusive?

Solution. We claim that $|a_j|^{1/j} = (2^{(-1)^j})^{1/j} \cdot \frac{1}{2} \to \frac{1}{2}$ as $j \to \infty$. This is because

$$\left(\frac{1}{2}\right)^{1/j} \leq (2^{(-1)^j})^{1/j} \leq 2^{1/j},$$

and so the Squeeze Theorem implies that $(2^{(-1)^j})^{1/j} \to 1$ since we have both $(\frac{1}{2})^{1/j} \to 1$ and $2^{1/j} \to 1$ as $j \to \infty$.

Since $\limsup |a_j|^{1/j} = \frac{1}{2} < 1$, the Root Test (Theorem 6.2.3) says that $\sum 2^{(-1)^j} (\frac{1}{2})^j$ converges.

7 Let $n \in \mathbb{N}$, and let S_1, S_2, ..., S_n be countable sets. Recall

$$S_1 \times S_2 \times \ldots \times S_n = \{(s_1, s_2, \ldots, s_n) \mid s_i \in S_i \text{ for all } 1 \leq i \leq n\}.$$

(a) Construct a one-to-one function $h\colon S_1 \times S_2 \times \ldots \times S_n \to \mathbb{N}$.

Solution. We mimic the proof of Proposition 1.3.4. Let $1 \leq i \leq n$. Since S_i is countable, there exists a one-to-one and onto function $f_i\colon S_i \to \mathbb{N}$. Let $p_1, p_2, \ldots, p_n$ be the first n prime numbers. We define function

$$h\colon S_1 \times S_2 \times \ldots \times S_n \to \mathbb{N}$$

by setting

$$h(s_1, s_2, \ldots, s_n) = p_1^{f_1(s_1)} \cdot p_2^{f_2(s_2)} \cdot \ldots \cdot p_n^{f_n(s_n)}.$$

The fact that h is one-to-one follows from The Fundamental Theorem of Arithmetic (Theorem 1.3.3), which says that prime factorizations are unique.

(b) Prove that $S_1 \times S_2 \times \ldots \times S_n$ is countable.

Solution. In (a) we constructed a one-to-one function $h\colon S_1 \times S_2 \times \ldots \times S_n \to \mathbb{N}$. This shows that $S_1 \times S_2 \times \ldots \times S_n$ has the cardinality of an infinite subset of the countable set $\mathbb{N}$. Since any any infinite subset of a countable set is countable (by Proposition 1.3.2), this shows that $S_1 \times S_2 \times \ldots \times S_n$ is countable.

8 Suppose ρ and σ are two metrics on a set $\mathcal{M}$. Suppose there are positive constants c_1 and c_2 such that for all $x, y \in \mathcal{M}$, we have

$$\rho(x,y) \leq c_1\sigma(x,y) \quad \text{and} \quad \sigma(x,y) \leq c_2\rho(x,y).$$

Prove that metric space $(\mathcal{M}, \rho)$ is complete if and only if $(\mathcal{M}, \sigma)$ is complete.

Solution. We will show that if $(\mathcal{M}, \rho)$ is complete, then so is $(\mathcal{M}, \sigma)$. The reverse direction is symmetric.

Suppose $\{x_n\}$ is a Cauchy sequence in $(\mathcal{M}, \sigma)$. We first show that $\{x_n\}$ is also a Cauchy sequence in $(\mathcal{M}, \rho)$. To see this, given $\epsilon > 0$ note there is an $N \in \mathbb{N}$ so that $n, m \geq N$ implies $\sigma(x_n, x_m) \leq \frac{\epsilon}{c_1}$, and hence $n, m \geq N$ implies

$$\rho(x_n, x_m) \leq c_1\sigma(x_n, x_m) \leq c_1\frac{\epsilon}{c_1} = \epsilon.$$

Since $\{x_n\}$ is a Cauchy sequence in the complete metric space $(\mathcal{M}, \rho)$, there is some $x \in \mathcal{M}$ so that $\rho(x_n, x) \to 0$ as $n \to \infty$. Hence given $\epsilon > 0$, there is an $N \in \mathbb{N}$ so that $n \geq N$ implies $\rho(x_n, x) \leq \frac{\epsilon}{c_2}$, and hence $n \geq N$ implies

$$\sigma(x_n, x) \leq c_2\rho(x_n, x) \leq c_2\frac{\epsilon}{c_2} = \epsilon.$$

We have shown $\sigma(x_n, x) \to 0$ as $n \to \infty$, meaning that $\{x_n\}$ converges to x in $(\mathcal{M}, \sigma)$. So $(\mathcal{M}, \sigma)$ is complete.

9 Let $\{a_n\}$ be a bounded sequence, and let $L \in \mathbb{R}$. Suppose that every convergent subsequence of $\{a_n\}$ has limit L. Prove that $\lim_{n\to\infty} a_n = L$.

