Math 147, Homework 4 Solutions
Due: May 1, 2012

1. On Homework 3 you constructed a smooth function $f : \mathbb{R}^1 \to \mathbb{R}^1$ with a dense set of critical values. Can you construct a smooth map $f : S^1 \to S^1$ whose critical values are dense?

   Solution.

   No, there is no such map.

   To see this, suppose $f\colon S^1 \to S^1$ is a smooth map. First we will show that the set of critical points of f is closed. Note that if $(x, y) \in S^1$ then $(-y, x) \in TS^1_{(x,y)}$. Consider the map $g\colon S^1 \to \mathbb{R}$ given by $g(x, y) = df_{(x,y)}(-y, x)$. Note that $df_{(x,y)}$ is the zero map if and only if $g(x, y) = 0$. Hence the set of critical points of f is equal to $g^{-1}(0)$. Since g is continuous and $\{0\}$ is closed, the set of critical points of f is closed.

   As a closed subset of the compact set S^1, the set of critical points of f is compact. The set of critical values is therefore compact, since it is the image of the compact set of critical points under the continuous map f. Hence the set of critical values of f is closed.

   If the set of critical values of f were dense, then as a closed and dense subset, the set of critical values would need to be all of S^1. This is not a set of measure zero, contradicting Sard's theorem. Hence the set of critical values of f cannot be dense.

2. (a) Give an example of a smooth manifold M and an embedding $f : M \to \mathbb{R}^k$ which is *not* proper.

   Solution.

   Let $M = (0, \infty) \subset \mathbb{R}$ and define $f\colon (0, \infty) \to \mathbb{R}$ by $f(x) = x$. Then f is clearly a smooth map that is a diffeomorphism onto its image, hence an embedding. However, f is not proper. For example, the set $[-1, 1]$ is compact, but $f^{-1}([-1, 1]) = (0, 1]$ is not compact.

   (b) Suppose $f : M \to \mathbb{R}^k$ is a proper map. Show that the map $g : M \to \mathbb{R}^1$ defined by $g(x) = |f(x)|$ is also proper.

   Solution.

   Let $\mathrm{abs}\colon \mathbb{R}^k \to \mathbb{R}^1$ be the map given by $\mathrm{abs}(y) = |y|$. We will show that abs is a proper map. If $K \subset \mathbb{R}^1$ is bounded then clearly so is $\mathrm{abs}^{-1}(K)$. Since abs is continuous, if $K \subset \mathbb{R}^1$ is closed then so is its preimage $\mathrm{abs}^{-1}(K)$. Thus if $K \subset \mathbb{R}^1$ is compact then so is $\mathrm{abs}^{-1}(K)$.

   Note that $g = \mathrm{abs} \circ f$. Hence if $K \subset \mathbb{R}^1$ is compact, then $\mathrm{abs}^{-1}(K)$ is compact since abs is proper, and $g^{-1}(K) = f^{-1}\big(\mathrm{abs}^{-1}(K)\big)$ is compact since f is proper. Hence g is proper.

   More generally, the composition of any two proper maps is proper.

3. Let $T = S^1 \times S^1$ be the torus and consider the smooth map $\pi : \mathbb{R}^2 \to T$ defined by:

$$\pi(x, y) = (\sin(2\pi x), \cos(2\pi x), \sin(2\pi y), \cos(2\pi y)).$$

For which lines L in $\mathbb{R}^2$ is the restriction of π to L a one-to-one immersion? For which lines L is $\pi(L)$ a smooth manifold? For which lines L is π an embedding?

Solution.

Let L be the line $y = mx + b$. We claim that π is one-to-one if and only if m is irrational. Let $(x, y) \neq (x', y')$ be two points in L. So $y - y' = m(x - x')$. We have $\pi(x, y) = \pi(x', y')$ if and only if the two conditions $x - x' \in \mathbb{Z}$ and $y - y' \in \mathbb{Z}$ are satisfied. These two conditions cannot both be satisfied if m is irrational, and hence π is one-to-one if m is irrational. Conversely, if $m = \frac{p}{q}$ is rational for $p, q \in \mathbb{Z}$, then these two conditions are satisfied by picking $x - x' = q$ and $y - y' = p$. So π is not one-to-one if m is rational.

We calculate

$$d\pi_{(x,y)} = \begin{bmatrix} 2\pi\cos(2\pi x) & 0 \\ -2\pi\sin(2\pi x) & 0 \\ 0 & 2\pi\cos(2\pi y) \\ 0 & -2\pi\sin(2\pi y) \end{bmatrix}.$$

This derivative is injective when restricted to any one-dimensional linear space in $\mathbb{R}^2$. Hence π is an immersion when restricted to any line L in $\mathbb{R}^2$.

In summary, the restriction of π to L is a one-to-one immersion if and only if the slope m is irrational.

If slope m is irrational then $\pi(L)$ is not a smooth manifold. To see this, let $z \in \pi(L)$. The intersection of $\pi(L)$ with any sufficiently small open set in $\mathbb{R}^4$ about z contains an infinite number of path-connected components. Hence no neighborhood of z in $\pi(L)$ is diffeomorphic to any Euclidean space, and so $\pi(L)$ is not a smooth manifold.

If slope m is rational then $\pi(L)$ is a smooth manifold. If $m = \frac{p}{q}$ where p and q are relatively prime, then $\pi(L)$ wraps regularly p times around one copy of S^1 and q times around the other. For any point $z \in \pi(L)$ there exists a sufficiently small ball about z whose intersection with $\pi(L)$ is diffeomorphic to the open line segment $(0, 1)$. Hence $\pi(L)$ is a 1-dimensional smooth manifold.

