Duke Math 431
Spring 2015

Lipschitz vs Uniform Continuity

In §3.2 #7, we proved that if f is Lipschitz continuous on a set $S \subseteq \mathbb{R}$ then f is uniformly continuous on S. The reverse is not true: a function may be uniformly continuous on a domain while not being Lipschitz continuous on that domain. Indeed, consider the following problem.

§3.2 #11. Show by example that a function can be uniformly continuous without being Lipschitz continuous. Hint: consider $f(x) = \sqrt{x}$.

Solution. By §3.2 #10, $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$, and hence it suffices to show that f is not Lipschitz continuous. Given any $M > 0$, choose $c = 0$ and $0 < x < \frac{1}{M^2}$, so that $\frac{1}{\sqrt{x}} > M$. Then we have

$$\frac{|f(x) - f(c)|}{|x - c|} = \frac{|\sqrt{x}|}{|x|} = \frac{1}{\sqrt{x}} > M.$$

Since M was arbitrary, this shows that f is not Lipschitz continuous.