

Jan 24 Homework Solutions
Math 151, Winter 2012

Chapter 3 Problems (pages 102-110)

Problem 12

A recent college graduate is planning to take the first three actuarial examinations in the coming summer. She will take the first actuarial exam in June. If she passes that exam, then she will take the second exam in July, and if she also passes that one, then she will take the third exam in September. If she fails an exam, then she is not allowed to take any others. The probability that she passes the first exam is .9. If she passes the first exam, then the conditional probability that she passes the second one is .8, and if she passes both the first and the second exams, then the conditional probability that she passes the third exam is .7.

- (a) What is the probability that she passes all three exams?

Let E_i denote the event where she passes the i -th exam. We have

$$P(\text{passes all exams}) = P(E_3 E_2 E_1) = P(E_3 | E_2 E_1) P(E_2 | E_1) P(E_1) = .7 \cdot .8 \cdot .9 = .504$$

by the multiplication rule.

- (b) Given that she did not pass all three exams, what is the conditional probability that she failed the second exam?

Note that the event that she failed the second exam is $E_1 E_2^c$, and the event that she did not pass all three exams is $(E_1 E_2 E_3)^c$. We compute

$$\begin{aligned} P(E_1 E_2^c | (E_1 E_2 E_3)^c) &= \frac{P(E_1 E_2^c \cap (E_1 E_2 E_3)^c)}{P((E_1 E_2 E_3)^c)} && \text{by definition} \\ &= \frac{P(E_1 E_2^c)}{P((E_1 E_2 E_3)^c)} && \text{since } E_1 E_2^c \subset (E_1 E_2 E_3)^c \\ &= \frac{P(E_2^c | E_1) P(E_1)}{P((E_1 E_2 E_3)^c)} \\ &= \frac{(1 - P(E_2 | E_1)) P(E_1)}{1 - P(E_1 E_2 E_3)} \\ &= \frac{(1 - .8) \cdot .9}{1 - .504} && \text{where we know the denominator from part (a)} \\ &\approx .3629. \end{aligned}$$

Problem 14

An urn initially contains 5 white and 7 black balls. Each time a ball is selected, its color is noted and it is replaced in the urn along with 2 other balls of the same color. Compute the probability that

- (a) the first 2 balls selected are black and the next 2 are white.

Let a string of k letters, B or W , denote the choice of the first k balls, black or white respectively. Then

$$P(BBWW) = P(BBWW|BBW)P(BBW|BB)P(BB|B)P(B).$$

$P(B) = 7/12$. If the first ball selected is black, then there are 5 white and 9 black balls, so $P(BB|B) = 9/14$. If the first two balls selected are black, then there are 5 white and 11 black balls, so $P(BBW|BB) = 5/16$. If the first three balls are BBW , then there are 7 white and 11 black balls, so $P(BBWW|BBW) = 7/18$. Hence

$$P(BBWW) = \frac{7}{18} \cdot \frac{5}{16} \cdot \frac{9}{14} \cdot \frac{7}{12} = \frac{35}{768}.$$

(b) *of the first 4 balls selected, exactly 2 are black.*

Using the same process as in part (a), we get that

$$\begin{aligned} & P(BBWW) + P(BWBW) + P(BWWB) \\ & + P(WBWB) + P(WBBW) + P(WWBB) \\ = & \frac{7}{18} \cdot \frac{5}{16} \cdot \frac{9}{14} \cdot \frac{7}{12} + \frac{7}{18} \cdot \frac{9}{16} \cdot \frac{5}{14} \cdot \frac{7}{12} + \frac{9}{18} \cdot \frac{7}{16} \cdot \frac{5}{14} \cdot \frac{7}{12} \\ & + \frac{9}{18} \cdot \frac{7}{16} \cdot \frac{7}{14} \cdot \frac{5}{12} + \frac{7}{18} \cdot \frac{9}{16} \cdot \frac{7}{14} \cdot \frac{5}{12} + \frac{9}{18} \cdot \frac{7}{16} \cdot \frac{7}{14} \cdot \frac{5}{12} \\ = & \frac{13230}{48384} \\ \approx & .2734. \end{aligned}$$

Problem 16

Ninety-eight percent of all babies survive delivery. However, 15 percent of all births involve Cesarean (C) sections, and when a C section is performed the baby survives 96 percent of the time. If the randomly chosen pregnant woman does not have a C section, what is the probability that her baby survives?

