CSU Math 301

Homework 5

Due Friday, October 11 at the beginning of class

Reading.
Sections 4.3, 6.1, 6.2, 6.3, 6.4

Remark. Make grammatically correct sentences by adding in just a few English words.

Problems.

1. When climbing a staircase, you can take either one or three stairs in a single step.
   (a) Write "Let S_n be the number of ways to climb a staircase with n stairs." Prove the recurrence relation $S_n = S_{n-1} + S_{n-3}$.
   (b) Write down what S_0, S_1, S_2 are (or alternatively, what S_1, S_2, S_3 are). No justification needed.
   (c) Use (a) and (b) to answer the following question. How many ways are there to climb a staircase with 12 stairs?

2. Let F_n be the n-th Fibonacci number. Prove that $F_1^2 + \ldots + F_n^2 = F_n F_{n+1}$ for all $n \geq 1$.

3. Define the *Lucas numbers* by $L_0 = 2$, $L_1 = 1$, and $L_{n+1} = L_n + L_{n-1}$ for $n \geq 1$. Find L_{10}.

4. Suppose it were true that the Lucas numbers satisfied the formula

$$L_n = c_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^n \quad \text{for some} \quad c_1, c_2 \in \mathbb{R}.$$

   Using the base cases $2 = L_0 = c_1 + c_2$ and $1 = L_1 = c_1 \frac{1+\sqrt{5}}{2} + c_2 \frac{1-\sqrt{5}}{2}$, solve for c_1 and c_2.

   Remark: Your task is to solve a system of two equations ($2 = c_1 + c_2$ and $1 = c_1 \frac{1+\sqrt{5}}{2} + c_2 \frac{1-\sqrt{5}}{2}$) in two unknowns ($c_1$ and c_2), i.e. solve for the intersection point of two lines. This is something you probably learned how to do in your first high school algebra class!

   Remark: What's the point of this exercise? In class on 10/2/19, we showed that $c_1(\frac{1+\sqrt{5}}{2})^n + c_2(\frac{1-\sqrt{5}}{2})^n$ satisfies the same recurrence relation as the Lucas numbers (or equivalently, the Fibonacci numbers). Hence when you find c_1 and c_2 satisfying the base cases for L_0 and L_1 in this exercise, you will have proven the theorem that gives an explicit formula for Lucas number $L_n = c_1(\frac{1+\sqrt{5}}{2})^n + c_2(\frac{1-\sqrt{5}}{2})^n$ as a function of n.