Duke Math 431 **Midterm 2** **March 27, 2015**

Name: ____

- This is Midterm 2 for Duke Math 431. Partial credit is available. No notes, books, calculators, or other electronic devices are permitted.
- Write proofs that consist of complete sentences, make your logic clear, and justify all conclusions that you make.
- Please sign below to indicate you accept the following statement:

  "I have abided with all aspects of the honor code on this examination."

Signature: ____

Problem	Total Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
Total	60	

1 (a) Give the precise definition of when a function $f\colon Dom(f) \to \mathbb{R}$ is uniformly continuous. (4 points)

Solution. Function $f\colon Dom(f) \to \mathbb{R}$ is uniformly continuous if for each $\epsilon > 0$ there is a $\delta > 0$ so that for all $x, c \in Dom(f)$, we have that $|x - c| \leq \delta$ implies $|f(x) - f(c)| \leq \epsilon$.

(b) Prove that the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$. (6 points)

Solution. This is analogous to homework problem 3.2 #9. Let $\epsilon = 1$. Given any $\delta > 0$, choose $n \in \mathbb{N}$ so that $n \geq \frac{1}{\delta}$, which gives $\frac{1}{n} \leq \delta$. Consider $x = n + \frac{1}{n}$ and $c = n$. Note that $|x - c| = |n + \frac{1}{n} - n| = \frac{1}{n} \leq \delta$ while

$$\begin{aligned} |f(x) - f(c)| &= |(n + \tfrac{1}{n})^2 - n^2| \\ &= |n^2 + 2 + \tfrac{1}{n^2} - n^2| \\ &= 2 + \tfrac{1}{n^2} \\ &> \epsilon. \end{aligned}$$

Hence f is not uniformly continuous on $\mathbb{R}$.

2 (a) Compute the fourth-order Taylor polynomial $T^{(4)}(x,0)$ for $f(x) = \cos x$ about $x = 0$. (4 points)

Solution. We compute

$$f'(x) = -\sin x, \quad f''(x) = -\cos x, \quad f'''(x) = \sin x, \quad f^{(4)}(x) = \cos x.$$

Hence

$$f(0) = 1, \quad f'(0) = -0, \quad f''(0) = -1, \quad f'''(0) = 0, \quad f^{(4)}(0) = 1.$$

This gives

$$T^{(4)}(x,0) = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4.$$

(b) Use Taylor's Theorem to prove $\lim_{x\to 0} \frac{\cos x - 1 + \frac{1}{2}x^2}{x^4}$ exists. (6 points)

Solution. This is analogous to homework problem 4.3 #9. We compute $f^{(5)}(x) = -\sin x$. Given $\epsilon > 0$, choose $\delta = (5!)\epsilon$. For $0 < |x| < \delta = (5!)\epsilon$, we have

$$\begin{aligned}
\left|\frac{\cos x - 1 + \frac{1}{2}x^2}{x^4} - \frac{1}{4!}\right| &= \left|\frac{(1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{\sin\xi}{5!}x^5) - 1 + \frac{1}{2}x^2}{x^4} - \frac{1}{4!}\right| \\
&\quad \text{for some } \xi \text{ between } 0 \text{ and } x \text{ by Taylor's Theorem (Theorem 4.3.1)} \\
&= \left|\frac{1}{4!} - \frac{\sin(\xi)x}{5!} - \frac{1}{4!}\right| \\
&= \left|\frac{\sin(\xi)x}{5!}\right| \\
&\le \left|\frac{x}{5!}\right| \quad \text{since } |\sin(\xi)| \le 1 \\
&\le \epsilon \quad \text{by choice of } \delta.
\end{aligned}$$

Hence $\lim_{x\to 0} \frac{\cos x - 1 + \frac{1}{2}x^2}{x^4} = \frac{1}{4}$.

3 Suppose $f\colon [0,1) \to \mathbb{R}$ is a continuously differentiable function that is not bounded, and for simplicity assume $f(0) = 0$. Prove that $f'\colon [0,1) \to \mathbb{R}$ is not bounded.