Let S be the event that a baby survives and let C be the event of a C section. Then

$$P(S|C^c) = \frac{P(SC^c)}{P(C^c)} = \frac{P(S) - P(SC)}{P(C^c)} = \frac{P(S) - P(S|C)P(C)}{1 - P(C)} = \frac{.98 - .96 \cdot .15}{1 - .15} \approx .9835.$$

Problem 21

A total of 500 married working couples were polled about their annual salaries, with the following information resulting. 212 couples reported both the husband and wife earned less than \$25,000, 198 couples reported only the husband earned more than \$25,000, 36 couples reported that only the wife earned more than \$25,000, and 54 couples reported that both the husband and wife earned more than \$25,000. If one of the couples is randomly chosen, what is

(a) *the probability that the husband earns less than \$25,000?*

$$\frac{\text{number of husbands earning less than \$25,000}}{\text{number of couples}} = \frac{212 + 36}{500} = \frac{248}{500}$$

- (b) *the conditional probability that the wife earns more than \$25,000 given that the husband earns more than this amount?*

$$\frac{\text{number couples with both earning more than \$25,000}}{\text{number of husbands earning more than \$25,000}} = \frac{54}{198 + 54} = \frac{54}{252}$$

- (c) *the conditional probability that the wife earns more than \$25,000 given that the husband earns less than this amount?*

$$\frac{\text{number couples where wife earns more than \$25,000 and husbands earns less}}{\text{number of husbands earning less than \$25,000}} \\ = \frac{36}{212 + 36} = \frac{36}{248}$$

Problem 33

On rainy days, Joe is late to work with probability .3; on nonrainy days, he is late with probability .1. With probability .7, it will rain tomorrow.

- (a) *Find the probability that Joe is early tomorrow.*

Let R be the event that it rains, and let E be the event that Joe is early. We have

$$P(E) = P(E|R)P(R) + P(E|R^c)P(R^c) = (1 - .3) \cdot .7 + (1 - .1) \cdot (1 - .7) = .76.$$

- (b) *Given that Joe was early, what is the conditional probability that it rained?*

By Bayes' formula, we have

$$P(R|E) = \frac{P(RE)}{P(E)} = \frac{P(E|R)P(R)}{P(E)} = \frac{(1 - .3) \cdot .7}{.76} \approx .6447.$$

Problem 37

- (a) *A gambler has a fair coin and a two-headed coin in his pocket. He selects one of the coins at random; when he flips it, it shows heads. What is the probability that it is the fair coin?*

By Bayes' formula, we have

$$P(\text{fair}|\text{heads}) = \frac{P(\text{heads}|\text{fair})P(\text{fair})}{P(\text{heads}|\text{fair})P(\text{fair}) + P(\text{heads}|\text{2-headed})P(\text{2-headed})} \\ = \frac{1/2 \cdot 1/2}{1/2 \cdot 1/2 + 1 \cdot 1/2} = \frac{1}{3}.$$

- (b) *Suppose that he flips the same coin a second time and, again, it shows heads. Now what is the probability that it is the fair coin?*

By Bayes' formula, we have

$$P(\text{fair}|HH) = \frac{P(HH|\text{fair})P(\text{fair})}{P(HH|\text{fair})P(\text{fair}) + P(HH|\text{2-headed})P(\text{2-headed})} = \frac{1/4 \cdot 1/2}{1/4 \cdot 1/2 + 1 \cdot 1/2} = \frac{1}{5}.$$

Alternatively, one can apply Example 5f.

- (c) Suppose that he flips the same coin a third time and it shows tails. Now what is the probability that it is the fair coin?

Since it is impossible to flip the two-headed coin and get tails, the current probability that it is the fair coin is 1.

Problem 44

Three prisoners are informed by their jailer that one of them has been chosen at random to be executed and the other two are to be freed. Prisoner A asks the jailer to tell him privately which of his fellow prisoners will be set free, claiming that there would be no harm in divulging this information because he already knows that at least one of the two will go free. The jailer refuses to answer this question, pointing out that if A knew which of his fellow prisoners were to be set free, then his own probability of being executed would rise from $1/3$ to $1/2$ because he would then be one of two prisoners. What do you think of the jailer's reasoning?

