Feb 7 Homework Solutions
Math 151, Winter 2012

Chapter 4 Problems (pages 172-179)

Problem 3

Three dice are rolled. By assuming that each of the $6^3 = 216$ possible outcomes is equally likely, find the probabilities attached to the possible values that X can take on, where X is the sum of the 3 dice.

The possible values of X are integers between $1+1+1=3$ and $6+6+6=18$.

To have $X=3$, we must have the three die be 1, 1, and 1. Hence

$$P\{X=3\} = \frac{1}{216}.$$

To have $X=4$, we must have the die read 1, 1, 2. This can happen in 3 different orders. Hence

$$P\{X=4\} = \frac{3}{216}.$$

To have $X=5$, we must have the die read 1, 1, 3 (3 orders) or 1, 2, 2 (3 orders). Hence

$$P\{X=5\} = \frac{3+3}{216} = \frac{6}{216}.$$

To have $X=6$, we must have the die read 1, 1, 4 (3 orders) or 1, 2, 3 (6 orders), or 2, 2, 2 (1 order). Hence

$$P\{X=6\} = \frac{3+6+1}{216} = \frac{10}{216}.$$

To have $X=7$, we must have the die read 1, 1, 5 (3 orders), 1, 2, 4 (6 orders), 1, 3, 3 (3 orders), or 2, 2, 3 (3 orders). Hence

$$P\{X=7\} = \frac{3+6+3+3}{216} = \frac{15}{216}.$$

To have $X=8$, we must have the die read 1, 1, 6 (3 orders), 1, 2, 5 (6 orders), 1, 3, 4 (6 orders), 2, 2, 4 (3 orders), or 2, 3, 3 (3 orders). Hence

$$P\{X=8\} = \frac{3+6+6+3+3}{216} = \frac{21}{216}.$$

To have $X=9$, we must have the die read 1, 2, 6 (6 orders), 1, 3, 5 (6 orders), 1, 4, 4 (3 orders), 2, 2, 5 (3 orders), 2, 3, 4 (6 orders), or 3, 3, 3 (1 order). Hence

$$P\{X=9\} = \frac{6+6+3+3+6+1}{216} = \frac{25}{216}.$$

To have $X=10$, we must have the die read 1, 3, 6 (6 orders), 1, 4, 5 (6 orders), 2, 2, 6 (3 orders), 2, 3, 5 (6 orders), 2, 4, 4 (3 orders), or 3, 3, 4 (3 orders). Hence

$$P\{X=10\} = \frac{6+6+3+6+3+3}{216} = \frac{27}{216}.$$

By symmetry (replacing each die roll with value i by a die roll with value $7 - i$), we see that

$$P\{X = 18\} = P\{X = 3\} = \frac{1}{216}.$$

$$P\{X = 17\} = P\{X = 4\} = \frac{3}{216}.$$

$$P\{X = 16\} = P\{X = 5\} = \frac{6}{216}.$$

$$P\{X = 15\} = P\{X = 6\} = \frac{10}{216}.$$

$$P\{X = 14\} = P\{X = 7\} = \frac{15}{216}.$$

$$P\{X = 13\} = P\{X = 8\} = \frac{21}{216}.$$

$$P\{X = 12\} = P\{X = 9\} = \frac{25}{216}.$$

$$P\{X = 11\} = P\{X = 10\} = \frac{27}{216}.$$

A good way to check our work is to note that

$$\sum_{i=3}^{18} P\{X = i\} = \frac{216}{216} = 1.$$

Problem 7

Suppose that a die is rolled twice. What are the possible values that the following random variables can take on:

(a) *the maximum value to appear in the two rolls?*

The possible values are $1, 2, 3, 4, 5, 6$.

(b) *the minimum value to appear in the two rolls?*

The possible values are $1, 2, 3, 4, 5, 6$.

(c) *the sum of the two rolls?*

The possible values are $2, 3, ..., 12$.

(d) *the value of the first roll minus the value of the second roll?*

The possible values are $-5, -4, ..., 4, 5$.

