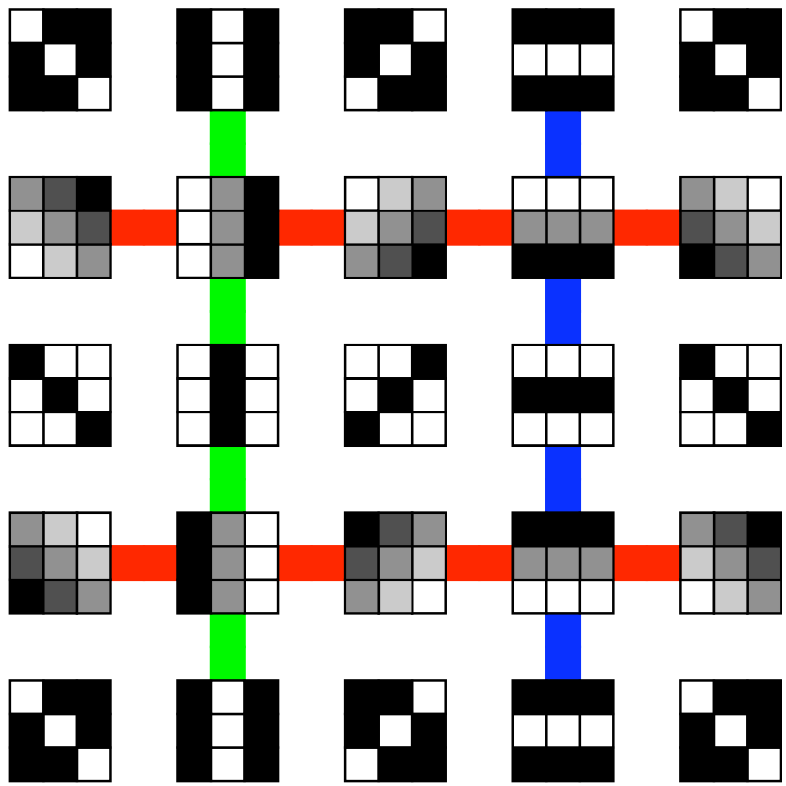
An Introduction to Applied and Computational Topology