Suppose (without loss of generality) that the jailer told Prisoner A that Prisoner B would be set free. Then by Bayes' formula, we have

$$P(A \text{ dies} | \text{told B}) = \frac{P(\text{told B} | A \text{ dies})P(A \text{ dies})}{P(\text{told B} | A \text{ dies})P(A \text{ dies}) + P(\text{told B} | B \text{ dies})P(B \text{ dies}) + P(\text{told B} | C \text{ dies})P(C \text{ dies})}.$$

If A is to die, he could be told either B is to be freed or C is to be freed, each with probability $1/2$. Thus $P(\text{told B} | A \text{ dies}) = 1/2$. If B is to die, A would not be told B is to be freed. Thus $P(\text{told B} | B \text{ dies}) = 0$. If C is to die, A must be told B is to be freed. Thus $P(\text{told B} | C \text{ dies}) = 1$. So we have

$$\begin{aligned} P(A \text{ dies} | \text{told B}) &= \frac{P(\text{told B} | A \text{ dies})P(A \text{ dies})}{P(\text{told B} | A \text{ dies})P(A \text{ dies}) + P(\text{told B} | B \text{ dies})P(B \text{ dies}) + P(\text{told B} | C \text{ dies})P(C \text{ dies})} \\ &= \frac{1/2 \cdot 1/3}{1/2 \cdot 1/3 + 0 \cdot 1/3 + 1 \cdot 1/3} \\ &= \frac{1}{3}. \end{aligned}$$

The jailer is wrong. Divulging this information will not change the probability that A will be executed.

Problem 49

Prostate cancer is the most common type of cancer found in males. As an indicator of whether a male has prostate cancer, doctors often perform a test that measures the level of the prostate-specific antigen (PSA) that is produced only by the prostate gland. Although PSA levels are indicative of cancer, the test is notoriously unreliable. Indeed, the probability that a noncancerous man will have an elevated PSA level is approximately .135, with this probability increasing to approximately .268 if the man does have cancer. If, on the basis of other factors, a physician is 70 certain that a male has prostate cancer, what is the conditional probability that he has the cancer given that

- (a) *the test indicated an elevated PSA level?*

Let C be the event that the man has cancer and let E be the event of elevated PSA levels. Then by Bayes' formula we have

$$P(C|E) = \frac{P(E|C)P(C)}{P(E|C)P(C) + P(E|C^c)P(C^c)} = \frac{.268 \cdot .7}{.268 \cdot .7 + .135 \cdot .3} \approx .8224.$$

- (b) *the test did not indicate an elevated PSA level?*

By Bayes' formula we have

$$P(C|E^c) = \frac{P(E^c|C)P(C)}{P(E^c|C)P(C) + P(E^c|C^c)P(C^c)} = \frac{(1 - .268) \cdot .7}{(1 - .268) \cdot .7 + (1 - .135) \cdot .3} \approx .6638.$$

Problem 57

A simplified model for the movement of the price of a stock supposes that on each day the stock's price either moves up 1 unit with probability p or moves down 1 unit with probability $1 - p$. The changes on different days are assumed to be independent.

- (a) *What is the probability that after 2 days the stock will be at its original price?*

Let a string of k letters denote what the stock did the first k days, with U denoting up and D denoting down. For the stock to return to its original price, we either have UD or DU , each of which occurs with probability $p(1 - p)$. Thus the probability that the stock is at its original price after two days is $2p(1 - p)$.

- (b) *What is the probability that after 3 days the stock's price will have increased by 1 unit?*

For the stock to increase 1 unit in 3 days, we either have UUD , UDU , or DUU , each of which occurs with probability $p^2(1 - p)$. Thus the probability that the stock is at its original price after two days is $3p^2(1 - p)$.

- (c) *Given that after 3 days the stock's price has increase by 1 unit, what is the probability that it went up on the first day?*

We want

$$\begin{aligned} & P(U \text{ on first day} | \text{increase 1 unit}) \\ &= \frac{P(U \text{ on first day and increase 1 unit})}{P(\text{increase 1 unit})} \\ &= \frac{P(UUD) + P(UDU)}{P(\text{increase 1 unit})} \\ &= \frac{2p^2(1 - p)}{3p^2(1 - p)} \quad \text{by part (b)} \\ &= \frac{2}{3}. \end{aligned}$$

Problem 1

Show that if $P(A) > 0$, then $P(AB|A) \geq P(AB|A \cup B)$.