Problem 12

In the game of Two-Finger Morra, 2 players show 1 or 2 fingers and simultaneously guess the number of fingers their opponent will show. If only one of the players guesses correctly, he wins an amount (in dollars) equal to the sum of the fingers shown by him and his opponent. If both players guess correctly or if neither guesses correctly, then no money is exchanged. Consider a specified player, and denote by X the amount of money he wins in a single game of Two-Finger Morra.

(a) *If each player acts independently of the other, and if each player makes his choice of the number of fingers he will hold up and the number he will guess that his opponent will hold up in such a way that each of the 4 possibilities is equally likely, what are the possible values of X and what are their associated probabilities?*

The possible values of X are 0 if both or neither player guesses correctly, 2, 3, or 4 if he guesses correctly and the other player doesn't, and -2, -3, or -4 if his opponent guesses correctly and he doesn't. We compute

$$\begin{aligned} & P\{X = 2\} \\ = \; & P(\text{he guesses 1})P(\text{he holds up 1})P(\text{opponent guesses 2})P(\text{opponent holds up 1}) \\ = \; & \frac{1}{2^4} \\ = \; & \frac{1}{16}. \end{aligned}$$

By similar computations we compute that

$$P\{X = -2\} = P\{X = 4\} = P\{X = -4\} = \frac{1}{16}$$

and

$$P\{X = 3\} = P\{X = -3\} = \frac{1}{8}.$$

Hence

$$\begin{aligned} & P\{X = 0\} \\ = \; & 1 - P\{X = 2\} - P\{X = -2\} - P\{X = 4\} - P\{X = -4\} - P\{X = 3\} - P\{X = -3\} \\ = \; & 1 - \frac{1}{16} - \frac{1}{16} - \frac{1}{16} - \frac{1}{16} - \frac{1}{8} - \frac{1}{8} \\ = \; & \frac{1}{2}. \end{aligned}$$

(b) *Suppose that each player acts independently of the other. If each player decides to hold up the same number of fingers that he guesses his opponent will hold up, and if each player is equally likely to hold up 1 or 2 fingers, what are the possible values of X and their associated probabilities?*

If both players guess the same, then both players guess correctly and $X = 0$. If both players guess differently, then both players guess wrong and $X = 0$. Thus the only possible value for X is 0 and $P\{X = 0\} = 1$.

Problem 15

The National Basketball Association (NBA) draft lottery involves the 11 teams that had the worst won-lost records during the year. A total of 66 balls are placed in an urn. Each of these balls is inscribed with the name of a team: Eleven have the name of the team with the worst record, 10 have the name of the team with the second-worst record, 9 have the name of the team with the third-worst record, and so on (with 1 ball having the name of the team with the 11th-worst record). A ball is then chosen at random and the team *whose name is on the ball is given the first pick in the draft of players about to enter the league. Another ball is then chosen, and if it "belongs" to a team different than the one that received the first draft pick, then the team to which it belongs receives the second draft pick. (If the ball belongs to the team receiving the first pick, then it is discarded and another one is chosen; this continues until the ball of another team is chose.) Finally, another ball is chosen and the team named on the ball (provided it is different from the previous two teams) receives the third draft pick. The remaining draft picks 4 through 11 are then awarded to the 8 teams that did not "win the lottery," in inverse order of their won-lost records. For instance, if the team with the worst record did not receive any of the 3 lottery picks, then that team would receive the fourth draft pick. Let X denote the draft pick of the team with the worst record. Find the probability mass function of X.*

Since the team with the worst record is either one of the 3 lottery picks or the fourth draft pick, X takes values $1, 2, 3, 4$ and $p(X = x) = 0$ for all other values of x.

