

Jan 17 Homework Solutions
Math 151, Winter 2012

Chapter 2 Problems (pages 50-54)

Problem 2

In an experiment, a die is rolled continually until a 6 appears, at which point the experiment stops. What is the sample space of this experiment? Let E_n denote the event that n rolls are necessary to complete the experiment. What points of the sample space are contained in E_n ? What is $(\bigcup_{n=1}^{\infty} E_n)^c$?

The sample space consists of the following sequences of die rolls. If the sequence has finite length, then its last entry must equal six and no other entry can be six. If the sequence is infinite, then no entry can equal six.

Even E_n contains all sequences in the sample space of length n .

Note that $\bigcup_{n=1}^{\infty} E_n$ is the event that the experiment terminates. Therefore, $(\bigcup_{n=1}^{\infty} E_n)^c$ is the event that the experiment never terminates, i.e. $(\bigcup_{n=1}^{\infty} E_n)^c$ is the set of sequences in the sample space that have infinite length.

Problem 5

A system is composed of 5 components, each of which is either working or failed. Consider an experiment that consists of observing the status of each component, and let the outcome of the experiment be given by the vector $(x_1, x_2, x_3, x_4, x_5)$, where x_i is equal to 1 if the component i is working and is equal to 0 if the component i is failed.

- (a) *How many outcomes are in the sample space of this experiment?*

Since there are 5 components, each individually a 1 or 0, there are $2^5 = 32$ outcomes in the sample space.

- (b) *Suppose that the system will work if components 1 and 2 are both working, or if components 3 and 4 are both working, or if components 1, 3, and 5 are all working. Let W be the event that the system will work. Specify all the outcomes in W .*

W consist of vectors $(1, 1, x_3, x_4, x_5)$, x_3, x_4, x_5 either 0 or 1, (8 outcomes), $(x_1, x_2, 1, 1, x_5)$, x_1, x_2, x_5 either 0 or 1, (6 additional outcomes), and $(1, 0, 1, 0, 1)$. So there are 15 outcomes in W .

- (c) *Let A be the event that components 4 and 5 are both failed. How many outcomes are contained in the event A ?*

A consist of vectors $(x_1, x_2, x_3, 0, 0)$ where x_1, x_2, x_3 are each individually a 1 or 0, so there are $2^3 = 8$ outcomes in A .

- (d) *Write out all the outcomes in the event AW .*

The system works with components 4 and 5 failing only for vectors of the form $(1, 1, x_3, 0, 0)$, where x_3 is either 1 or 0 (2 outcomes). So the outcomes in the event AW are $(1, 1, 1, 0, 0)$ and $(1, 1, 0, 0, 0)$.

Problem 8

Suppose that A and B are mutually exclusive events for which $P(A) = .3$ and $P(B) = .5$. What is the probability that

- (a) either A or B occurs?

Since A and B are mutually exclusive, $P(A \cup B) = P(A) + P(B) = .3 + .5 = .8$.

- (b) A occurs but B does not?

Since A and B are mutually exclusive, $P(AB^c) = P(A) = .3$.

- (c) both A and B occur?

Since A and B are mutually exclusive, A and B cannot both occur, so $P(AB) = 0$.

Problem 12

An elementary school is offering 3 language classes: one in Spanish, one in French, and one in German. These classes are open to any of the 100 students in the school. There are 28 students in the Spanish class, 26 in the French class, and 16 in the German class. There are 12 students that are in both Spanish and French, 4 that are in both Spanish and German, and 6 that are in both French and German. In addition, there are 2 students taking all three classes.

- (a) If a student is chosen randomly, what is the probability that he or she is not in any of the language classes?

Let S , F , and G denote the events of taking Spanish, French, and German, respectively. By the inclusion-exclusion identity, Proposition 4.4, we have

$$\begin{aligned} P(S \cup F \cup G) &= P(S) + P(F) + P(G) - P(SF) - P(SG) - P(FG) + P(SFG) \\ &= \frac{28}{100} + \frac{26}{100} + \frac{16}{100} - \frac{12}{100} - \frac{4}{100} - \frac{6}{100} + \frac{2}{100} \\ &= \frac{50}{100}. \end{aligned}$$

Thus the probability of not taking any of the classes is

$$P((S \cup F \cup G)^c) = 1 - P(S \cup F \cup G) = 1 - \frac{50}{100} = \frac{50}{100}.$$

- (b) If a student is chosen randomly, what is the probability that he or she is taking exactly one language class?