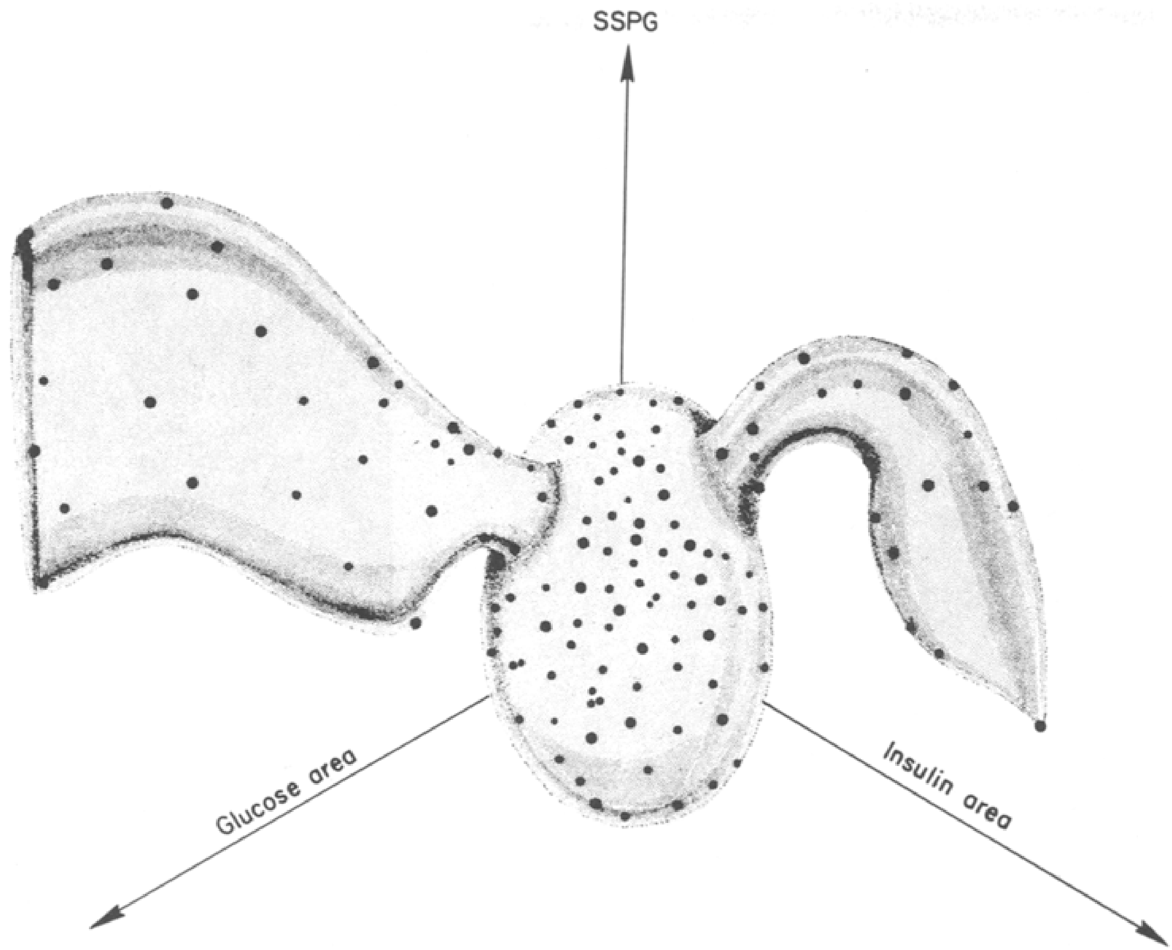


Henry Adams and Joshua Mirth
Colorado State University

Datasets have shapes

Example: Diabetes study

145 points in 5-dimensional space

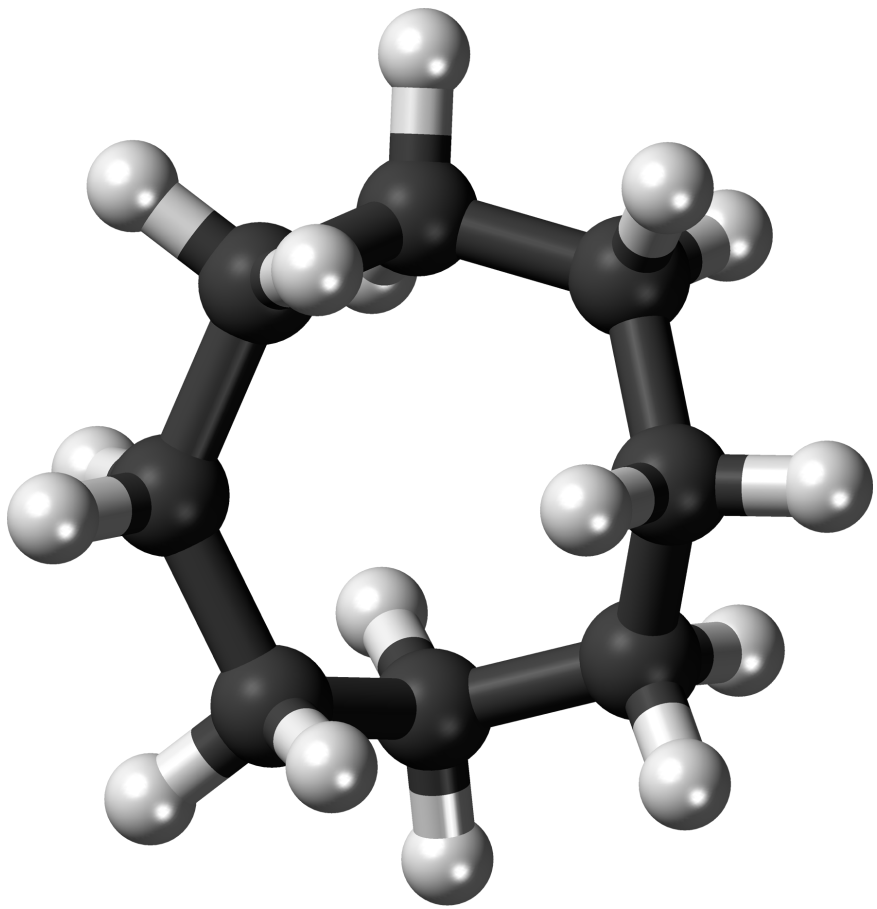


An Attempt to Define the Nature of Chemical Diabetes Using a Multidimensional Analysis by G. M. Reaven and R. G. Miller, 1979.

Datasets have shapes

Example: Cyclo-Octane (C_8H_{16}) data

1,000,000+ points in 72-dimensional space



Non-Manifold Surface Reconstruction from High Dimensional Point Cloud Data by Shawn Martin and Jean-Paul Watson, 2010.

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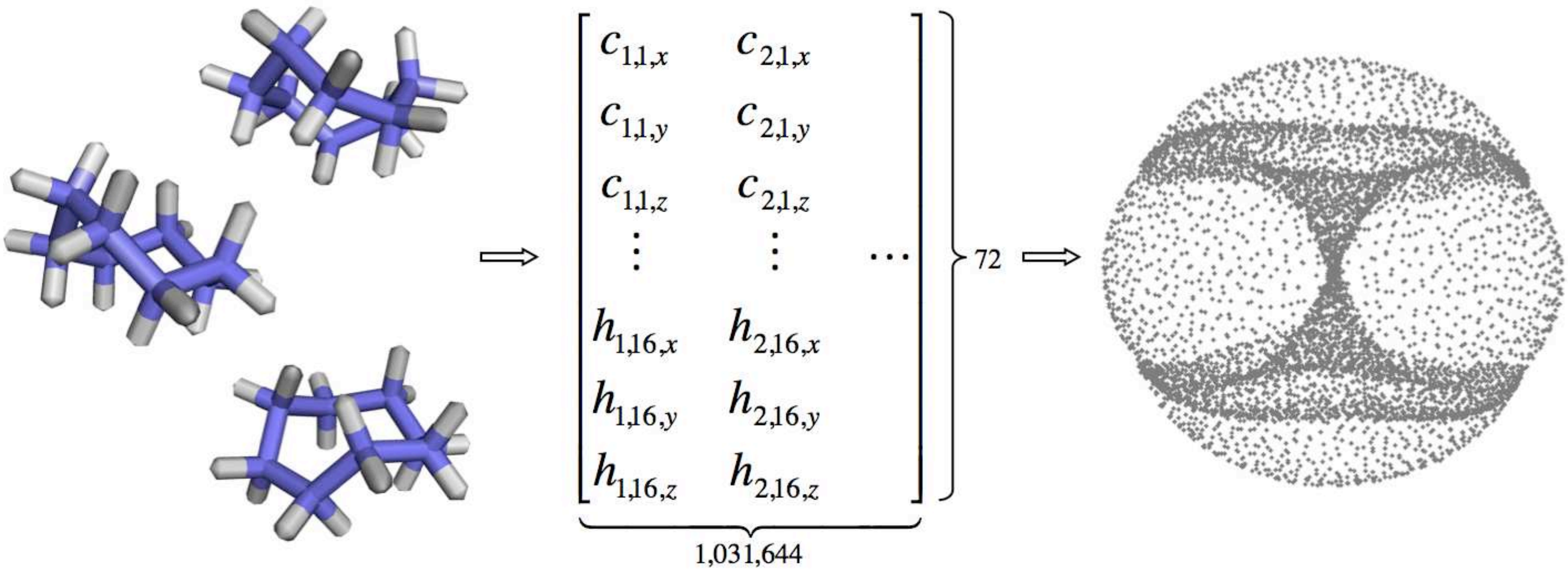

