Feb 21 Homework Solutions
Math 151, Winter 2012

Chapter 5 Problems (pages 224-227)

Problem 5

A filling station is supplied with gasoline once a week. If its weekly volume of sales in thousands of gallons is a random variable with probability density function

$$f(x) = \begin{cases} 5(1-x)^4 & 0 < x < 1, \\ 0 & \textit{otherwise} \end{cases}$$

what must the capacity of the tank be so that the probability of the supply's being exhausted in a given week is .01*?*

We want to find the capacity a so that $P\{X \leq a\} \geq 1 - .01 = .99$. For $0 \leq a \leq 1$, we have

$$P\{X \leq a\} = \int_0^a 5(1-x)^4 dx = 1 - (1-a)^5,$$

Hence we need

$$\begin{aligned} &P\{X \leq a\} \geq .99 \\ \Longrightarrow &1 - (1-a)^5 \geq .99 \\ \Longrightarrow &(1-a)^5 \leq .01 \\ \Longrightarrow &a \geq 1 - (.01)^{1/5} \approx 0.6019 \end{aligned}$$

So the capacity of the tank must be at least 6019 gallons.

Problem 6

Compute $E[X]$ if X has a density function given by

(a)

$$f(x) = \begin{cases} \frac{1}{4}xe^{-x/2} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = \int_{-\infty}^{\infty} xf(x)dx = \int_0^{\infty} \frac{1}{4}x^2 e^{-x/2} dx = \left(-\frac{1}{2}x^2 - 2x - 4\right)e^{-x/2}\Big|_0^{\infty} = 4.$$

(b)

$$f(x) = \begin{cases} c(1-x^2) & -1 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$E[X] = \int_{-\infty}^{\infty} xf(x)dx = \int_{-1}^{1} cx(1-x^2)dx = 0$$

since $f(x)$ is an even function. Note that the solution does not depend on c.

(c)

$$f(x) = \begin{cases} 5x^{-2} & x > 5 \\ 0 & x \leq 5 \end{cases}$$

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx = \int_{5}^{\infty} 5x^{-1} dx = 5 \log(x)\Big|_5^{\infty} = \infty.$$

Problem 10

Trains headed for destination A arrive at the train station at 15-minute intervals starting at 7 a.m., whereas trains headed for destination B arrive at 15-minute intervals starting at 7:05 a.m.

(a) *If a certain passenger arrives at the station at a time uniformly distributed between 7 and 8 a.m. and then gets on the first train that arrives, what proportion of the time does he or she go to destination A?*

The passenger takes the train to destination A if he arrives between 7:05 and 7:15 a.m, between 7:20 and 7:30 a.m, between 7:35 and 7:45 a.m, or between 7:50 and 8 a.m. Thus the passenger takes the train to destination A with probability $(10 + 10 + 10 + 10)/60 = 40/60 = 2/3$. In other words, the passenger takes the train to destination A two-thirds of the time.

(b) *What if the passenger arrives at a time uniformly distributed between 7:10 and 8:10 a.m.?*

The passenger takes the train to destination A if he arrives between 7:10 and 7:15 a.m, between 7:20 and 7:30 a.m, between 7:35 and 7:45 a.m, between 7:50 and 8 a.m, or between 8:05 and 8:10 a.m. Thus the passenger takes the train to destination A with probability $(5 + 10 + 10 + 10 + 5)/60 = 40/60 = 2/3$. In other words, the passenger still takes the train to destination A two-thirds of the time.

Problem 11

A point is chosen at random on a line segment of length L. Interpret this statement, and find the probability that the ratio of the shorter to the longer segment is less than $1/4$.

