1. On Homework 3 you constructed a smooth function $f : \mathbb{R}^1 \to \mathbb{R}^1$ with a dense set of critical values. Can you construct a smooth map $f : S^1 \to S^1$ whose critical values are dense?

*Solution.*

No, there is no such map.

To see this, suppose $f : S^1 \to S^1$ is a smooth map. First we will show that the set of critical points of $f$ is closed. Note that if $(x, y) \in S^1$ then $(-y, x) \in TS^1_{(x,y)}$. Consider the map $g : S^1 \to \mathbb{R}$ given by $g(x, y) = df_{(x,y)}(-y, x)$. Note that $df_{(x,y)}$ is the zero map if and only if $g(x, y) = 0$. Hence the set of critical points of $f$ is equal to $g^{-1}(0)$. Since $g$ is continuous and $\{0\}$ is closed, the set of critical points of $f$ is closed.

As a closed subset of the compact set $S^1$, the set of critical points of $f$ is compact. The set of critical values is therefore compact, since it is the image of the compact set of critical points under the continuous map $f$. Hence the set of critical values of $f$ is closed.

If the set of critical values of $f$ were dense, then as a closed and dense subset, the set of critical values would need to be all of $S^1$. This is not a set of measure zero, contradicting Sard’s theorem. Hence the set of critical values of $f$ cannot be dense.

2. (a) Give an example of a smooth manifold $M$ and an embedding $f : M \to \mathbb{R}^k$ which is *not* proper.

*Solution.*

Let $M = (0, \infty) \subset \mathbb{R}$ and define $f : (0, \infty) \to \mathbb{R}$ by $f(x) = x$. Then $f$ is clearly a smooth map that is a diffeomorphism onto its image, hence an embedding. However, $f$ is not proper. For example, the set $[-1, 1]$ is compact, but $f^{-1}([-1, 1]) = (0, 1]$ is not compact.

(b) Suppose $f : M \to \mathbb{R}^k$ is a proper map. Show that the map $g : M \to \mathbb{R}^1$ defined by $g(x) = |f(x)|$ is also proper.

*Solution.*

Let abs: $\mathbb{R}^k \to \mathbb{R}^1$ be the map given by abs$(y) = |y|$. We will show that abs is a proper map. If $K \subset \mathbb{R}^1$ is bounded then clearly so is abs$^{-1}(K)$. Since abs is continuous, if $K \subset \mathbb{R}^1$ is closed then so is its preimage abs$^{-1}(K)$. Thus if $K \subset \mathbb{R}^1$ is compact then so is abs$^{-1}(K)$.

Note that $g = \text{abs} \circ f$. Hence if $K \subset \mathbb{R}^1$ is compact, then abs$^{-1}(K)$ is compact since abs is proper, and $g^{-1}(K) = f^{-1}(\text{abs}^{-1}(K))$ is compact since $f$ is proper. Hence $g$ is proper.

More generally, the composition of any two proper maps is proper.
3. Let $T = S^1 \times S^1$ be the torus and consider the smooth map $\pi : \mathbb{R}^2 \to T$ defined by:

$$\pi(x, y) = (\sin(2\pi x), \cos(2\pi x), \sin(2\pi y), \cos(2\pi y)).$$

For which lines $L$ in $\mathbb{R}^2$ is the restriction of $\pi$ to $L$ a one-to-one immersion? For which lines $L$ is $\pi(L)$ a smooth manifold? For which lines $L$ is $\pi$ an embedding?

Solution.

Let $L$ be the line $y = mx + b$. We claim that $\pi$ is one-to-one if and only if $m$ is irrational. Let $(x, y) \neq (x', y')$ be two points in $L$. So $y - y' = m(x - x')$. We have $\pi(x, y) = \pi(x', y')$ if and only if the two conditions $x - x' \in \mathbb{Z}$ and $y - y' \in \mathbb{Z}$ are satisfied. These two conditions cannot both be satisfied if $m$ is irrational, and hence $\pi$ is one-to-one if $m$ is irrational. Conversely, if $m = \frac{p}{q}$ is rational for $p, q \in \mathbb{Z}$, then these two conditions are satisfied by picking $x - x' = q$ and $y - y' = p$. So $\pi$ is not one-to-one if $m$ is rational.