Solution. Suppose for a contradiction that f' were bounded, meaning that there is some $M \in \mathbb{R}$ with $|f'(x)| \leq M$ for all $x \in [0,1)$. Then for any $x \in [0,1)$, Part I of the Fundamental Theorem of Caluculus (Theorem 4.2.4) gives

$$\begin{aligned}
|f(x)| &= |f(x) - f(0)| && \text{since } f(0) = 0 \\
&= \left| \int_0^x f'(t)dt \right| \\
&\leq \int_0^x |f'(t)|dt && \text{by Theorem 3.3.5} \\
&\leq \int_0^x M dt \\
&= Mx \\
&< M && \text{since } x \in [0,1).
\end{aligned}$$

This shows that f is bounded, a contradiction. Hence it must be the case that f' is not bounded.

4 Define $f\colon [0,3] \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } 1 < x \leq 2 \\ 2 & \text{if } 2 < x \leq 3. \end{cases}$$

Prove that f is Riemann integrable.

Solution. Consider the partition P given by $0 < \frac{1}{N} < \frac{2}{N} < \ldots < \frac{3N-1}{N} < 3N$ with $N \in \mathbb{N}$. Note

$$\begin{aligned} U_P(f) - L_P(f) &= \sum_{i=1}^{3N} (M_i - m_i)\frac{1}{N} \\ &= \frac{1}{N}\sum_{i=1}^{3N}(M_i - m_i) \\ &= \frac{1}{N}\Big(\Big(\sum_{i=1}^{N}\frac{1}{N}\Big) + 1\Big) \text{ since we have } M_i - m_i = \frac{1}{N} \text{ for } 1 \leq i \leq N, \\ &\qquad \text{since } M_{2n+1} - m_{2n+1} = 1, \text{ and since all other terms are zero} \\ &= \frac{1}{N}(1+1) \\ &= \frac{2}{N}. \end{aligned}$$

Given any $\epsilon > 0$, choose $N \in \mathbb{N}$ with $\mathbb{N} \geq \frac{2}{\epsilon}$, which gives $U_P(f) - L_P(f) \leq \epsilon$. Hence by Lemma 3 in §3 we have shown that f is Riemann integrable.

5 Let $f\colon \mathbb{R} \to \mathbb{R}$ via

$$f(x) = \begin{cases} x^2 \cos(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

(a) For $x \neq 0$, compute $f'(x)$.

Solution. All three parts of this problem are analogous to 4.1 #12.

$$\begin{aligned} f'(x) &= 2x\cos(1/x) + x^2(-\sin(1/x))(\tfrac{-1}{x^2}) \\ &= 2x\cos(1/x) + \sin(1/x). \end{aligned}$$

(b) Compute $f'(0)$.

Solution.

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h\to 0} \frac{h^2\cos(1/h)}{h} \\ &= \lim_{h\to 0} h\cos(1/h) \\ &= 0 && \text{by the Squeeze Theorem, or since } |\cos(1/h)| \leq 1. \end{aligned}$$

(c) Prove f' is not continuous at $x = 0$.

Solution. Let $x_n = \frac{1}{\pi/2 + 2\pi n}$. Note that $x_n \to 0$ as $n \to \infty$ with $x_n \neq 0$. We have

$$\lim_{n\to\infty} f'(x_n) = \lim_{n\to\infty} 2x_n\cos(1/x_n) + \sin(1/x_n) = \lim_{n\to\infty} 1 = 1 \neq 0 = f'(0).$$

This shows that f' is not continuous at $x = 0$.

6 State Rolle's Theorem (along with its hypotheses) for a function $g\colon [a,b] \to \mathbb{R}$.
State the Mean Value Theorem (along with its hypotheses) for a function $f\colon [a,b] \to \mathbb{R}$.
Use Rolle's Theorem to prove the Mean Value Theorem.

Solution.

Rolle's Theorem: If $g\colon [a,b] \to \mathbb{R}$ is continuous and differentiable and $g(a) = 0 = g(b)$, then there is a point c satisfying $a < c < b$ such that $f'(c) = 0$.

Mean Value Theorem: If $f\colon [a,b] \to \mathbb{R}$ is continuous and differentiable, then there is a point c satisfying $a < c < b$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

To prove the Mean Value theorem, define the function

$$g(x) = f(x) - \Big(f(a) + \frac{f(b)-f(a)}{b-a}(x-a)\Big).$$

Note that $g(a) = 0 = g(b)$. Hence we apply Rolle's Theorem to g to get a point c with $a < c < b$ satisfying $g'(c) = 0$. Taking the derivative of $g(x)$ at c, this gives

$$0 = g'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}.$$

This means $f'(c) = \frac{f(b)-f(a)}{b-a}$ as desired.