A point is chosen at random on this line means that we can represent the chosen point by a continuous random variable X taking values between 0 and L. So X is the distance from a fixed endpoint of the line segment to the chosen point. The random variable X has a uniform probability density function $f(x) = 1/L$ for $0 \leq x \leq L$. If we chose the point at X, the line is divided into two pieces of length X and $L - X$. The ratio of the shorter to the longer segment is the smaller number of $X/(L - X)$ or $(L - X)/X$. We have $X/(L - X) < 1/4 \Leftrightarrow X < L/5$ and $(L - X)/X < 1/4 \Leftrightarrow X > 4L/5$. Note that the events $X < L/5$ and $X > 4L/5$ are disjoint. Hence the probability that the ratio of the shorter to the longer segment is less than 1/4 is

$$\begin{aligned} P\{X/(L-X) < 1/4 \text{ or } (L-X)/X < 1/4\} &= P\{X < L/5 \text{ or } X > 4L/5\} \\ &= P\{X < L/5\} + P\{X > 4L/5\} \\ &= \int_0^{L/5} (1/L)dx + \int_{4L/5}^{L} (1/L)dx \\ &= 1/5 + 1/5 \\ &= 2/5. \end{aligned}$$

Problem 15

If X is a normal random variable with parameters $\mu = 10$ and $\sigma^2 = 36$, compute

(a) $P\{X > 5\}$.

Recall that $P\{X \leq a\} = \Phi((a-\mu)/\sigma) = \Phi((a-10)/6)$ where $\Phi(x)$ is the cumulative distribution function of the standard normal random variable. Using Table 5.1 on page 201, we have

$$P\{X > 5\} = 1 - P\{X < 5\} = 1 - \Phi(-5/6) = \Phi(5/6) \approx \Phi(.83) \approx .7967.$$

(b) $P\{4 < X < 16\}$.

$$P\{4 < X < 16\} = P\{X < 16\} - P\{X \leq 4\} = \Phi(1) - \Phi(-1) = 2\Phi(1) - 1 \approx 2(.8413) - 1 = .6826.$$

(c) $P\{X < 8\}$.

$$P\{X < 8\} = \Phi(-1/3) = 1 - \Phi(1/3) \approx 1 - \Phi(.33) \approx 1 - .6293 = .3707$$

(d) $P\{X < 20\}$.

$$P\{X < 20\} = \Phi(5/3) \approx \Phi(1.67) = .9525.$$

(e) $P\{X > 16\}$.

$$P\{X > 16\} = 1 - P\{X < 16\} = 1 - \Phi(1) \approx 1 - .8413 = .1587.$$

Problem 16

The annual rainfall (in inches) in a certain region is normally distributed with $\mu = 40$ and $\sigma = 4$. What is the probability that, starting with this year, it will take over 10 years before a year occurs having a rainfall of over 50 inches? What assumptions are you making?

Assume the annual rainfall in different years are independent. Let X denote the annual rainfall in a given year and recall that $P\{X \leq a\} = \Phi((a-\mu)/\sigma) = \Phi((a-40)/4)$ where $\Phi(x)$ is the cumulative distribution function of the standard normal random variable. The probability that it will take over 10 years before a year occurs having a rainfall of over 50 inches is

$$(P\{X \leq 50\})^{10} = (\Phi((50-40)/4))^{10} = (\Phi(2.5))^{10} \approx (.9938)^{10} \approx .9397.$$

Problem 23

One thousand independent rolls of a fair die will be made. Compute an approximation to the probability that the number 6 will appear between 150 and 200 times inclusively. If the number 6 appears exactly 200 times, find the probability that number 5 will appear less than 150 times.

Let X be the number of times the number 6 appears. Then X is a binomial random variable with $p = 1/6$ and $n = 1000$. By the DeMoivre-Laplace limit theorem,

$$\begin{aligned}
P\{150 \le X \le 200\} &= P\left\{\frac{150 - np}{\sqrt{np(1-p)}} \le \frac{X - np}{\sqrt{np(1-p)}} \le \frac{200 - np}{\sqrt{np(1-p)}}\right\} \\
&= P\left\{\frac{150 - 1000/6}{\sqrt{1250/9}} \le \frac{X - np}{\sqrt{np(1-p)}} \le \frac{200 - 1000/6}{\sqrt{1250/9}}\right\} \\
&\approx P\left\{-1.4142 \le \frac{X - np}{\sqrt{np(1-p)}} \le 2.8284\right\} \\
&\approx \Phi(2.8284) - \Phi(-1.4142) \\
&\approx .9977 - (1 - .9207) \\
&= .9184.
\end{aligned}$$

