Duke Math 431 **Practice Final** **April 19, 2015**

Name: ____________________________________

- This is the Practice Final for Duke Math 431. Partial credit is available. No notes, books, calculators, or other electronic devices are permitted.
- Write proofs that consist of complete sentences, make your logic clear, and justify all conclusions that you make.
- Please sign below to indicate you accept the following statement:

  "I have abided with all aspects of the honor code on this examination."

Signature: ____

Problem	Total Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
Total	110	

1 (a) State the precise definition of when a function $\rho\colon \mathcal{M} \times \mathcal{M} \to [0, \infty)$ is a metric.

(b) Let $(\mathcal{M}_1, \rho_1)$ and $(\mathcal{M}_2, \rho_2)$ be metric spaces. Prove that $(\mathcal{M}_1 \times \mathcal{M}_2, \rho)$ is a metric space, where $\rho\colon (\mathcal{M}_1 \times \mathcal{M}_2) \times (\mathcal{M}_1 \times \mathcal{M}_2) \to [0, \infty)$ is defined by the formula

$$\rho\big((x_1, x_2), (y_1, y_2)\big) = \rho_1(x_1, y_1) + \rho_2(x_2, y_2).$$

2 Let p and q be integers, $q \neq 0$. Suppose that $f(x) = x^{p/q}$ is differentiable for $x > 0$. Prove that $\frac{d}{dx} x^{p/q} = \frac{p}{q} x^{p/q-1}$.
Hint: differentiate $f(x)^q = x^p$.

3 Let f_n, f, and g be functions defined on $[a, b]$. Suppose that g is continuous.

(a) Prove that if $f_n \to f$ pointwise, then $gf_n \to gf$ pointwise.

(b) Prove that if $f_n \to f$ uniformly, then $gf_n \to gf$ uniformly.

4 Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers such that $\{a_n\}$ converges to a limit $a \in \mathbb{R}$ and $\{b_n\}$ is bounded. Prove that

$$\limsup_{n\to\infty}(a_n + b_n) = \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

5 Let $f\colon (0,\infty) \to \mathbb{R}$ be defined by $f(x) = \frac{1}{x^2}$.

(a) Prove that f is continuous on $(0,\infty)$.

(b) Prove that f is not uniformly continuous on $(0,\infty)$.

6 Consider the series $\sum_{j=1}^{\infty} 2^{(-1)^j} (\frac{1}{2})^j$.

(a) Use the comparison test to determine whether this series converges or diverges.

(b) Does the ratio test determine whether this series converges or diverges, or is it inconclusive?

(c) Does the root test determine whether this series converges or diverges, or is it inconclusive?

7 Let $n \in \mathbb{N}$, and let S_1, S_2, ..., S_n be countable sets. Recall

$$S_1 \times S_2 \times \ldots \times S_n = \{(s_1, s_2, \ldots, s_n) \mid s_i \in S_i \text{ for all } 1 \leq i \leq n\}.$$

(a) Construct a one-to-one function $h\colon S_1 \times S_2 \times \ldots \times S_n \to \mathbb{N}$.

(b) Prove that $S_1 \times S_2 \times \ldots \times S_n$ is countable.

8 Suppose ρ and σ are two metrics on a set $\mathcal{M}$. Suppose there are positive constants c_1 and c_2 such that for all $x, y \in \mathcal{M}$, we have

$$\rho(x,y) \leq c_1 \sigma(x,y) \quad \text{and} \quad \sigma(x,y) \leq c_2 \rho(x,y).$$

Prove that metric space $(\mathcal{M}, \rho)$ is complete if and only if $(\mathcal{M}, \sigma)$ is complete.

9 Let $\{a_n\}$ be a bounded sequence, and let $L \in \mathbb{R}$. Suppose that every convergent subsequence of $\{a_n\}$ has limit L. Prove that $\lim_{n\to\infty} a_n = L$.

[10] Suppose that $f\colon [a,b] \to \mathbb{R}$ is continuous. Suppose $c \in (a,b)$ is a point where f achieves its maximum. Prove that if f is differentiable at c, then $f'(c) = 0$.
Remark: I am not asking you to say "this is a theorem from the book" (in fact Theorem 4.2.1); I'm asking you to prove this theorem.

11 For the following true and false questions, you do not need to explain your answer at all. Just write "True" or "False".

(a) True or false: Every monotone increasing sequence either converges to a finite limit or diverges to infinity.

(b) True or false: If a bounded sequence $\{a_n\}$ has exactly one limit point d, then sequence $\{a_n\}$ converges to d.

(c) Let $f_n \colon [0,1] \to \mathbb{R}$ be a sequence of continuously differentiable functions and let $f \colon [0,1] \to \mathbb{R}$ be a function. If $f_n \to f$ uniformly then f is continuously differentiable.

(d) True or false: The function $f \colon [0,1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

is Riemann integrable.

(e) True or false: For $f, g \in C[0,1]$, let $\rho_1(f,g) = \|f - g\|_1$. Let $\{f_n\}$ be a sequence of functions in $C[0,1]$. If $\{f_n\}$ converges in the metric space $(C[0,1], \rho_1)$ to some function $f \in C[0,1]$, then $\int_0^1 f_n(x)\, dx \to \int_0^1 f(x)\, dx$ as $n \to \infty$.