Solution 1. Suppose for a contradiction that $\{a_n\}$ does not have limit L. Hence there exists an $\epsilon > 0$ such that for any $N \in \mathbb{N}$, there exists some $n > N$ with $|a_n - L| > \epsilon$. This allows us to define $n_1 < n_2 < n_3 \dots$ such that $|a_{n_k} - L| > \epsilon$ for all $k \in \mathbb{N}$. Since $\{a_{n_k}\}$ is a bounded sequence, by Bolzano-Weierstrass it has a convergent subsequence, which clearly cannot converge to L. This is a contradiction, and so it must be that $\lim_{n\to\infty} a_n = L$.

Solution 2. By Proposition 2.6.1 there is a subsequence converging to any limit point of $\{a_n\}$, and it follows from the hypothesis that L is the only limit point of $\{a_n\}$. Let P be the set of all limit points of $\{a_n\}$; in this case we have $P = \{L\}$. By homework 6.1 #9 we know that $\limsup a_n = \sup P = L$, and $\liminf a_n = \inf P = L$. Since the $\limsup$ and $\liminf$ are equal, Corollary 6.1.2 says that $\lim_{n\to\infty} a_n = L$.

10 Suppose that $f\colon [a,b] \to \mathbb{R}$ is continuous. Suppose $c \in (a,b)$ is a point where f achieves its maximum. Prove that if f is differentiable at c, then $f'(c) = 0$.
Remark: I am not asking you to say "this is a theorem from the book" (in fact Theorem 4.2.1); I'm asking you to prove this theorem.

Solution. Suppose for a contradiction that $f'(c) \neq 0$. We will do the case when $f'(c) > 0$; the case when $f'(c) < 0$ is analogous. Since $\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = f'(c)$, there exists a $\delta > 0$ so that

$$\frac{f(c+h)-f(c)}{h} \geq \frac{f'(c)}{2} \quad \text{if } 0 < |h| \leq \delta.$$

Thus for $0 < h \leq \delta$ we have

$$f(c+h) \geq f(c) + h\frac{f'(c)}{2} > f(c).$$

This contradicts the hypothesis that f achieves its maximum at c. Hence it must be the case that $f'(c) = 0$.

11 For the following true and false questions, you do not need to explain your answer at all. Just write "True" or "False".

(a) True or false: Every monotone increasing sequence either converges to a finite limit or diverges to infinity.

Solution. True. Indeed, if the monotone increasing sequence is bounded then Theorem 2.4.3 says it converges, and if it is not bounded then page 50 explains why it diverges to infinity.

(b) True or false: If a bounded sequence $\{a_n\}$ has exactly one limit point d, then sequence $\{a_n\}$ converges to d.

Solution. True. The assumption of boundedness is crucial here. Note that 6.1 #9 says that $\limsup a_n = d = \liminf a_n$, and hence $\lim_{n\to\infty} a_n = d$ by Corollary 6.1.2.

(c) Let $f_n \colon [0,1] \to \mathbb{R}$ be a sequence of continuously differentiable functions and let $f \colon [0,1] \to \mathbb{R}$ be a function. If $f_n \to f$ uniformly then f is continuously differentiable.

Solution. False. For a counterexample, draw a picture of continuously differentiable functions that converge uniformly to $|x - \frac{1}{2}|$, which is not differentiable at $x = \frac{1}{2}$.

(d) True or false: The function $f \colon [0,1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

is Riemann integrable.

Solution. True. Here's a sketch of a proof. We'll use Lemma 3 in §3.3. Let $\epsilon > 0$; we must bound $U_P(f) - L_P(f)$ by ϵ. Include $[0, \frac{\epsilon}{2}]$ as a subinterval in our partition. If N is such that $\frac{1}{N+1} \le \frac{\epsilon}{2}$, then f has only N discontinuities in the interval $[\frac{\epsilon}{2}, 1]$, located at $\frac{1}{1}$, $\frac{1}{2}$, $\ldots$, $\frac{1}{N}$. Place a subinterval of width at most $\frac{\epsilon}{2N}$ around each such discontinuity. Then for this partition we have

$$U_P(f) - L_P(f) \le 1 \cdot \frac{\epsilon}{2} + \sum_{i=1}^{N} 1 \cdot \frac{\epsilon}{2N} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

(e) True or false: For $f, g \in C[0,1]$, let $\rho_1(f,g) = \|f-g\|_1$. Let $\{f_n\}$ be a sequence of functions in $C[0,1]$. If $\{f_n\}$ converges in the metric space $(C[0,1], \rho_1)$ to some function $f \in C[0,1]$, then $\int_0^1 f_n(x)\,dx \to \int_0^1 f(x)\,dx$ as $n \to \infty$.

Solution. True. Note that f_n converging to f in $(C[0,1], \rho_1)$ means

$$\rho_1(f_n, f) = \int_0^1 |f_n(x) - f(x)|\,dx \to 0 \quad \text{as} \quad n \to \infty,$$

which implies $\int_0^1 (f_n(x) - f(x))\,dx \to 0$ as $n \to \infty$, which implies $\int_0^1 f_n(x)\,dx \to \int_0^1 f(x)\,dx$ as $n \to \infty$.