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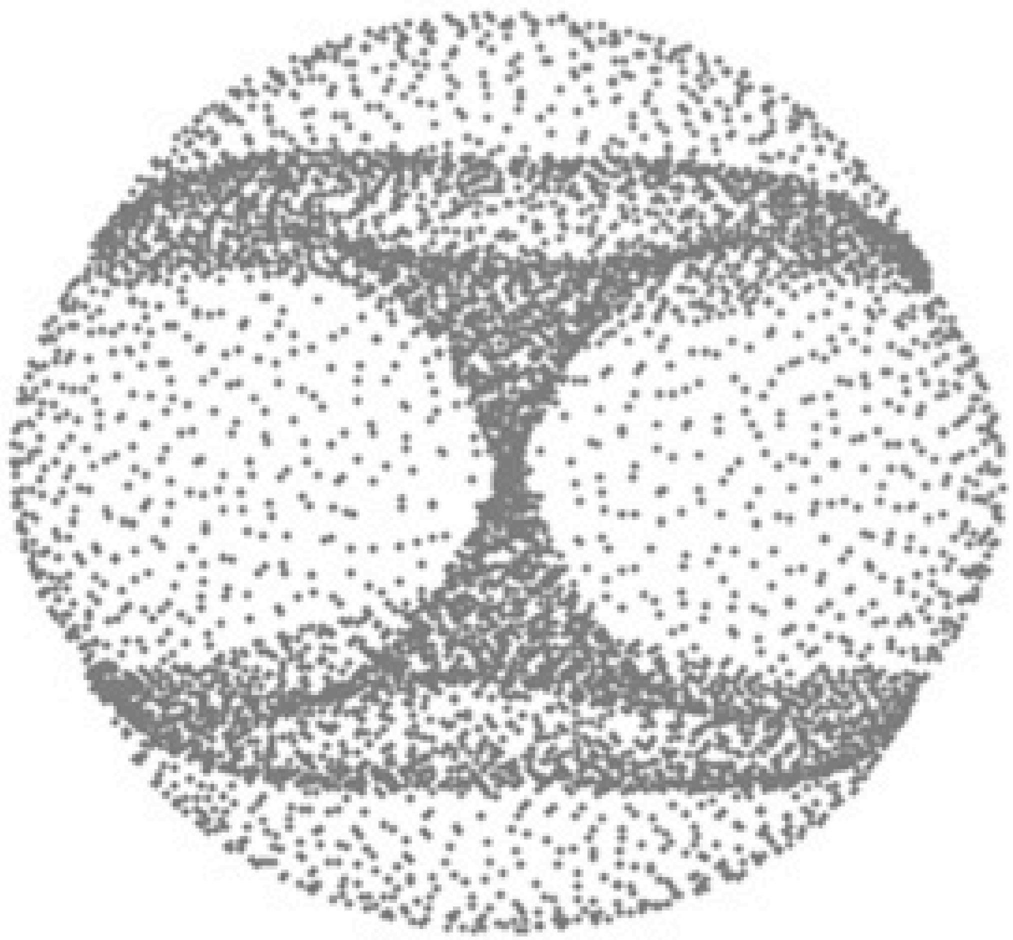
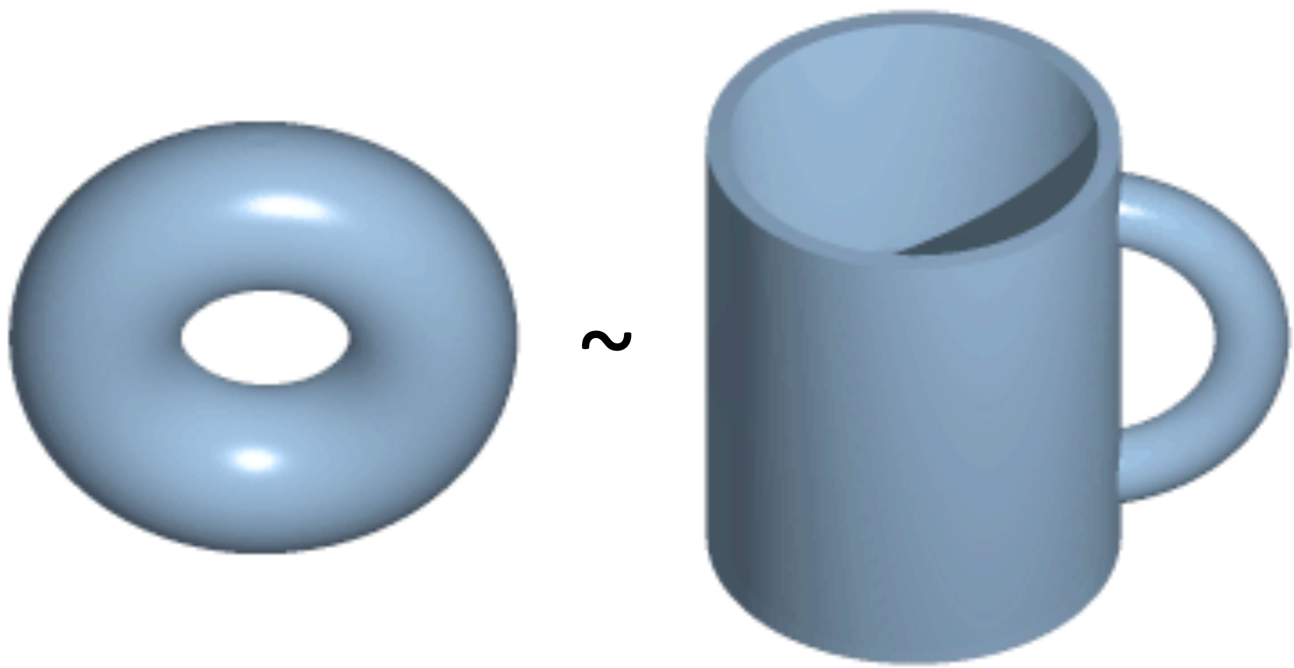
Datasets have shapes

What shape is this?

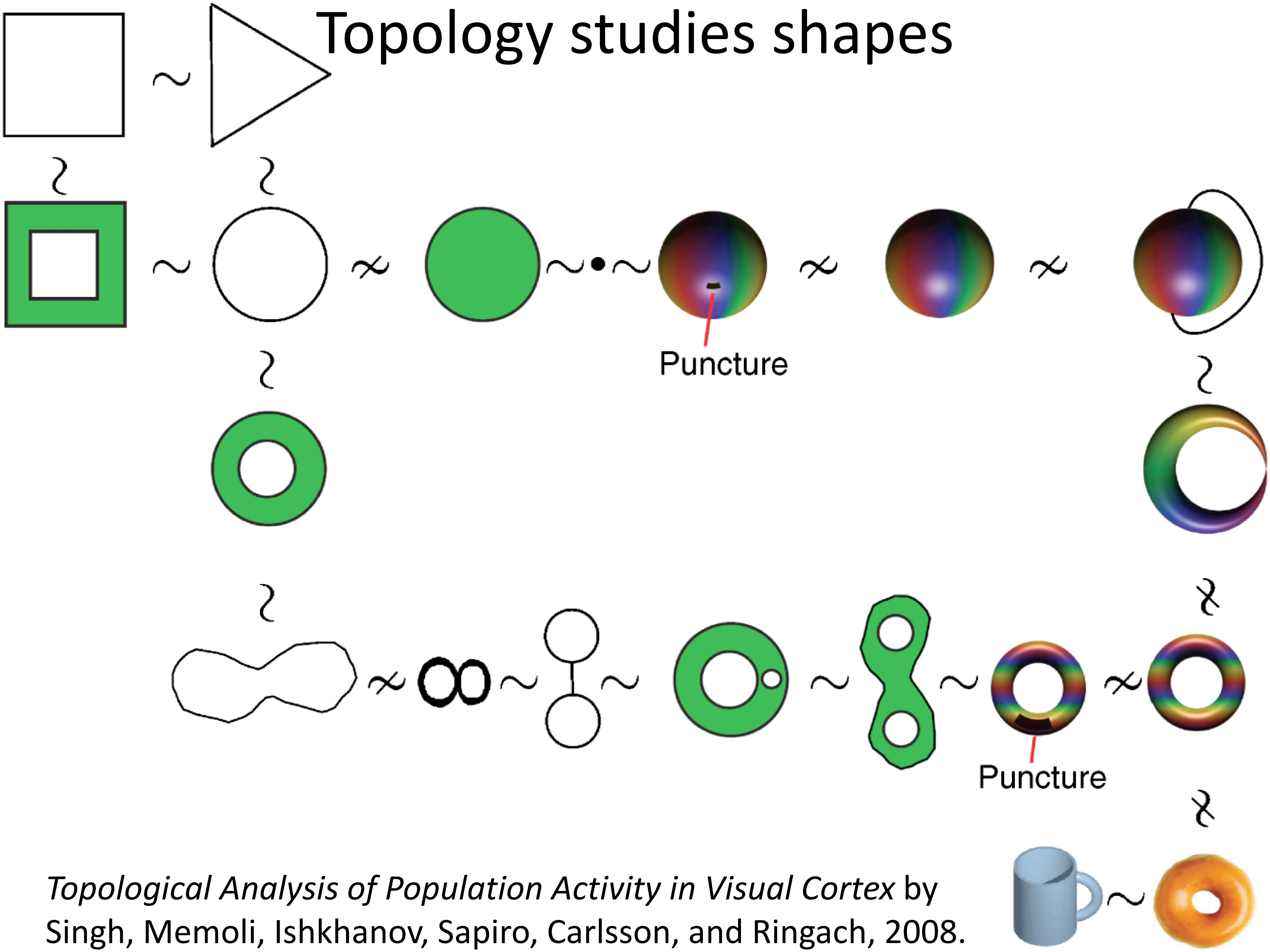


Topology studies shapes

A donut and coffee mug are “homotopy equivalent”, and considered to be the same shape. You can bend and stretch (but not tear) one to get the other.