Given that the number 6 appears exactly 200 times, the number 5 appears less than 150 times if it does so in the remaining 800 trials. Represent the number of times the number 5 appears in the remaining 800 trials by a binomial random variable Y with $p = 1/5$ and $n = 800$, since in the remaining 800 trials only the numbers between 1 and 5 appear. Thus, given that the number 6 appears exactly 200 times, the probability that the number 5 will appear less than 150 times is

$$\begin{aligned}
P\{Y < 150\} &= P\{0 \le Y < 150\} \\
&= P\left\{\frac{0 - np}{\sqrt{np(1-p)}} \le \frac{Y - np}{\sqrt{np(1-p)}} \le \frac{150 - np}{\sqrt{np(1-p)}}\right\} \\
&= P\left\{\frac{0 - 160}{\sqrt{128}} \le \frac{Y - np}{\sqrt{np(1-p)}} \le \frac{150 - 160}{\sqrt{128}}\right\} \\
&= P\left\{-14.1421 \le \frac{Y - np}{\sqrt{np(1-p)}} \le -.8839\right\} \\
&\approx \Phi(-.8839) - \Phi(-14.1421) \\
&\approx (1 - .8106) - 0 \\
&= .1894.
\end{aligned}$$

Problem 34

Jones figures that the total number of thousands of miles that an auto can be driven before it would need to be junked is an exponential random variable with parameter $1/20$. *Smith has a used car that he claims has been driven only 10,000 miles. If Jones purchases the car, what is the probability that she would get at least 20,000 additional miles out of* *it? Repeat under the assumption that the lifetime mileage of the car is not exponentially distributed but rather is (in thousands of miles) uniformly distributed over* $(0, 40)$.

Let X denote the total number of thousands of miles that the auto is driven before it needs to be junked. We want to compute $P(\{X \geq 30\}|\{X \geq 10\})$. Assuming X is an exponential random variable with parameter $1/20$, we have

$$\begin{aligned} P\{X \geq 30\} &= \int_{30}^{\infty} \tfrac{1}{20} e^{-x/20} dx = \int_{3/2}^{\infty} e^{-t} dt = e^{-3/2}, \\ P\{X \geq 10\} &= \int_{10}^{\infty} \tfrac{1}{20} e^{-x/20} dx = \int_{1/2}^{\infty} e^{-t} dt = e^{-1/2}, \\ P(\{X \geq 30\}|\{X \geq 10\}) &= P\{X \geq 30\}/P\{X \geq 10\} = e^{-3/2}/e^{-1/2} = e^{-1} \approx .3678. \end{aligned}$$

Note that we could have saved some work here by using the memoryless property of the exponential random variable.

Assuming X is uniformly distributed over $(0, 40)$, we have

$$\begin{aligned} P\{X \geq 30\} &= \int_{30}^{40} \tfrac{1}{40} dx = 1/4, \\ P\{X \geq 10\} &= \int_{10}^{40} \tfrac{1}{40} dx = 3/4, \\ P(\{X \geq 30\}|\{X \geq 10\}) &= P\{X \geq 30\}/P\{X \geq 10\} = (1/4)/(3/4) = 1/3. \end{aligned}$$

Problem 35

The lung cancer hazard rate $\lambda(t)$ *of a* t*-year-old male smoker is such that*

$$\lambda(t) = .027 + .00025(t-40)^2 \textit{ for } t \geq 40.$$

Assuming that a 40-year-old male smoker survives all other hazards, what is the probability that he survives without contracting lung cancer to

(a) age 50?

Let X denote the life-time of the smoker. Then

$$P(\{X > 50\}|\{X > 40\}) = \frac{1 - F(50)}{1 - F(40)}.$$

We know from equation (5.4) on page 213 that

$$F(x) = 1 - \exp\left(-\int_0^x \lambda(t) dt\right).$$

Hence the conditional probability that he survives to age 50 without cancer is

$$
\begin{aligned}
P(\{X > 50\}|\{X > 40\}) &= \frac{1 - F(50)}{1 - F(40)} \\
&= \frac{\exp\left(-\int_0^{50} \lambda(t)dt\right)}{\exp\left(-\int_0^{40} \lambda(t)dt\right)} \\
&= \exp\left(-\int_{40}^{50} \lambda(t)dt\right) \\
&= \exp\left(-\int_{40}^{50} (.027 + .00025(t-40)^2)dt\right) \\
&= \exp\left(-.027 \cdot 10 - \frac{.00025(10)^3}{3}\right) \\
&\approx .7023.
\end{aligned}
$$

(b) age 60?

