

Math 151 - Midterm Exam - Winter 2012

The exam will be due in class on Tuesday, February 14. Justify all your work carefully, and carry out all computations completely wherever possible. This is an open book exam, but do not consult each other or Internet sources.

1. The following is a strategy for deciding who goes first in a game which is to be played by N players. Each of the players flips a fair coin. If either (a) all coins come out the same or (b) the number of heads and tails are identical, the flips are repeated. If not, then the players in the majority (i.e. those who flipped heads if there were more heads than tails and vice versa) drop out and the remaining players repeat the process, until only one player is left, and he/she goes first.
 - (a) Suppose there are 3 players. What is the probability that the number of flips is greater than 2?
 - (b) Again with 3 players, what is the expected value of the number of flips?
 - (c) Suppose there are 4 players. What is the expected value of the number of flips?
 - (d) Again with 4 players, what is the probability that the process ends?
2. Suppose three numbers are chosen uniformly at random from among the integers between 0 and 100 inclusive.
 - (a) What is the probability that the numbers add up to 100?
 - (b) What is the probability that two of the integers are equal, conditioned on the requirement that the three integers add up to 100?
 - (c) Given again that the sum of the integers is 100, what is the probability that all three are equal?
 - (d) Given that the sum of all the integers is 100, what is the probability that the first number drawn is a 7?

3. Let $S = \{1, \dots, n\}$. Suppose that subsets A and B are chosen independently from S so that A and B are equally likely to be any of the 2^n subsets of S (including both the empty set and S itself).
- What is the probability that $A \subseteq B$?
 - What is the probability that $A \cap B = \emptyset$?
 - What is the expected value of $\#(A \cap B)$, where $\#(A \cap B)$ is the number of elements in the set $A \cap B$?
4. Suppose that we have two tests, T_1 and T_2 , for a disease. A positive test result is indicative of the presence of the disease. The following table gives the probabilities that a healthy patient or a patient with the disease will receive a positive test result under test T_1 or T_2 .

Probability of a positive test result

	<i>Healthy</i>	<i>Disease</i>
T_1	10%	95%
T_2	8%	90%

We assume the tests are independent, so that the events of T_1 and T_2 being positive are independent. Assume also that 5% of the population has the disease.

- Find the conditional probabilities $P(\text{Healthy}|T_i \text{ positive})$ for $i = 1, 2$.
 - Find the conditional probability $P(\text{Healthy}|T_1 \text{ and } T_2 \text{ are both negative})$.
 - Find the conditional probability $P(\text{Disease}|T_1 \text{ and } T_2 \text{ are both positive})$.
5. Let X be a binomial random variable with number of trials equal to n and probability of success p . What value of p maximizes $P\{X = k\}$, for $k = 0, 1, \dots, n$?

6. We suppose that a point (x, y) is chosen from the circular region $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$, in such a way that the probability that the point is chosen from any region in D is proportional to its area.
- What is the probability that $y \geq \frac{1}{2}$?
 - What is the conditional probability that $y \leq 0$ given that $x \geq \frac{1}{2}$?
 - What is the probability that (x, y) lies inside the square S given by $-\frac{1}{\sqrt{2}} \leq x, y \leq \frac{1}{\sqrt{2}}$?
 - What is the conditional probability that $x + y \geq 1$ given that (x, y) is in S ?
7. Let $p(n) = \frac{2}{3^n}$ for any integer $n \geq 1$. Let X denote the random variable $X(n) = n$.
- Show that p defines a probability distribution on the set $\{1, 2, \dots, n, \dots\}$.
 - Determine the expected value of X .
 - Determine the variance of X .
 - What is the conditional probability that n is divisible by 4 given that it is divisible by 2?
8. Suppose that 10,000 monkeys are typing at typewriters, and that 4,000 of them are typing at 100 characters per minute and 6,000 of them are typing at 200 characters per minute. After how many years does it become likely that one of the monkeys will have typed “Macbeth”?

Comments:

- Assume that “Macbeth” is a document with 300,000 alphabetic characters. A monkey must type “Macbeth” from start to finish. For a monkey to successfully type “Macbeth”, there can be no typos within the 300,000 characters of “Macbeth”, but the text before or after “Macbeth” can be gibberish.

- Your answer should only be an estimate, and will require assumptions about the document, so different estimates will be possible. State clearly all assumptions you make.