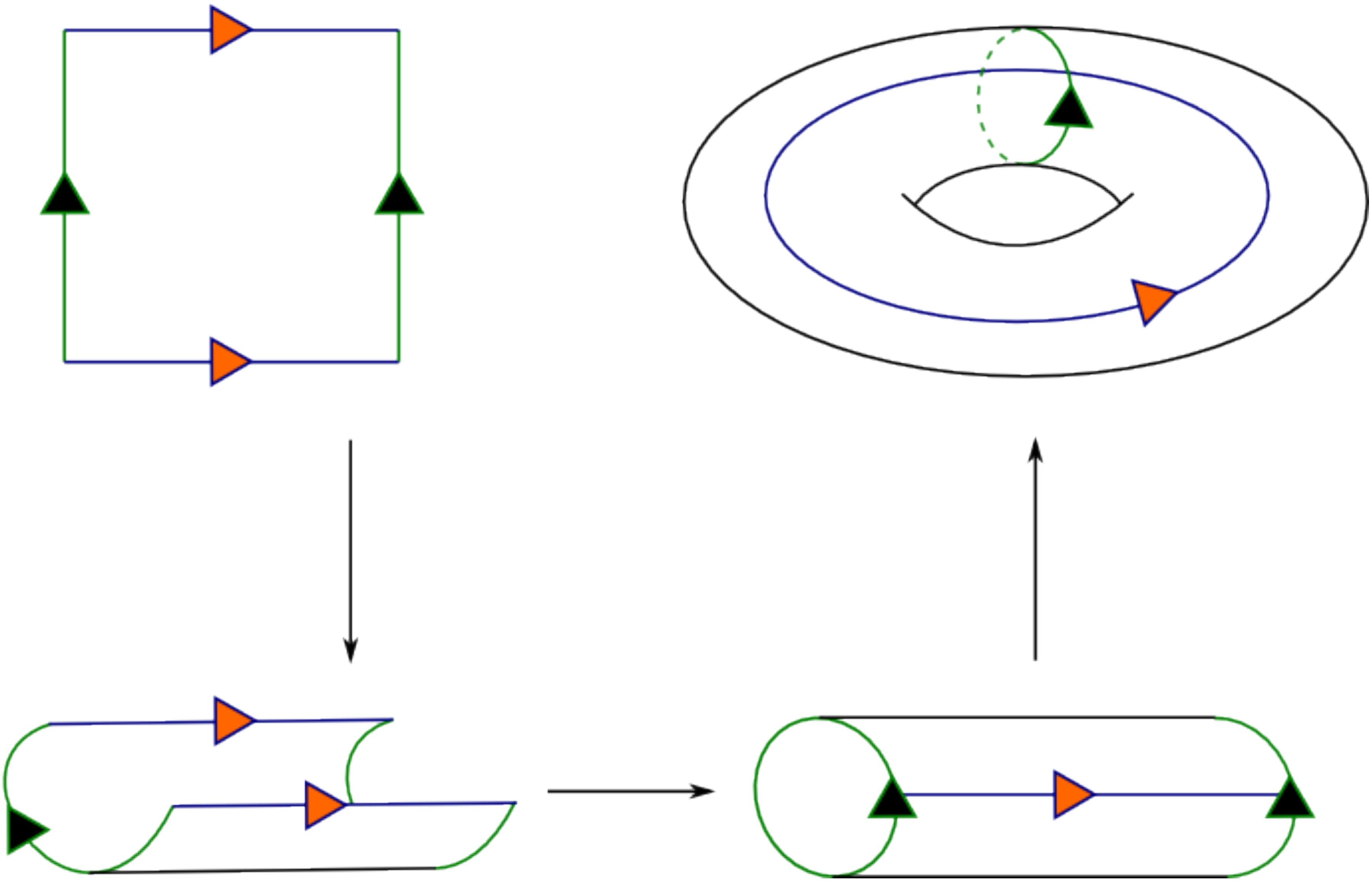
Topology studies shapes



Topological Analysis of Population Activity in Visual Cortex by Singh, Memoli, Ishkhanov, Sapiro, Carlsson, and Ringach, 2008.

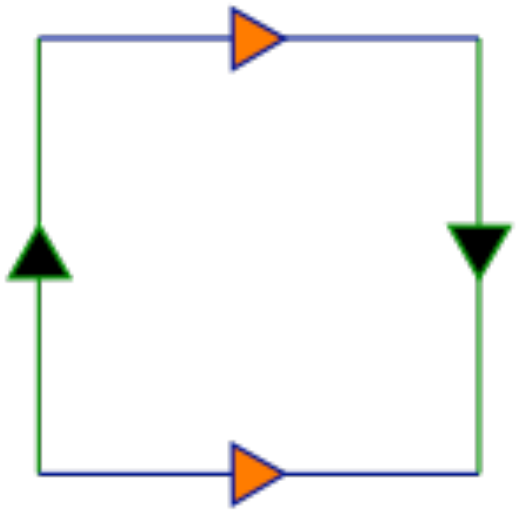
Topology studies shapes

Torus



Topology studies shapes

Klein bottle



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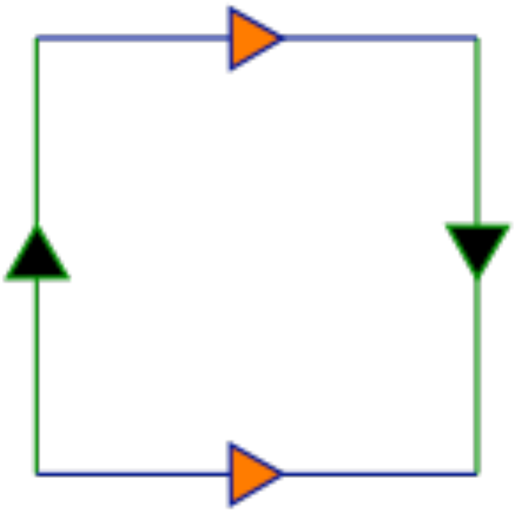

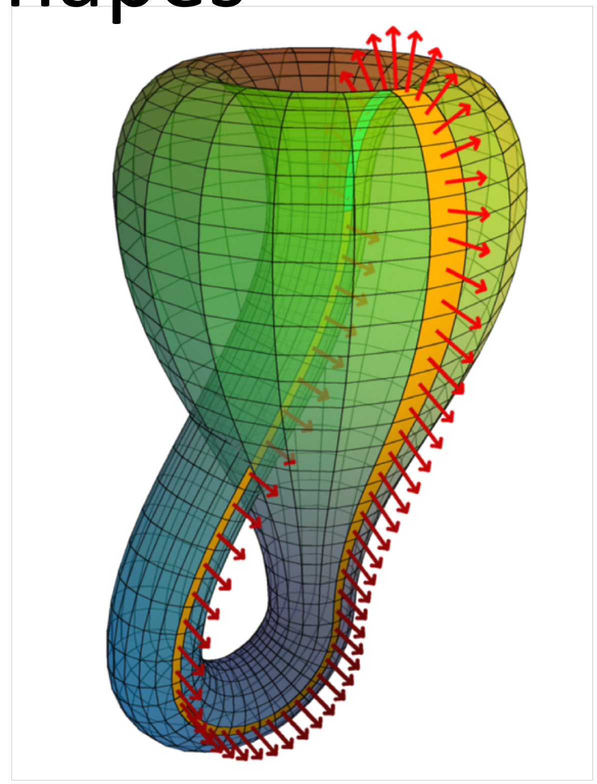

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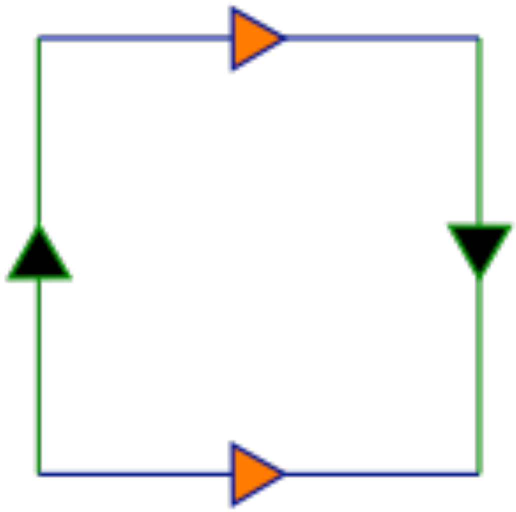