We calculate

$$d\pi_{(x,y)} = \begin{bmatrix}
2\pi \cos(2\pi x) & 0 \\
-2\pi \sin(2\pi x) & 0 \\
0 & 2\pi \cos(2\pi y) \\
0 & -2\pi \sin(2\pi y)
\end{bmatrix}.$$ 

This derivative is injective when restricted to any one-dimensional linear space in $\mathbb{R}^2$. Hence $\pi$ is an immersion when restricted to any line $L$ in $\mathbb{R}^2$.

In summary, the restriction of $\pi$ to $L$ is a one-to-one immersion if and only if the slope $m$ is irrational.

If slope $m$ is irrational then $\pi(L)$ is not a smooth manifold. To see this, let $z \in \pi(L)$. The intersection of $\pi(L)$ with any sufficiently small open set in $\mathbb{R}^4$ about $z$ contains an infinite number of path-connected components. Hence no neighborhood of $z$ in $\pi(L)$ is diffeomorphic to any Euclidean space, and so $\pi(L)$ is not a smooth manifold.

If slope $m$ is rational then $\pi(L)$ is a smooth manifold. If $m = \frac{p}{q}$ where $p$ and $q$ are relatively prime, then $\pi(L)$ wraps regularly $p$ times around one copy of $S^1$ and $q$ times around the other. For any point $z \in \pi(L)$ there exists a sufficiently small ball about $z$ whose intersection with $\pi(L)$ is diffeomorphic to the open line segment $(0, 1)$. Hence $\pi(L)$ is a 1-dimensional smooth manifold.

In summary, $\pi(L)$ is a smooth manifold if and only if slope $m$ is rational.

Map $\pi$ is an embedding for no lines $L$. If slope $m$ is rational then $\pi$ is not one-to-one and hence $\pi$ is not an embedding. If slope $m$ is irrational then $\pi(L)$ is not a smooth manifold and hence $\pi$ is not an embedding.

4. Prove the Whitney Immersion Theorem: Every $m$-dimensional manifold $M \subset \mathbb{R}^k$ admits an immersion in $\mathbb{R}^{2m}$.

Solution.
We will modify the proof of the theorem on page 51 of Guillemin and Pollack: instead of using both their map $h$ and $g$, we will only use $g$ but not $h$. This will allow us to produce an immersion in $\mathbb{R}^{2m}$, though the immersion need not be injective.

We shall produce a linear projection $\mathbb{R}^{k} \to \mathbb{R}^{2m}$ that restricts to an immersion of $M$. Proceeding inductively, we prove that if $f: M \to \mathbb{R}^{n}$ is an immersion with $n > 2k$, then there exists a unit vector $a \in \mathbb{R}^{n}$ such that the orthogonal complement of $a$ is still an immersion. Now the complement $H = \{b \in \mathbb{R}^{n} : b \perp a\}$ is an $n - 1$ dimensional vector subspace of $\mathbb{R}^{n}$, hence isomorphic to $\mathbb{R}^{n-1}$; thus we obtain an immersion into $\mathbb{R}^{n-1}$.

Define the map $g: T(M) \to \mathbb{R}^{n}$ by $g(x, v) = df_{x}(v)$. That this map is smooth follows from problem #8. Since $n > 2k$, Sard’s theorem implies that there exists a point $a \in \mathbb{R}^{n}$ that’s not in the image of $g$, and note that $a \neq 0$ since 0 is in the image of $g$.

