CSU Math 366

Homework 4

Due Friday, March 26, anytime, on Gradescope.

Remark. Your answers should be briefly explained. If you're only writing math symbols, then you're not explaining things — make grammatically correct sentences by adding in just a few English words.

Reading. Read Chapters 9, 10, and 11.

Problems.

1. (a) Prove that if G is a group and a is an element in G, then a has a unique inverse in G.

   Remark: Use multiplicative notation.

   (b) Re-write the above proof in additive notation, following exactly the same steps you did above.

   Remark: 0 *is a good name for the identity when using additive notation. Additive notation is typically used only when the group G is commutative, but don't assume that G is commutative here — it's not needed.*

2. Let α be the permutation $\alpha = (1246)(45)(263)(1892673)(37) \in S_9$. Write both α and its inverse, α^{-1}, in disjoint cycle form.

3. Let $\{R_0, R_{90}, R_{180}, R_{270}\}$ be the group of rotations of the square. You may take it as a fact that the function $\phi\colon \mathbb{Z}/4\mathbb{Z} \to \{R_0, R_{90}, R_{180}, R_{270}\}$ defined by $\phi(j) = R_{90\cdot j}$ is bijective (which means both one-to-one and onto). Prove that ϕ is an isomorphism.

4. If n is a positive integer, then $U(n)$ denotes the *group of units modulo* n. The elements of $U(n)$ are all of the numbers from 1 up to $n-1$ that are relatively prime to n (i.e. that share no common divisors with n). The binary operation is multiplication modulo n.

   For example, the elements of $U(8)$ are $U(8) = \{1, 3, 5, 7\}$. The Cayley table or multiplication table for $U(8)$ is:

	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

The Klein 4 group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (also denoted $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$) has as its elements $(0,0), (0,1), (1,0), (1,1)$ with its binary operation given by component-wise addition in $\mathbb{Z}/2\mathbb{Z}$. So its Cayley table is

	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$(0,0)$	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$(0,1)$	$(0,1)$	$(0,0)$	$(1,1)$	$(1,0)$
$(1,0)$	$(1,0)$	$(1,1)$	$(0,0)$	$(0,1)$
$(1,1)$	$(1,1)$	$(1,0)$	$(0,1)$	$(0,0)$

The function $\phi\colon U(8) \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ defined by $\phi(1) = (1,1)$, $\phi(3) = (1,0)$, $\phi(5) = (0,1)$, and $\phi(7) = (0,0)$ is bijective. Nevertheless, show that ϕ is not an isomorphism.

Hint: You should do this by finding specific elements $a, b \in U(8)$ *such that* $\phi(ab) \neq \phi(a) + \phi(b)$.

Remark: Other bijective functions between these groups are indeed isomorphisms.