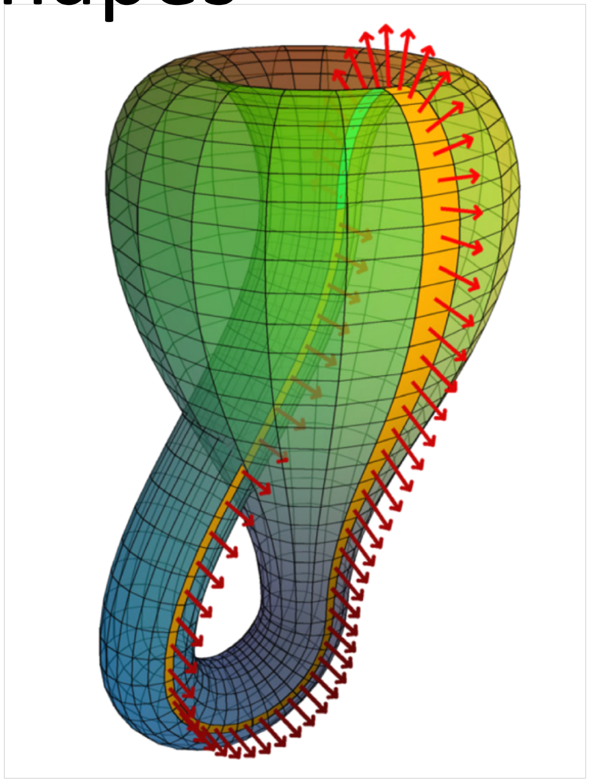

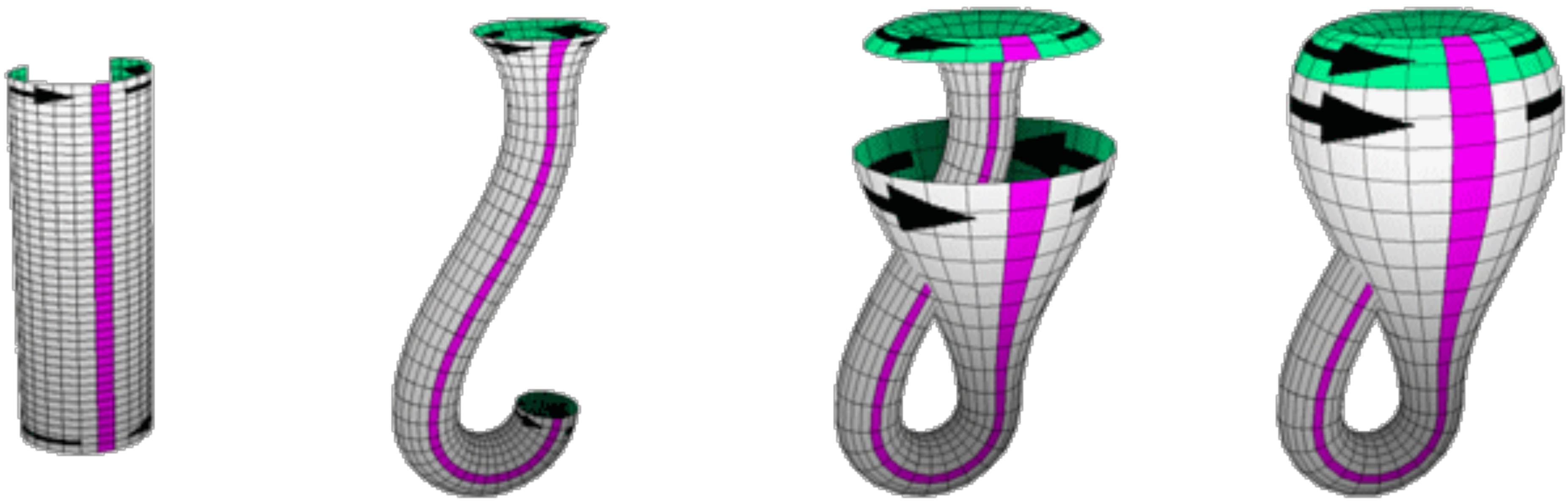

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