Duke Math 431 **Midterm 1** **February 16, 2015**

Name: ____

- This is Midterm 1 for Duke Math 431. Partial credit is available. No notes, books, calculators, or other electronic devices are permitted.
- Write proofs that consist of complete sentences, make your logic clear, and justify all conclusions that you make.
- Please sign below to indicate you accept the following statement:

  "I have abided with all aspects of the honor code on this examination."

Signature: ____

Problem	Total Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
Total	60	

1 (a) Give a precise definition of when a sequence $\{a_n\}$ of real numbers is a Cauchy sequence.

Solution. Sequence $\{a_n\}$ is a Cauchy sequence if, for any $\epsilon > 0$, there exists some $N \in \mathbb{N}$ so that $|a_n - a_m| \leq \epsilon$ if $n, m \geq N$.

(b) Give a precise definition of when a function $f : S \to T$ is one-to-one (also called injective).

Solution 1. Function $f : S \to T$ is one-to-one if for each $t \in \mathrm{Ran}(f)$ there is only one $s \in S$ so that $f(s) = t$.

Solution 2. Function $f : S \to T$ is one-to-one if for any $s_1, s_2 \in S$, the equality $f(s_1) = f(s_2)$ implies $s_1 = s_2$.

2 Prove that the sequence $\{a_n\}$ given by $a_n = \sqrt{2 + \frac{1}{n}}$ converges to a limit.

Solution. Given $\epsilon > 0$, choose $N \in \mathbb{N}$ so that $N \geq \frac{1}{\epsilon}$. Then note $n \geq N$ implies

$$\begin{aligned}
|a_n - \sqrt{2}| &= \left|\sqrt{2 + \frac{1}{n}} - \sqrt{2}\right| \\
&= \left|\frac{\left(\sqrt{2 + \frac{1}{n}} - \sqrt{2}\right)\left(\sqrt{2 + \frac{1}{n}} + \sqrt{2}\right)}{\sqrt{2 + \frac{1}{n}} + \sqrt{2}}\right| \\
&= \left|\frac{2 + \frac{1}{n} - 2}{\sqrt{2 + \frac{1}{n}} + \sqrt{2}}\right| \\
&= \frac{\frac{1}{n}}{\sqrt{2 + \frac{1}{n}} + \sqrt{2}} \\
&\leq \frac{1}{n} \\
&\leq \frac{1}{N} \\
&\leq \epsilon && \text{by choice of } N.
\end{aligned}$$

Hence we have shown that $\{a_n\}$ converges to the limit $\sqrt{2}$.

3 (a) Prove that if q and r are rational numbers, then their product qr is rational. *(You may use without comment that the product of two integers is an integer.)*

Solution. Let $q = \frac{a}{b}$ and $r = \frac{c}{d}$ for integers a, $b \neq 0$, c, and $d \neq 0$. Then $qr = \frac{a}{b} \cdot \frac{c}{d} = \frac{ab}{cd}$ is rational.

(b) Prove that if $q \neq 0$ is rational and r is irrational, then their product qr is irrational.

Solution. Suppose for a contradiction that qr were rational, hence $qr = \frac{a}{b}$ for integers a and $b \neq 0$. Since $q \neq 0$ is rational we can let $q = \frac{c}{d}$ for integers $c \neq 0$ and $d \neq 0$. Then we'd have $r = (qr)/q = \frac{a}{b}/\frac{c}{d} = \frac{ad}{bc}$ rational, a contradiction. Hence qr must be irrational.

4 Prove that if $\{a_n\}$ converges to a limit $a \in \mathbb{R}$, then $\{a_n\}$ is a Cauchy sequence.
(I am not asking you to say "This is a proposition from our book or from class"; I am asking you to give a proof of this proposition from the definitions.)

Solution. Let $\epsilon > 0$ be given. Since $a_n \to a$, there exists an integer N so that $n \geq N$ implies $|a_n - a| \leq \frac{\epsilon}{2}$. Therefore if $n, m \geq N$, then we have

$$\begin{aligned}
|a_n - a_m| &= |(a_n - a) + (a - a_m)| \\
&\leq |a_n - a| + |a - a_m| && \text{by the triangle inequality} \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Hence $\{a_n\}$ is a Cauchy sequence.

5 (a) (3 points). Give a precise definition of when a number b is an upper bound for a set S of real numbers.

Solution. Number b is an upper bound for set S if $x \leq b$ for all $x \in S$.

(b) (7 points). Let S be a set of real numbers and let $\{a_n\}$ be a convergent sequence with $a_n \to a$. Prove that if a_n is an upper bound for S for each n, then a is an upper bound for S.

Solution. Suppose for a contradiction that a is not an upper bound for S. Then there is some element $x \in S$ with $x > a$. Choose $\epsilon > 0$ with $\epsilon < x - a$. Since $a_n \to a$, there exists some integer N so that $|a_n - a| \leq \epsilon < x - a$ for all $n \geq N$. Hence

$$a_N - a \leq |a_N - a| < x - a$$

implies $a_N < x$. This contradicts the fact that a_N is an upper bound for S. Hence it must be the case that a is an upper bound for S.

6 For the following true and false questions, you do not need to explain your answer at all. Just write "True" or "False".

(a) True or false: There exists a function $f\colon \mathbb{R} \to \mathbb{Q}$ from the set of real numbers to the set of rational numbers which is onto (i.e. surjective).

Solution. True. Consider the function $f\colon \mathbb{R} \to \mathbb{Q}$ defined by

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

(b) True or false: If a sequence $\{a_n\}$ is bounded, then $\{a_n\}$ has a limit point.

Solution. True. Sine $\{a_n\}$ is bounded it has a convergent subsequence by the Bolzano-Weierstrass Theorem (Theorem 2.6.2), and hence it has a limit point by Proposition 2.6.1 (which says that d is a limit point of $\{a_n\}$ if and only if there exists a subsequence $\{a_{n_k}\}$ converging to d).

(c) True or false: If $\{a_n\}$ is a sequence of rational numbers and $a_n \to a$, then a is a rational number.

Solution. False. Consider the sequence of rational numbers given on page 47 of the book which converges to the irrational number π.

(d) True or false: If S is a bounded set and $\sup S$ is its least upper bound, then $\sup S \in S$.

Solution. False. Consider $S = [0, 1)$, which has $\sup S = 1$ but $1 \notin S$.

(e) True or false: If some subsequence $\{a_{n_k}\}$ of a sequence $\{a_n\}$ has $d \in \mathbb{R}$ as a limit point, then sequence $\{a_n\}$ also has d as a limit point.

Solution. True. Since $\{a_{n_k}\}$ has d as a limit point, that means that for any $\epsilon > 0$ and $K \in \mathbb{N}$ there exists a $k \geq K$ so that $|a_{n_k} - d| \leq \epsilon$. It is then not hard to see that for any $\epsilon > 0$ and $N \in \mathbb{N}$ there exists an $n \geq N$ so that $|a_n - d| \leq \epsilon$. (Indeed, just pick K so that $n_K \geq N$.) Hence $\{a_n\}$ has d as a limit point.