Duke Math 431 **Midterm 1** **February 16, 2015**

Name: ______________________________

- This is Midterm 1 for Duke Math 431. Partial credit is available. No notes, books, calculators, or other electronic devices are permitted.
- Write proofs that consist of complete sentences, make your logic clear, and justify all conclusions that you make.
- Please sign below to indicate you accept the following statement:

  "I have abided with all aspects of the honor code on this examination."

Signature: ______________________________________

Problem	Total Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
Total	60	

$\boxed{1}$ (a) Give a precise definition of when a sequence $\{a_n\}$ of real numbers is a Cauchy sequence.

(b) Give a precise definition of when a function $f\colon S \to T$ is one-to-one (also called injective).

$\boxed{2}$ Prove that the sequence $\{a_n\}$ given by $a_n = \sqrt{2 + \frac{1}{n}}$ converges to a limit.

$\boxed{3}$ (a) Prove that if q and r are rational numbers, then their product qr is rational.
(You may use without comment that the product of two integers is an integer.)

(b) Prove that if $q \neq 0$ is rational and r is irrational, then their product qr is irrational.

$\boxed{4}$ Prove that if $\{a_n\}$ converges to a limit $a \in \mathbb{R}$, then $\{a_n\}$ is a Cauchy sequence.
(I am not asking you to say "This is a proposition from our book or from class"; I am asking you to give a proof of this proposition from the definitions.)

$\boxed{5}$ (a) (3 points). Give a precise definition of when a number b is an upper bound for a set S of real numbers.

(b) (7 points). Let S be a set of real numbers and let $\{a_n\}$ be a convergent sequence with $a_n \to a$. Prove that if a_n is an upper bound for S for each n, then a is an upper bound for S.

$\boxed{6}$ For the following true and false questions, you do not need to explain your answer at all. Just write "True" or "False".

(a) True or false: There exists a function $f\colon \mathbb{R} \to \mathbb{Q}$ from the set of real numbers to the set of rational numbers which is onto (i.e. surjective).

(b) True or false: If a sequence $\{a_n\}$ is bounded, then $\{a_n\}$ has a limit point.

(c) True or false: If $\{a_n\}$ is a sequence of rational numbers and $a_n \to a$, then a is a rational number.

(d) True or false: If S is a bounded set and $\sup S$ is its least upper bound, then $\sup S \in S$.

(e) True or false: If some subsequence $\{a_{n_k}\}$ of a sequence $\{a_n\}$ has $d \in \mathbb{R}$ as a limit point, then sequence $\{a_n\}$ also has d as a limit point.