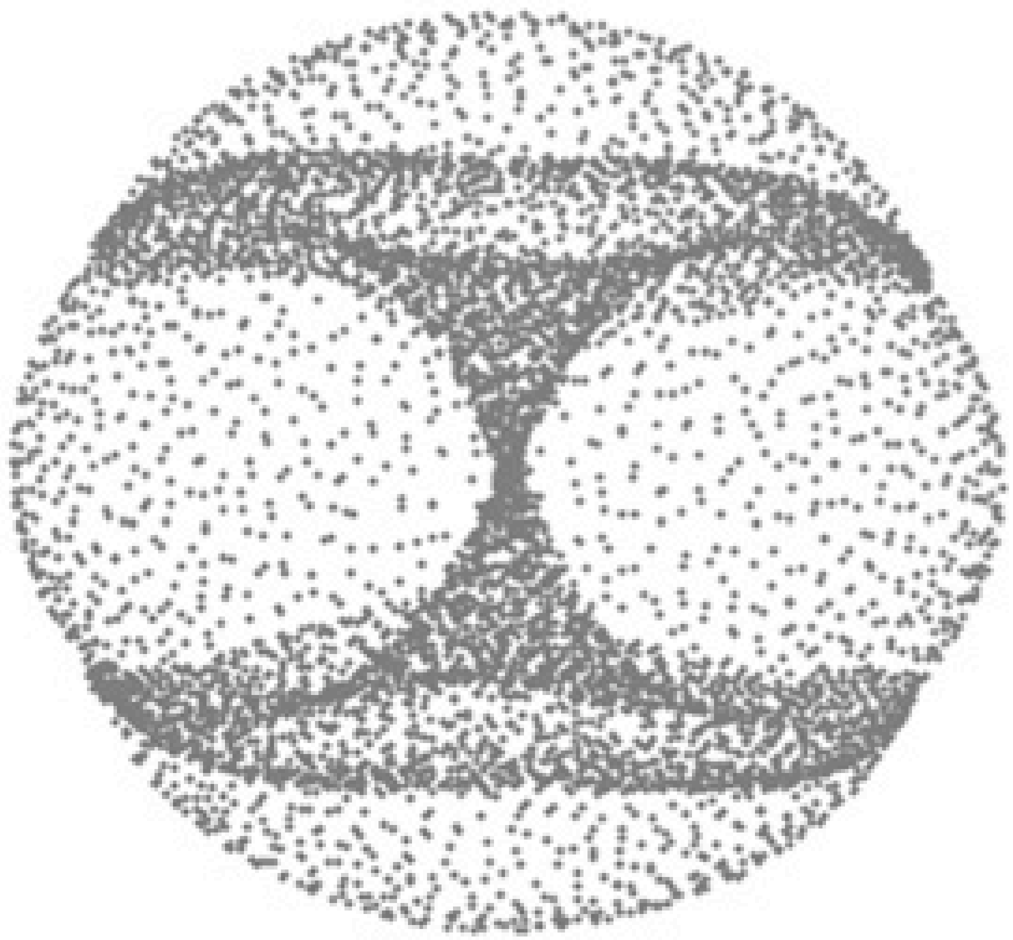
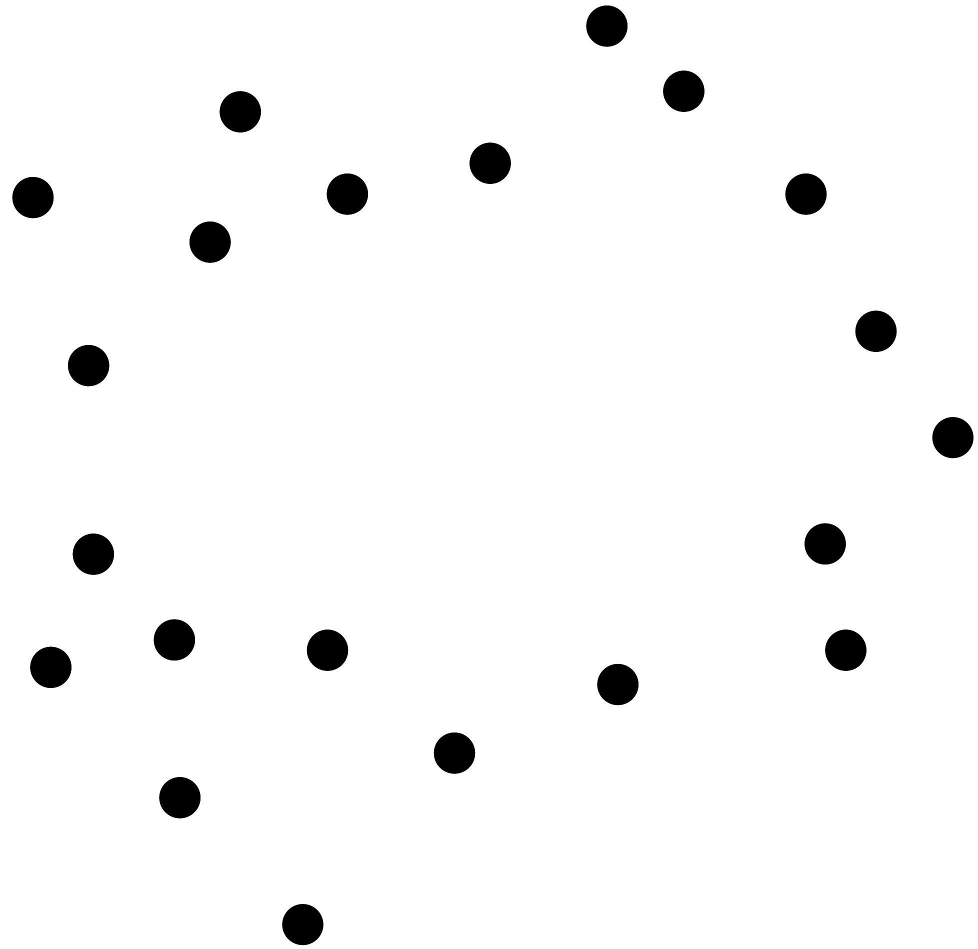
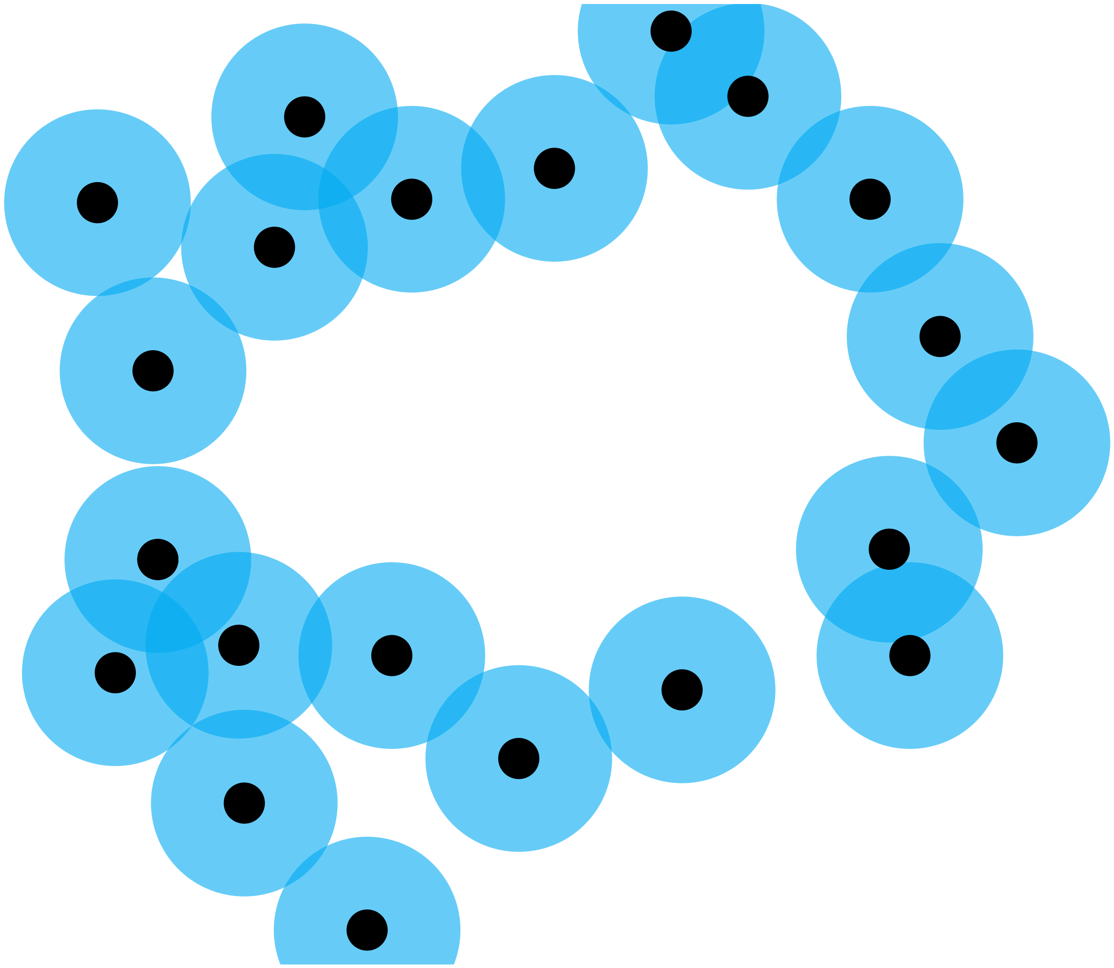