We present two different solutions to this problem.

First solution: Since there are 2 students taking all 3 classes, this means that there are

- $12 - 2 = 10$ students in Spanish and French but not German,
- $4 - 2 = 2$ students in both Spanish and German but not French, and
- $6 - 2 = 4$ students in both French and German but not Spanish.

Hence we have $10 + 2 + 4 = 16$ students taking exactly two language classes, and 2 students taking exactly 3 language classes. We essentially computed in part (a) that there are 50 students taking at least one language class. So the number of students taking exactly one language class is $50 - 16 - 2 = 32$. Hence the probability that a random student is taking exactly one language class is $\frac{32}{100}$.

Second solution: In the sum

$$\#S + \#F + \#G - 2\#SF - 2\#SG - 2\#FG + 3\#SFG,$$

students taking only one language class are counted once, students taking two language classes are counted $1 \cdot 2 - 2 = 0$ times, and students taking all three language classes are counted $1 \cdot 3 - 2 \cdot 3 + 3 = 0$ times. Thus the probability of taking exactly one language class is

$$\begin{aligned} & P(S) + P(F) + P(G) - 2P(SF) - 2P(SG) - 2P(FG) + 3P(SFG) \\ = & \frac{28}{100} + \frac{26}{100} + \frac{16}{100} - 2 \cdot \frac{12}{100} - 2 \cdot \frac{4}{100} - 2 \cdot \frac{6}{100} + 3 \cdot \frac{2}{100} \\ = & \frac{32}{100}. \end{aligned}$$

- (c) *If 2 students are chosen randomly, what is the probability that at least 1 is taking a language class?*

We essentially computed in part (a) that there are 50 students taking at least one language class. Let E_i , $i = 1, 2$, denote the event that the i -th student is taking a language class. By inclusion-exclusion, Proposition 4.4, we have

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 E_2) = \frac{50}{100} + \frac{50}{100} - \frac{50 \cdot 49}{100 \cdot 99} = \frac{149}{99}.$$

Problem 15

If it is assumed that all $\binom{52}{5}$ poker hands are equally likely, what is the probability of being dealt

- (a) *a flush? (A hand is said to be a flush if all 5 cards are of the same suit.)*

There are 4 choices for the suit and $\binom{13}{5}$ choices for the cards in that suit. Hence the number of different flush hands is $4 \cdot \binom{13}{5}$, and the probability of a flush is

$$\frac{4 \cdot \binom{13}{5}}{\binom{52}{5}} = .001981.$$

- (b) *one pair? (This occurs when the cards have denominations a, a, b, c, d where a, b, c and d are all distinct.)*

There are 13 choices for the value of a , $\binom{4}{2}$ choices for the suits of a , $\binom{12}{3}$ choices for the values of b, c , and d , 4 choices for the suit of b , 4 choices for the suit of c , and 4 choices for the suit of d . Hence the probability of a one pair is

$$\frac{13 \cdot \binom{4}{2} \cdot \binom{12}{3} \cdot 4 \cdot 4 \cdot 4}{\binom{52}{5}} = .422569.$$

- (c) *two pairs?* (This occurs when the cards have denominations a, a, b, b, c where $a, b,$ and c are all distinct.)

There are $\binom{13}{2}$ choices for the values of a and b , $\binom{4}{2}$ choices for the suits of a , $\binom{4}{2}$ choices for the suits of b , and 44 choices for c from the remaining cards. Thus the probability of a two pair is

$$\frac{\binom{13}{2} \cdot \binom{4}{2} \cdot \binom{4}{2} \cdot 44}{\binom{52}{5}} = .047539.$$

- (d) *three of a kind?* (This occurs when the cards have denominations a, a, a, b, c where $a, b,$ and c are all distinct.)

