Math 151 - Winter 2012 - Final Exam

The exam will be due by 5:00 PM on Wednesday, March 21. It is to be turned in at my office (383L). You may use the book, and books in the library, but not internet resources.

1. An urn contains n white balls and m black balls. We now remove two batches of balls from the urn. In the first batch, we remove p random balls from the urn without replacement, so that the urn now contains $n+m-p$ balls. Let X denote the number of white balls removed in the first batch of p balls. In the second batch, we remove q random balls from the urn without replacement, so that the urn now contains $n+m-p-q$ balls. Let Y denote the number of white balls removed in the second batch of q balls. We assume that $n, m, p, q \geq 1$ and that $p+q \leq \min(m, n)$.

   (a) Calculate $\mathrm{Var}(X)$ and $\mathrm{Var}(Y)$.

   (b) Calculate $\mathrm{Cov}(X, Y)$.

   (c) Is $\mathrm{Cov}(X, Y)$ positive or negative? Explain why we should have expected it to be either positive or negative even before we performed any of the calculations in part (b).

2. There are n windows at a ticket office. Currently, n customers are being served at the n windows and k customers are waiting in a single line. The amount of time that it takes for a customer to be served at a ticket window is exponentially distributed with parameter λ. As soon as a customer at a window is finished being served, the customer leaves the ticket office. The next person in line immediately starts being served at the vacated window. No new customers ever enter the ticket office, and this process continues until all $n+k$ customers have left the ticket office.

   Let's index the customers. Label the n customers currently being served as $1, \ldots, n$. Label the k customers in line as $n+1, \ldots, n+k$, with the first person in line being labeled $n+1$ and the last person in line being labeled $n+k$. Let X be the random variable that is equal to the index of the last customer to leave the ticket office. So for $1 \leq i \leq n+k$, if customer number i is the last person to leave the ticket office, then $X = i$. If two or more people are simultaneously the last people to leave the ticket office, then we set $X = 0$. But note that $P\{X = 0\} = 0$.

   (a) Suppose $n = 3$ and $k = 1$. So there are 3 customers currently being served at the 3 windows, and there is 1 customer in line. Find $P\{X = 4\}$. That is, find the probability that the single customer initially waiting line is the last customer to leave the ticket office.

   (b) Suppose $n = 3$ and $k = 2$. So there are 3 customers currently being served at the 3 windows, and there are 2 customers in line. Find $P\{X = 4\}$. That is, find the probability that the first of the two customers in line is the last to leave the ticket office.

   (c) Now let n and k be arbitrary positive integers. Find $P\{X = i\}$ for all values $1 \leq i \leq n+k$. That is, for each of the $n+k$ customers, find the probability that customer i is the last to leave the ticket office.

3. The number of passengers who get on a bus at a loading station is a Poisson random variable with mean 25. There are $n \geq 1$ stations after the loading station. Each passenger is equally likely to get off at any one of the remaining n stations, independently of where the other passengers disembark. The bus does not pick up any new passengers, and the bus only stops at those stations where some of its passengers need to get off. Compute the expected number of stops that the bus will make until it has unloaded all of its passengers.

4. Let $X_1, X_2, \ldots, X_n$ be continuous random variables that are independent and identically distributed.

   (a) Find $P\{X_1 < X_2 \mid X_2 = \min(X_2, X_3, \ldots, X_n)\}$.

   (b) Find $P\{X_1 < X_3 \mid X_2 = \min(X_2, X_3, \ldots, X_n)\}$.

   (c) Find $P\{X_1 < X_2 \mid X_2 = \max(X_2, X_3, \ldots, X_n)\}$.

   (d) Find $P\{X_1 < X_3 \mid X_2 = \max(X_2, X_3, \ldots, X_n)\}$.

5. Let X and Y be random continuous variables with joint probability density function

$$f(x, y) = \begin{cases} \frac{1}{x^2 y^2} & \text{if } x \geq 1 \text{ and } y \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Find $P\{|Y - X| > 3\}$.

   (b) Let Z be the random variable $Z = X^2 Y$. Find the probability density function $f_Z(z)$ for the random variable Z.

6. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables. We say that a record value occurs at time j, $j \leq n$, if $X_j \geq X_i$ for all $1 \leq i \leq j$.

   (a) Show that

$$E[\text{number of record values}] = \sum_{j=1}^{n} \frac{1}{j}.$$

   (b) Show that

$$\text{Var}[\text{number of record values}] = \sum_{j=1}^{n} \frac{(j-1)}{j^2}.$$

7. The best quadratic predictor of Y with respect to X is $a + bX + cX^2$, where a, b, and c are chosen to minimize $E[(Y - (a + bX + cX^2))^2]$. Determine a, b, and c.

8. Let $\{X_i\}_{i=1}^{N}$ denote a collection of identically and independently distributed Poisson random variables, with parameter λ. Find an approximation of the distribution of values of $\sum_{i=1}^{N} X_i$, where N is large.

9. An airplane flight can carry 120 passengers. A passenger with a ticket arrives for the flight with probability $p = 0.95$. Assume the passengers behave independently.

(a) Suppose that the airline sells tickets to n passengers (which may be more or less than 120). For a given flight, what kind of distribution models the number of passengers who arrive for the flight?

(b) Suppose that the airline sells tickets to 130 passengers. Find an approximate value for the probability that more than 120 passengers arrive.

(c) If the airline wants the probability of a full flight to be at least 0.5, how many tickets should they sell?

10. Suppose you are given three coins (C_1, C_2, and C_3), with probabilities p_1, p_2, and p_3 of heads, respectively. You pick a coin at random and toss it four times.

    (a) What is the probability that you obtain four heads?

    (b) What is the conditional probability that you have selected coin 1 given that you have tossed 4 heads?