Since $A \subset A \cup B$, we have that $P(A) \leq P(A \cup B)$. It follows that

$$\frac{P(AB)}{P(A)} \geq \frac{P(AB)}{P(A \cup B)}$$

and therefore

$$P(AB|A) = \frac{P(AB)}{P(A)} \geq \frac{P(AB)}{P(A \cup B)} = P(AB|A \cup B).$$

Problem 4

A ball is in any one of n boxes and is in the i -th box with probability P_i . If the ball is in box i , a search of that box will uncover it with probability α_i . Show that the conditional probability that the ball is in box j , given that a search of box i did not uncover it, is

$$\frac{P_j}{1 - \alpha_i P_i} \text{ if } j \neq i \qquad \frac{(1 - \alpha_i)P_i}{1 - \alpha_i P_i} \text{ if } j = i.$$

First consider the case that $j \neq i$. By Bayes' formula we have

$$\begin{aligned} & P(\text{ball in box } j | \text{not found in box } i) \\ &= \frac{P(\text{not found in box } i | \text{ball in box } j)P(\text{ball in box } j)}{\sum_{k=1}^n P(\text{not found in box } i | \text{ball in box } k)P(\text{ball in box } k)} \\ &= \frac{1 \cdot P_j}{(1 - \alpha_i)P_i + \sum_{k \neq i} 1 \cdot P_k} \\ &= \frac{P_j}{\sum_{k=1}^n P_k - \alpha_i P_i} \quad \text{after rearranging the denominator} \\ &= \frac{P_j}{1 - \alpha_i P_i}, \end{aligned}$$

where the last step follows from the fact that the ball being in the k -th box is an exhaustive collection of mutually exclusive events and thus $\sum_{k=1}^n P_k = 1$.

If $i = j$, the only thing that changes is the numerator. We obtain

$$\begin{aligned} & P(\text{ball in box } i | \text{not found in box } i) \\ &= \frac{P(\text{not found in box } i | \text{ball in box } i)P(\text{ball in box } i)}{\sum_{k=1}^n P(\text{not found in box } i | \text{ball in box } k)P(\text{ball in box } k)} \\ &= \frac{(1 - \alpha_i) \cdot P_i}{(1 - \alpha_i)P_i + \sum_{k \neq i} 1 \cdot P_k} \\ &= \frac{(1 - \alpha_i)P_i}{\sum_{k=1}^n P_k - \alpha_i P_i} \quad \text{after rearranging the denominator} \\ &= \frac{(1 - \alpha_i)P_i}{1 - \alpha_i P_i} \quad \text{since } \sum_{k=1}^n P_k = 1. \end{aligned}$$

Problem 7

- (a) *An urn contains n white and m black balls. The balls are withdrawn one at a time until only those of the same color are left. Show that, with probability $n/(n + m)$, they are all white. (HINT: Imagine that the experiment continues until all the balls are removed and consider the last ball withdrawn.)*

Note that all the black balls are removed before all the white balls if and only if the last ball removed is white. Each of the $n + m$ balls is equally likely to be the last ball, and there are n different ways the last ball could be white. Hence, the probability that all remaining balls are white when there is only one color left is $n/(n + m)$.

- (b) *A pond contains 3 distinct species of fish, which we will call the Red, Blue, and Green fish. There are r Red, b Blue, and g Green fish. Suppose that the fish are removed from the pond in a random order. (That is, each selection is equally likely to be any of the remaining fish.) What is the probability that the Red fish are the first species to become extinct from the pond? (HINT: Write $P\{R\} = P\{RBG\} + P\{RGB\}$, and compute the probabilities on the right by first conditioning on the last species to be removed.)*

Let $\{RBG\}$ be the event that the Red fish are the first to become extinct, the Blue fish are the second, and the Green fish are the third. Let events with other orders, such as $\{RGB\}$, be defined analogously. Note that

$$P(\text{Red are first species to become extinct}) = P\{RBG\} + P\{RGB\}.$$

We will compute $P\{RBG\}$ as

$$P\{RBG\} = P\{RBG|G \text{ is last}\}P(G \text{ is last}).$$

By part (a), $P(G \text{ is last}) = g/(r + b + g)$. To compute $P\{RBG|G \text{ is last}\}$, we can ignore the Green fish and consider the probability that the Blue fish are last amongst the Red and Blue fish only. By part (a), we get $P\{RBG|G \text{ is last}\} = b/(r + b)$. Thus

$$\begin{aligned} P\{RBG\} &= P\{RBG|G \text{ is last}\}P(G \text{ is last}) \\ &= \frac{b}{r + b} \cdot \frac{g}{r + b + g} \\ &= \frac{bg}{(r + b)(r + b + g)}. \end{aligned}$$

By symmetry (or a similar computation), we have that

$$P\{RGB\} = \frac{gb}{(r + g)(r + b + g)}.$$

Hence

$$\begin{aligned} P(\text{Red are first species to become extinct}) &= P\{RBG\} + P\{RGB\} \\ &= \frac{bg}{(r + b)(r + b + g)} + \frac{gb}{(r + g)(r + b + g)}. \end{aligned}$$