In summary, $\pi(L)$ is a smooth manifold if and only if slope m is rational.

Map π is an embedding for no lines L. If slope m is rational then π is not one-to-one and hence π is not an embedding. If slope m is irrational then $\pi(L)$ is not a smooth manifold and hence π is not an embedding.

4. Prove the *Whitney Immersion Theorem*: Every m-dimensional manifold $M \subset \mathbb{R}^k$ admits an immersion in $\mathbb{R}^{2m}$.

Solution.

We will modify the proof of the theorem on page 51 of Guillemin and Pollack: instead of using both their map h and g, we will only use g but not h. This will allow us to produce an immersion in $\mathbb{R}^{2m}$, though the immersion need not be injective.

We shall produce a linear projection $\mathbb{R}^k \to \mathbb{R}^{2m}$ that restricts to an immersion of M. Proceeding inductively, we prove that if $f\colon M \to \mathbb{R}^n$ is an immersion with $n > 2k$, then there exists a unit vector $a \in \mathbb{R}^n$ such that the orthogonal complement of a is still an immersion. Now the complement $H = \{b \in \mathbb{R}^n : b \perp a\}$ is an $n-1$ dimensional vector subspace of $\mathbb{R}^n$, hence isomorphic to $\mathbb{R}^{n-1}$; thus we obtain an immersion into $\mathbb{R}^{n-1}$.

Define the map $g\colon T(M) \to \mathbb{R}^n$ by $g(x,v) = df_x(v)$. That this map is smooth follows from problem #8. Since $n > 2k$, Sard's theorem implies that there exists a point $a \in \mathbb{R}^n$ that's not in the image of g, and note that $a \neq 0$ since 0 is in the image of g.

Let π be the projection of $\mathbb{R}^n$ onto the orthogonal complement H of a. We will show that $\pi \circ f\colon M \to H$ is an immersion. For suppose not. Then there is some nonzero vector $v \in T_x(M)$ with $d(\pi \circ f)_x(v) = 0$. Because π is linear, the chain rule yields $d(\pi \circ f)_x = \pi \circ df_x$. Thus $\pi \circ df_x(v) = 0$, so $df_x(v) = ta$ for some scalar t. Because f is an immersion, $t \neq 0$. Thus $g(x, \frac{1}{t}v) = a$, contradicting the choice of a. Hence $\pi \circ f\colon M \to H$ is an immersion.

So by induction we have produced an immersion of M in $\mathbb{R}^{2m}$.

5. Give an example of a smooth map $f : D^2 \to D^2$ with no fixed point on the interior of D^2.

   Solution.

   Let $f\colon D^2 \to D^2$ be the constant map defined by $f(x) = (1,0)$ for all $x \in D^2$. Note that f is smooth because it can be extended to a smooth constant map given by the same formula on all of $\mathbb{R}^2$. Clearly $(1,0)$ is the only fixed point of f, and since $(1,0)$ is not in the interior of D^2, we have found a smooth map with no fixed point on the interior of D^2.

6. Show that TS^1 is diffeomorphic to $S^1 \times \mathbb{R}$.

   Solution.

   Note that if $(x,y) \in S^1$ then $(-y,x) \in TS^1_{(x,y)}$. Moreover, we have $TS^1_{(x,y)} = \{(-ty, tx) \mid t \in \mathbb{R}\}$. Hence we have

$$TS^1 = \{(x, y, -ty, tx) \in S^1 \times \mathbb{R}^2 \mid t \in \mathbb{R}\}.$$

   Define $f\colon S^1 \times \mathbb{R} \to TS^1$ by $f(x,y,t) = (x, y, -ty, tx)$. Note that f is a bijection. Map f is smooth because it extends to a smooth map given by the same formula on all of $\mathbb{R}^2 \times \mathbb{R}$. We can check the inverse of f is given by $f^{-1}\colon TS^1 \to S^1 \times \mathbb{R}$ defined by $f^{-1}(x, y, -ty, tx) = (x, y, t)$ extends to a smooth function $\mathbb{R}^2 \times \mathbb{R}^2$ defined by $(x,y,z,w) \mapsto (x, y, xw - yz)$, and since this extended map is smooth, so is f^{-1}. So $f\colon S^1 \times \mathbb{R} \to TS^1$ is a diffeomorphism.

7. Prove Brouwer's Theorem for continuous maps on $[-1, 1]$ directly, without using regular values.

   Solution.

   Suppose $f\colon [-1, 1] \to [-1, 1]$ is a continuous map. We must show that f has a fixed point x with $f(x) = x$.

   We may assume that $f(-1) > -1$, for otherwise -1 is a fixed point. Similarly, we may assume $f(1) < 1$. Consider the function $g\colon [-1, 1] \to \mathbb{R}$ defined by $g(x) = f(x) - x$. Note that g is continuous, that $g(1) < 0$, and that $g(-1) > 0$. By the intermediate value theorem there exists a point $x \in [-1, 1]$ with $g(x) = 0$, which means that $f(x) - x = 0$ and $f(x) = x$.

8. Let $f : M \to N$ be a smooth map between manifolds and define $df : TM \to TN$ by:

$$df(x, v) = (f(x), df_x(v)).$$

   Show that df is a smooth map between manifolds. Show that if f is a diffeomorphism, then so is df.

   Solution.

   See the third paragraph on page 50 of Guillemin and Pollack for a proof that df is a smooth map between manifolds. See the fourth paragraph of page 50 for a proof that if f is a diffeomorphism, then so is df.