We calculate

$$p(X = 1) = \frac{11}{66} = \frac{1}{6}.$$

We calculate

$$\begin{aligned} P\{X = 2\} &= \sum_{i=2}^{11} P(\{X = 2\}|\text{team } i \text{ chosen first})P(\text{team } i \text{ chosen first}) \\ &= \sum_{i=2}^{11} \frac{11}{66 - (12 - i)} \cdot \frac{12 - i}{66} \\ &= \sum_{i=2}^{11} \frac{12 - i}{6(54 + i)} \\ &\approx 0.1556. \end{aligned}$$

We calculate

$$\begin{aligned} &P\{X = 3\} \\ &= \sum_{i=2}^{11} P(\{X = 3\}|\text{team } i \text{ chosen first})P(\text{team } i \text{ chosen first}) \\ &= \sum_{i=2}^{11} P(\{X = 3\}|\text{team } i \text{ chosen first}) \cdot \frac{12 - i}{66} \\ &= \sum_{i=2}^{11} \left[\sum_{j \neq i} P(\{X = 3\}|j \text{ chosen second } \& \ i \text{ chosen first})P(j \text{ chosen second}|i \text{ chosen first}) \right] \cdot \frac{12 - i}{66} \\ &= \sum_{i=2}^{11} \left[\sum_{j \neq i} \frac{11}{66 - (12 - i) - (12 - j)} \cdot \frac{12 - j}{66 - (12 - i)} \right] \cdot \frac{12 - i}{66} \\ &= \sum_{i=2}^{11} \left[\sum_{j \neq i} \frac{11}{42 + i + j} \cdot \frac{12 - j}{54 + i} \right] \cdot \frac{12 - i}{66} \\ &\approx 0.1435. \end{aligned}$$

Finally,

$$\begin{aligned} P\{X=4\} &= 1 - P\{X=1\} - P\{X=2\} - P\{X=3\} \\ &\approx 1 - \frac{1}{6} - 0.1556 - 0.1435 \\ &\approx 0.5342. \end{aligned}$$

Problem 23

You have \$1000, and a certain commodity presently sells for \$2 per ounce. Suppose that after one week the commodity will sell for either \$1 or \$4 per ounce, with these two possibilities being equally likely.

(a) *If your objective is to maximize the expected amount of money that you possess at the end of the week, what strategy should you employ?*

Let X be the amount of money you possess at the end of the week. Let k be the number of items you buy to begin with. After a week, in order to maximize X, you must sell all k items. Thus

$$E[X] = \sum_x xP\{X=x\} = (1000 - 2k + k)\cdot\tfrac{1}{2} + (1000 - 2k + 4k)\cdot\tfrac{1}{2} = 1000 + \tfrac{1}{2}k.$$

This function is maximized when k is as large as possible. Thus the best strategy is to buy 500 items initially and to sell them all a week later.

(b) *If you objective is to maximize the expected amount of the commodity that you posses at the end of the week, what strategy should you employ?*

Let Y be the amount of the commodity that you posses at the end of the week. Let k be the number of items you buy to begin with. After a week, in order to maximize Y, you must buy as many items as possible. If the price at the end of the week is \$1, then you will be able to buy $1000 - 2k$ items. If the price at the end of the week is \$4, then you will be able to buy

$$\left\lfloor \frac{1000-2k}{4} \right\rfloor$$

items, i.e. the largest integer not greater than $\frac{1000-2k}{4}$.

Thus

$$\begin{aligned} E[Y] &= \sum_y yP\{Y=y\} \\ &= (k + 1000 - 2k)\cdot\frac{1}{2} + \left(k + \left\lfloor \frac{1000-2k}{4} \right\rfloor\right)\cdot\frac{1}{2} \\ &= \left(1000 + \left\lfloor \frac{1000-2k}{4} \right\rfloor\right)\cdot\frac{1}{2} \end{aligned}$$

This function is maximized when k is as small as possible. Thus the best strategy is to buy no items initially and to buy as many items as possible a week later.

Problem 25
Two coins are to be flipped. The first coin will land on heads with probability .6, the second with probability .7. Assume that the results of the flips are independent, and let X equal the total number of heads that result.