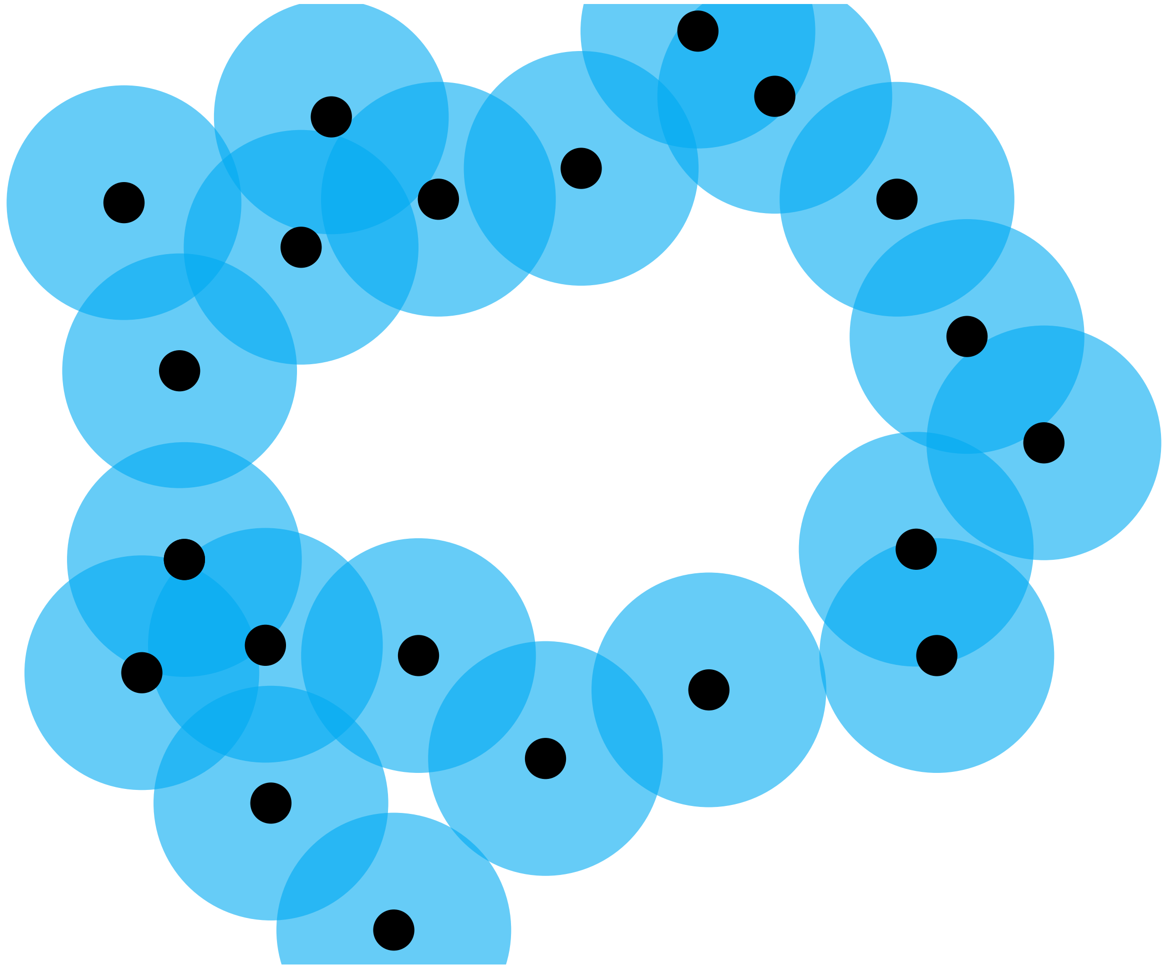
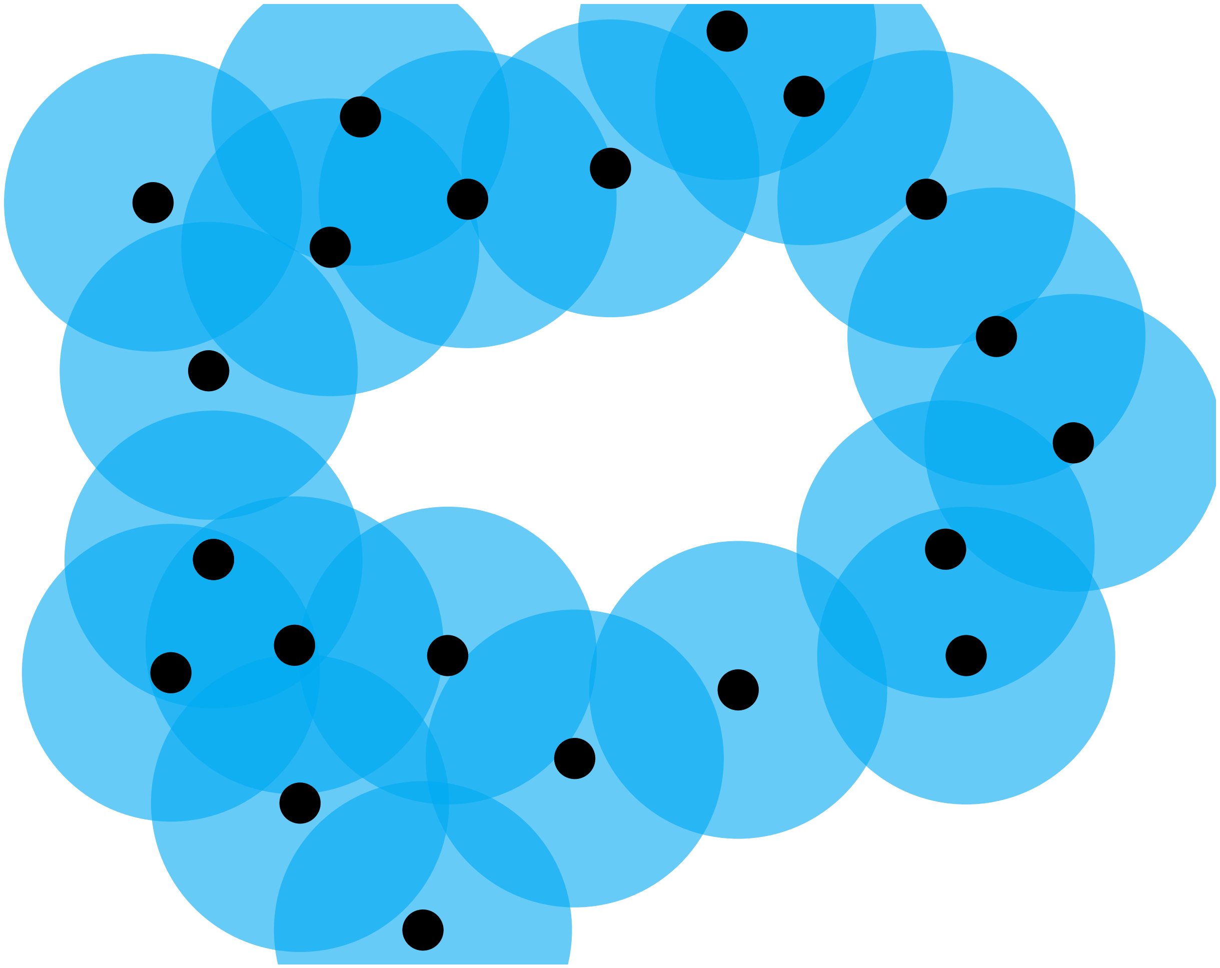
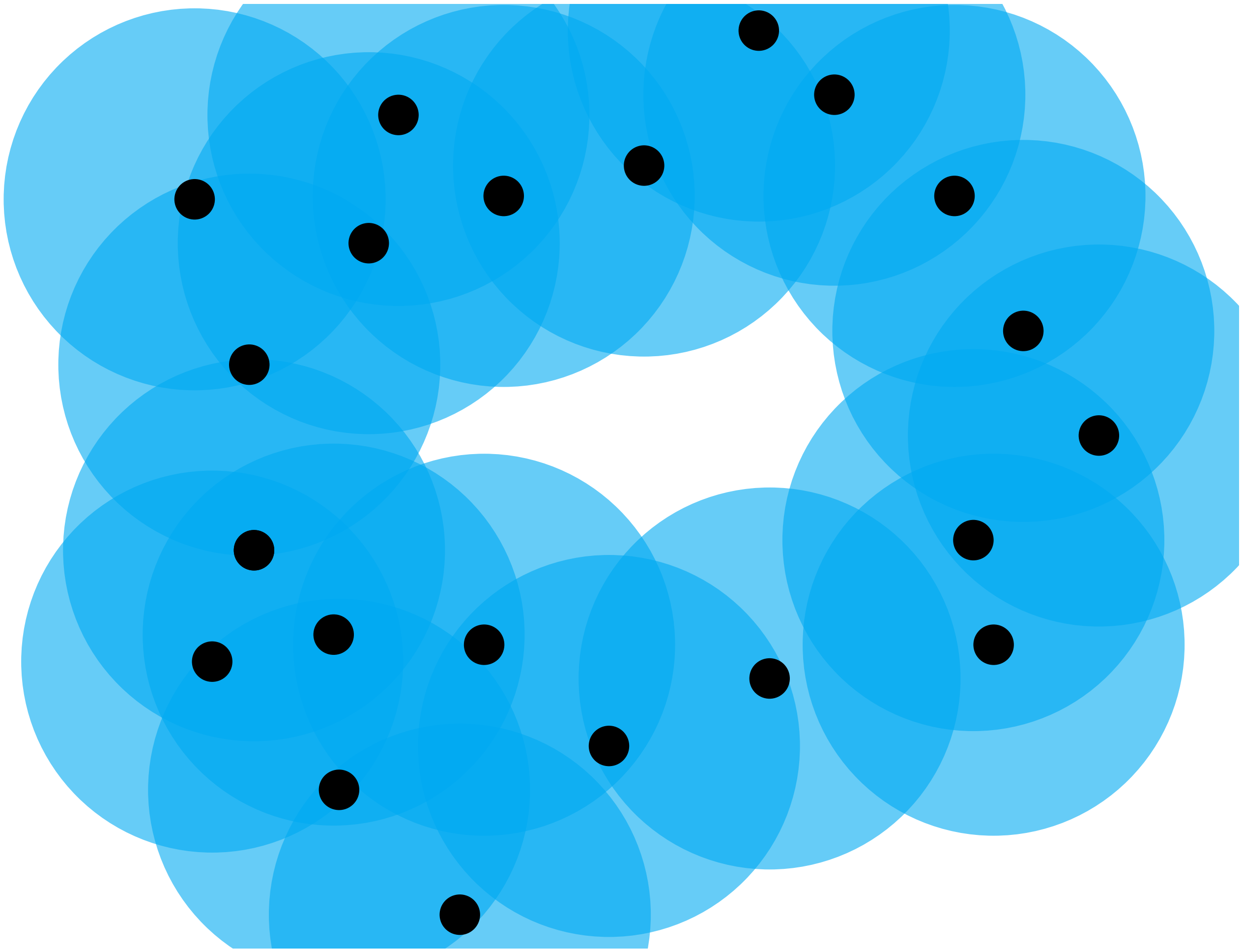