There are 13 choices for the value of a , $\binom{4}{3}$ choices for the suits of a , $\binom{12}{2}$ choices for the values of b and c , 4 choices for the suit of b , and 4 choices for the suit of c . Thus the probability of a three of a kind is

$$\frac{13 \cdot \binom{4}{3} \cdot \binom{12}{2} \cdot 4 \cdot 4}{\binom{52}{5}} = .021128.$$

- (e) *four of a kind?* (This occurs when the cards have denominations $a, a, a, a, b.$)

There are 13 choices for the value of a , $\binom{4}{4}$ choices for the suits of a , and 48 choices for b from the remaining cards. Thus the probability of a four of a kind is

$$\frac{13 \cdot \binom{4}{4} \cdot 48}{\binom{52}{5}} = .00024.$$

Problem 18

Two cards are randomly selected from an ordinary playing deck. What is the probability that they form a blackjack? That is, what is the probability that one of the cards is an ace and the other one is either a ten, a jack, a queen, or a king?

There are 4 possible aces and $4 \cdot 4 = 16$ tens, jacks, queens, and kings, so there are $4 \cdot 16 = 64$ ways to select black jack. There are $\binom{52}{2}$ ways to choose two cards. Therefore the probability of black jack is $\frac{64}{\binom{52}{2}} = .048265$.

Problem 30

The chess clubs of two schools consist of, respectively, 8 and 9 players. Four members from each club are randomly chosen to participate in a contest between two schools. The chosen players from one team are then randomly paired with those from the other team, and each pairing plays a game of chess. Suppose that Rebecca and her sister Elise are on the chess clubs at different schools. What is the probability that

(a) *Rebecca and Elise will be paired?*

The probability that Rebecca will be chosen to represent her school is $\frac{4}{8}$. The probability that Elise will be chosen to represent her school is $\frac{4}{9}$. Assuming that both Rebecca and Elise are chosen, the probability that they will be paired is $\frac{1}{4}$. Therefore, the probability that Rebecca and Elise will be paired is

$$\frac{4}{8} \cdot \frac{4}{9} \cdot \frac{1}{4} = \frac{1}{18}.$$

(b) *Rebecca and Elise will be chosen to represent their schools but will not play each other?*

The probability that Rebecca will be chosen to represent her school is $\frac{4}{8}$. The probability that Elise will be chosen to represent her school is $\frac{4}{9}$. Assuming that both Rebecca and Elise are chosen, the probability that they will not be paired is $\frac{3}{4}$. Therefore, the probability that Rebecca and Elise will be chosen to represent their schools but will not play each other is

$$\frac{4}{8} \cdot \frac{4}{9} \cdot \frac{3}{4} = \frac{1}{6}.$$

(c) *either Rebecca or Elise will be chosen to represent her school?*

Let R be the event that Rebecca is chosen to represent her school and let E be the event that Elise is chosen to represent her school. We are interested in calculating $P(R \cup E)$. By the inclusion-exclusion identity, Proposition 4.4, we have

$$\begin{aligned} P(R \cup E) &= P(R) + P(E) - P(RE) \\ &= \frac{4}{8} + \frac{4}{9} - \left(\frac{4}{8} \cdot \frac{4}{9}\right) \\ &= \frac{13}{18}. \end{aligned}$$

Problem 32

A group of individuals containing b boys and g girls is lined up in random order; that is, each of the $(b + g)!$ permutations is assumed to be equally likely. What is the probability that the person in the i -th position, $1 \leq i \leq b + g$, is a girl?

The number of ways to choose a girl for the i -th position is g and there are $(b + g - 1)!$ ways to order the people in the remaining positions. Therefore the probability that the person in the i -th position is a girl is

$$\frac{g \cdot (b + g - 1)!}{(b + g)!} = \frac{g}{b + g}.$$

An easier way of doing this problem is to note that all $b + g$ people are equally likely to end up in the i -th position, and g of the people are girls. Hence the probability that

the person in the i -th position is a girl is $\frac{g}{b+g}$.