Let $\pi$ be the projection of $\mathbb{R}^{n}$ onto the orthogonal complement $H$ of $a$. We will show that $\pi \circ f: M \to H$ is an immersion. For suppose not. Then there is some nonzero vector $v \in T_{x}(M)$ with $d(\pi \circ f)_{x}(v) = 0$. Because $\pi$ is linear, the chain rule yields $d(\pi \circ f)_{x} = \pi \circ df_{x}$. Thus $\pi \circ df_{x}(v) = 0$, so $df_{x}(v) = ta$ for some scalar $t$. Because $f$ is an immersion, $t \neq 0$. Thus $g(x, \frac{1}{t}v) = a$, contradicting the choice of $a$. Hence $\pi \circ f: M \to H$ is an immersion.

So by induction we have produced an immersion of $M$ in $\mathbb{R}^{2m}$.

5. Give an example of a smooth map $f: D^{2} \to D^{2}$ with no fixed point on the interior of $D^{2}$.

Solution.

Let $f: D^{2} \to D^{2}$ be the constant map defined by $f(x) = (1, 0)$ for all $x \in D^{2}$. Note that $f$ is smooth because it can be extended to a smooth constant map given by the same formula on all of $\mathbb{R}^{2}$. Clearly $(1, 0)$ is the only fixed point of $f$, and since $(1, 0)$ is not in the interior of $D^{2}$, we have found a smooth map with no fixed point on the interior of $D^{2}$.

6. Show that $TS^{1}$ is diffeomorphic to $S^{1} \times \mathbb{R}$.

Solution.

Note that if $(x, y) \in S^{1}$ then $(-y, x) \in TS^{1}(x,y)$. Moreover, we have $TS^{1}_{(x,y)} = \{(ty, tx) \mid t \in \mathbb{R}\}$. Hence we have

$$TS^{1} = \{(x, y, -ty, tx) \mid x \in S^{1} \times \mathbb{R}^{2} \mid t \in \mathbb{R}\}.$$

Define $f: S^{1} \times \mathbb{R} \to TS^{1}$ by $f(x, y, t) = (x, y, -ty, tx)$. Note that $f$ is a bijection. Map $f$ is smooth because it extends to a smooth map given by the same formula on all of $\mathbb{R}^{2} \times \mathbb{R}$. We can check the inverse of $f$ is given by $f^{-1}: TS^{1} \to S^{1} \times \mathbb{R}$ defined by $f^{-1}(x, y, -ty, tx) = (x, y, t)$ extends to a smooth function $\mathbb{R}^{2} \times \mathbb{R}^{2}$ defined by $(x, y, z, w) \mapsto (x, y, xw - yz)$, and since this extended map is smooth, so is $f^{-1}$. So $f: S^{1} \times \mathbb{R} \to TS^{1}$ is a diffeomorphism.
7. Prove Brouwer’s Theorem for continuous maps on $[-1, 1]$ directly, without using regular values.

   Solution.

   Suppose $f: [-1, 1] \to [-1, 1]$ is a continuous map. We must show that $f$ has a fixed point $x$ with $f(x) = x$.

   We may assume that $f(-1) > -1$, for otherwise $-1$ is a fixed point. Similarly, we may assume $f(1) < 1$. Consider the function $g: [-1, 1] \to \mathbb{R}$ defined by $g(x) = f(x) - x$. Note that $g$ is continuous, that $g(1) < 0$, and that $g(-1) > 0$. By the intermediate value theorem there exists a point $x \in [-1, 1]$ with $g(x) = 0$, which means that $f(x) - x = 0$ and $f(x) = x$.

8. Let $f: M \to N$ be a smooth map between manifolds and define $df: TM \to TN$ by:

   $df(x, v) = (f(x), df_x(v))$.

   Show that $df$ is a smooth map between manifolds. Show that if $f$ is a diffeomorphism, then so is $df$.

   Solution.

   See the third paragraph on page 50 of Guillemin and Pollack for a proof that $df$ is a smooth map between manifolds. See the fourth paragraph of page 50 for a proof that if $f$ is a diffeomorphism, then so is $df$. 

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