(a) *Find* $P\{X = 1\}$.

$$\begin{aligned} P\{X=1\} &= P(\text{1st heads and 2nd tails } \cup \text{ 1st tails and 2nd heads}) \\ &= P(\text{1st heads and 2nd tails}) + P(\text{1st tails and 2nd heads}) \\ &\qquad\qquad \text{since the two events are disjoint} \\ &= P(\text{1st heads})P(\text{2nd tails}) + P(\text{1st tails})P(\text{2nd heads}) \\ &\qquad\qquad \text{since the coin flips are independent} \\ &= .6 \cdot .3 + .4 \cdot .7 \\ &= 0.46 \end{aligned}$$

(b) *Determine* $E[X]$.

First we calculate

$$\begin{aligned} P\{X=0\} &= P(\text{1st tails and 2nd tails}) \\ &= P(\text{1st tails})P(\text{2nd tails}) \\ &= .4 \cdot .3 \\ &= 0.12 \end{aligned}$$

and

$$\begin{aligned} P\{X=2\} &= P(\text{1st heads and 2nd heads}) \\ &= P(\text{1st heads})P(\text{2nd heads}) \\ &= .6 \cdot .7 \\ &= 0.42 \end{aligned}$$

Hence

$$\begin{aligned} E[X] &= \sum_x xP\{X=x\} \\ &= 0 \cdot 0.12 + 1 \cdot 0.46 + 2 \cdot 0.42 \\ &= 1.3. \end{aligned}$$

Chapter 4 Theoretical Exercises (pages 179-183)

Problem 2
If X has distribution function F, what is the distribution function of e^X*?*

Since the exponential function is increasing, the events $\{e^X \leq x\}$ and $\{X \leq \log(x)\}$ are identical. Hence the distribution function of e^X is

$$P\{e^X \leq x\} = P\{X \leq \log(x)\} = F(\log(x)).$$

Problem 7

Let X be a random variable having expected value μ and variance σ^2. Find the expected value and variance of

$$Y = \frac{X-\mu}{\sigma}.$$

By Corollary 4.1,

$$E[Y] = E\left[\frac{X-\mu}{\sigma}\right] = \frac{E[X]-\mu}{\sigma} = \frac{\mu-\mu}{\sigma} = 0.$$

Also,

$$\begin{aligned}
\text{Var}(Y) &= E[Y^2] - (E[Y])^2 \\
&= E[Y^2] \\
&= E\left[\frac{(X-\mu)^2}{\sigma^2}\right] \\
&= E\left[\frac{X^2 - 2\mu X + \mu^2}{\sigma^2}\right] \\
&= \frac{E[X^2] - 2\mu E[X] + \mu^2}{\sigma^2} \\
&= \frac{E[X^2] - 2\mu^2 + \mu^2}{\sigma^2} \\
&= \frac{E[X^2] - \mu^2}{\sigma^2} \\
&= \frac{E[X^2] - (E[X])^2}{\sigma^2} \\
&= \frac{\text{Var}(X)}{\sigma^2} \\
&= 1.
\end{aligned}$$

So Y has expected value 0 and variance 1.

Problem 12

There are n components lined up in a linear arrangement. Suppose that each component independently functions with probability p. What is the probability that no 2 neighboring components are both nonfunctional?
Hint: Condition on the number of defective components and use the results of Example 4c of Chapter 1.

Let X denote the number of nonfunctional components and let E denote the event that no two neighboring components are both nonfunctional. Recall from Example 4c in Chapter 1 that given n components with m nonfunctional, there are $\binom{n-m+1}{m}$ possible orderings in which no two neighboring components are both nonfunctional, provided $n+1 \geq 2m$. If $n+1 < 2m$, then there must be a pair of neighboring nonfunctional components. Thus

$$\begin{aligned}
P(E) &= \sum_{m=0}^{n} P(E|X=m)P(X=m) \\
&= \sum_{0 \leq m \leq (n+1)/2} P(E|X=m)P(X=m) \\
&= \sum_{0 \leq m \leq (n+1)/2} \frac{\binom{n-m+1}{m}}{\binom{n}{m}} \cdot \binom{n}{m} p^{n-m}(1-p)^m \\
&= \sum_{0 \leq m \leq (n+1)/2} \binom{n-m+1}{m} p^{n-m}(1-p)^m.
\end{aligned}$$