Problem 38

There are n socks, 3 of which are red, in a drawer. What is the value of n if, when 2 of the socks are chosen randomly, the probability that they are both red is $\frac{1}{2}$?

The probability of choosing two red socks is

$$\frac{3}{n} \cdot \frac{2}{n-1} = \frac{6}{n(n-1)}.$$

Solving $\frac{6}{n(n-1)} = \frac{1}{2}$ yields $n^2 - n = 12$, or $(n-4)(n+3) = 0$. This implies that $n = 4$ since n must be positive. Hence there are 4 socks in the drawer.

Problem 42

Two dice are thrown n times in succession. Compute the probability that double 6 appears at least once. How large need n be to make this probability at least $\frac{1}{2}$?

Note that the probability that a double 6 appears at least once is equal to one minus the probability that double 6 never appears. The probability that double 6 never appears in n die rolls is $\left(\frac{35}{36}\right)^n$. Hence the probability that a double 6 appears at least once is

$$1 - \left(\frac{35}{36}\right)^n.$$

Consider the equation

$$1 - \left(\frac{35}{36}\right)^n \geq \frac{1}{2}.$$

After rearranging, we get

$$\left(\frac{35}{36}\right)^n \leq \frac{1}{2}.$$

Taking the logarithm of both sides, we get

$$n \log\left(\frac{35}{36}\right) \leq \log\left(\frac{1}{2}\right).$$

We divide by $\log\left(\frac{35}{36}\right)$, which is negative, to get

$$n \geq \log\left(\frac{1}{2}\right) / \log\left(\frac{35}{36}\right) = 24.6051.$$

So the probability of throwing at least one double 6 is at least $\frac{1}{2}$ if $n \geq 25$.

Section 2 Theoretical Exercises (page 55)

Problem 9

Suppose that an experiment is performed n times. For any event E of the sample space, let $n(E)$ denote the number of times that event E occurs and define $f(E) = n(E)/n$. Show that $f(\cdot)$ satisfies Axioms 1, 2, and 3.

- (1) Axiom 1: Since an event can occur in only a positive number of trials and in no more than n trials, we have $0 \leq n(E) \leq n$. Dividing by n , we get so $0 \leq n(E)/n = f(E) \leq 1$. Hence $f(\cdot)$ satisfies Axiom 1.
- (2) Axiom 2: Since the experiment has n outcomes, $f(S) = n(S)/n = n/n = 1$. Hence $f(\cdot)$ satisfies Axiom 2.
- (3) Axiom 3: Suppose E_1, E_2, E_3, \dots is a sequence of mutually exclusive events. Note that because the events E_1, E_2, E_3, \dots are mutually exclusive, the number $n(\bigcup_{i=1}^{\infty} E_i)$ of times that the union occurs is equal to the sum $\sum_{i=1}^{\infty} n(E_i)$ of the number of times that each event occurs. We have

$$\begin{aligned}
 f\left(\bigcup_{i=1}^{\infty} E_i\right) &= \frac{n\left(\bigcup_{i=1}^{\infty} E_i\right)}{n} && \text{by definition of } f \\
 &= \frac{\sum_{i=1}^{\infty} n(E_i)}{n} && \text{because the events are mutually exclusive} \\
 &= \sum_{i=1}^{\infty} \frac{n(E_i)}{n} \\
 &= \sum_{i=1}^{\infty} f(E_i) && \text{by definition of } f
 \end{aligned}$$

Hence $f(\cdot)$ satisfies Axiom 3.

Problem 15

An urn contains M white and N black balls. If a random sample of size r is chosen, what is the probability that it contains exactly k white balls?

There are $\binom{M}{k}$ ways to choose k white balls and $\binom{N}{r-k}$ ways to choose the remaining $r-k$ black balls. Hence there are $\binom{M}{k} \cdot \binom{N}{r-k}$ samples of size r containing exactly k white balls. Overall, there are $\binom{M+N}{r}$ samples of size r . Hence the probability that the urn contains exactly k white balls is

$$\frac{\binom{M}{k} \cdot \binom{N}{r-k}}{\binom{M+N}{r}}.$$