CSU Math 366

Homework 2

Due Friday, February 7 at the beginning of class

Reading. Chapter 3

Remark. Make grammatically correct sentences by adding in just a few English words.

Problems.

A binary operation $\diamond \colon G \to G$ is said to be *commutative* if $a \diamond b = b \diamond a$ for all $a, b \in G$. Some binary operations are commutative, but many are not!

A group G is said to be *commutative*, or equivalently *abelian*, if its underlying binary operation is commutative. Some groups are commutative, but many are not!

1. In this exercise we will check that it is possible for a binary operation to be commutative but not associative. Define the binary operation $\diamond \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ on the real numbers by $x \diamond y = \frac{x+y}{2}$. Since $x \diamond y = \frac{x+y}{2} = \frac{y+x}{2} = y \diamond x$, we see that $\diamond$ is commutative. Show that $\diamond$ is not associative, as follows:

   - Compute a formula for $(x \diamond y) \diamond z$, and compute a formula for $x \diamond (y \diamond z)$.
   - Say why it is sometimes possible to have $(x \diamond y) \diamond z \neq x \diamond (y \diamond z)$; for example you could pick specific choices of $x, y, z \in \mathbb{R}$ to show this.

   Remark: Other binary operations on $\mathbb{R}$ that are commutative but not associative include $x \diamond y = xy + 1$ or $x \diamond y = |x - y|$.

2. List all the elements of $U(8)$, write down the Cayley table (multiplication table) for this group, say what the identity in this group is, and say which elements are inverses in this group.

3. Recall from class or our book that $\mathbb{GL}(2, \mathbb{R})$ is the group of all 2×2 matrices with non-zero determinant, under the group operation of matrix multiplication. Show that $\mathbb{GL}(2, \mathbb{R})$ is not commutative.

4. Exercise 22 from our book: Let G be a group with the property that for any $x, y, z \in G$, we have that $xy = zx$ implies $y = z$. Prove that G is commutative (that is, that for all $a, b \in G$, we have $ab = ba$).

   Hint 1: The first sentence of your proof should be something like "Let a and b be arbitrary elements of G." The last sentence of your proof should be something like "Since $ab = ba$, we have shown that G is commutative."

Hint 2: Since we are trying to show $ab = ba$, what is a reasonable choice for y and for z? Then, once you have a good choice in mind for y and for z, is there a choice of x that makes $xy = zx$ true?