Using the same steps as in part (a), the conditional probability that he survives to age 60 without cancer is

$$
\begin{aligned}
P(\{X > 60\}|\{X > 40\}) &= \frac{1 - F(60)}{1 - F(40)} \\
&= \frac{\exp\left(-\int_0^{60} \lambda(t)dt\right)}{\exp\left(-\int_0^{40} \lambda(t)dt\right)} \\
&= \exp\left(-\int_{40}^{60} \lambda(t)dt\right) \\
&= \exp\left(-\int_{40}^{60} (.027 + .00025(t-40)^2)dt\right) \\
&= \exp\left(-.027 \cdot 20 - \frac{.00025(20)^3}{3}\right) \\
&\approx .2992.
\end{aligned}
$$

Section 5 Theoretical Exercises (page 227-229)

Problem 5

Use the result that, for a nonnegative random variable Y,

$$E[Y] = \int_0^\infty P\{Y > t\}dt$$

to show that, for a nonnegative random variable X,

$$E[X^n] = \int_0^\infty nx^{n-1}P\{X > x\}dx.$$

By letting $Y = X^n$, we know that

$$E[X^n] = \int_0^\infty P\{X^n > t\}dt.$$

Making the change of variables $t = x^n$ and $dt = nx^{n-1}dx$, we get that

$$E[X^n] = \int_0^\infty nx^{n-1}P\{X^n > x^n\}dx.$$

Since $u \mapsto u^n$ is an increasing function of $u \geq 0$, the events $\{X^n > x^n\}$ and $\{X > x\}$ are identical. So we've shown

$$E[X^n] = \int_0^\infty nx^{n-1}P\{X > x\}dx = \int_0^\infty nx^{n-1}P\{X > t\}dt,$$

as required.

Problem 8

Let X be a random variable that takes on values bewteen 0 and c. That is, $P\{0 \leq X \leq c\} = 1$. Show that $\mathrm{Var}(X) \leq c^2/4$.

Since X be a random variable that takes on values bewteen 0 and c, we know that $f(x) = 0$ if $x \leq 0$ or $x \geq c$. Thus

$$E[X^2] = \int_0^c x^2 f(x)dx \leq \int_0^c cxf(x)dx = cE[X].$$

Hence

$$\mathrm{Var}(X) = E[X^2] - E[X]^2 \leq cE[X] - E[X]^2 = c^2(\alpha - \alpha^2),$$

where $\alpha = E[X]/c$. We know $\alpha - \alpha^2$ attains its maximum value of 1/4 at $\alpha = 1/2$, so

$$\mathrm{Var}(X) \leq c^2(\alpha - \alpha^2)|_{\alpha=1/2} = c^2(1/2 - 1/4) = c^2/4.$$

Problem 10

Let $f(x)$ denote the probability density function of a normal random variable with mean μ and variance σ^2. Show that $\mu - \sigma$ and $\mu + \sigma$ are points of inflection of this function. That is, show that $f''(x) = 0$ when $x = \mu - \sigma$ or $x = \mu + \sigma$.

Recall that

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}.$$

Taking the derivative with respect to x twice, we get

$$f'(x) = \frac{-(x-\mu)}{\sqrt{2\pi}\sigma^3} e^{-(x-\mu)^2/2\sigma^2}.$$

and

$$f''(x) = \frac{-\sigma^2 + (x-\mu)^2}{\sqrt{2\pi}\sigma^5} e^{-(x-\mu)^2/2\sigma^2}.$$

Thus $f''(x) = 0 \Leftrightarrow -\sigma^2 + (x-\mu)^2 = 0 \Leftrightarrow x = \mu - \sigma$ or $x = \mu + \sigma$, as claimed. So $\mu - \sigma$ and $\mu + \sigma$ are the points of inflection of this function.