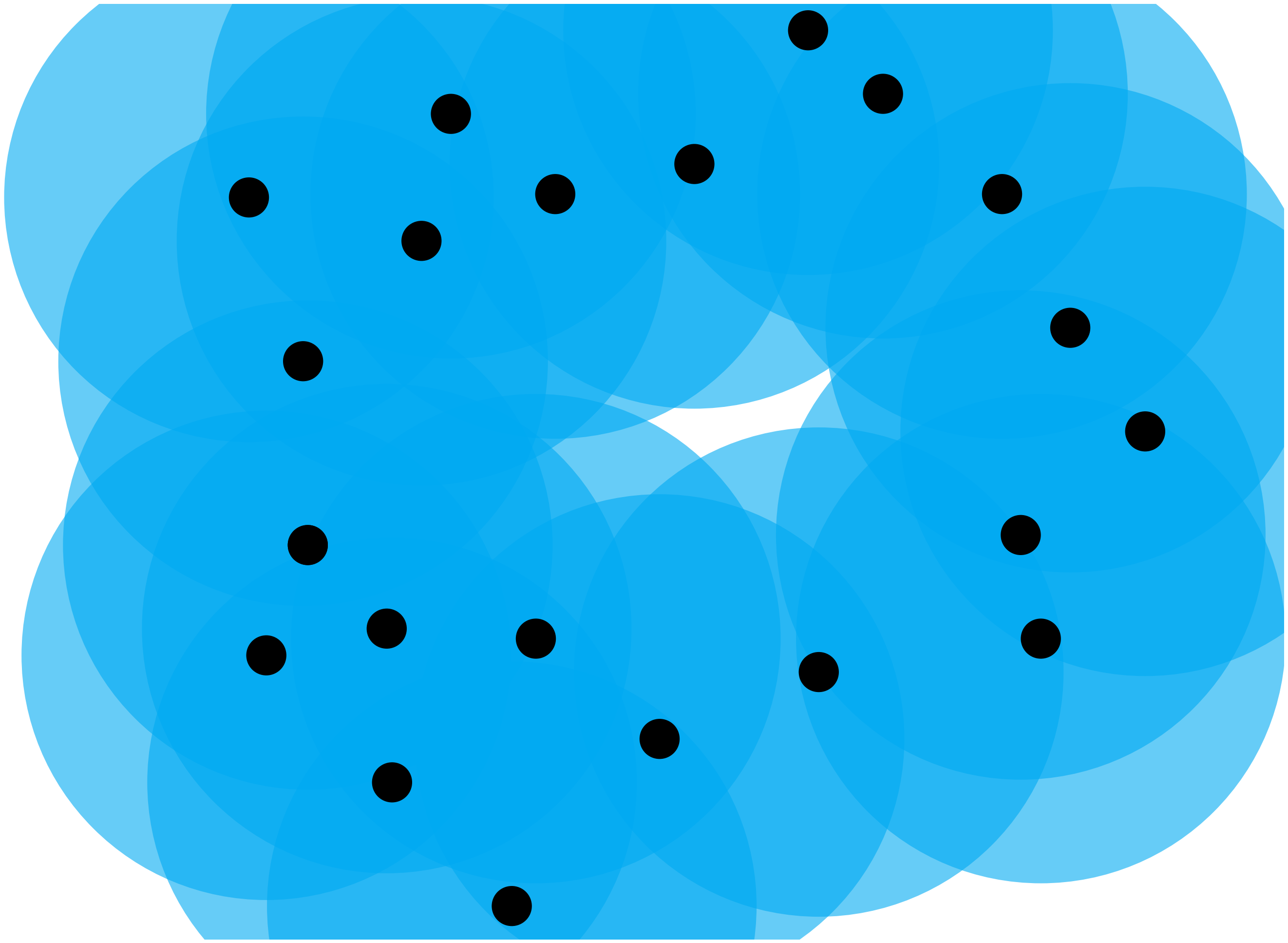


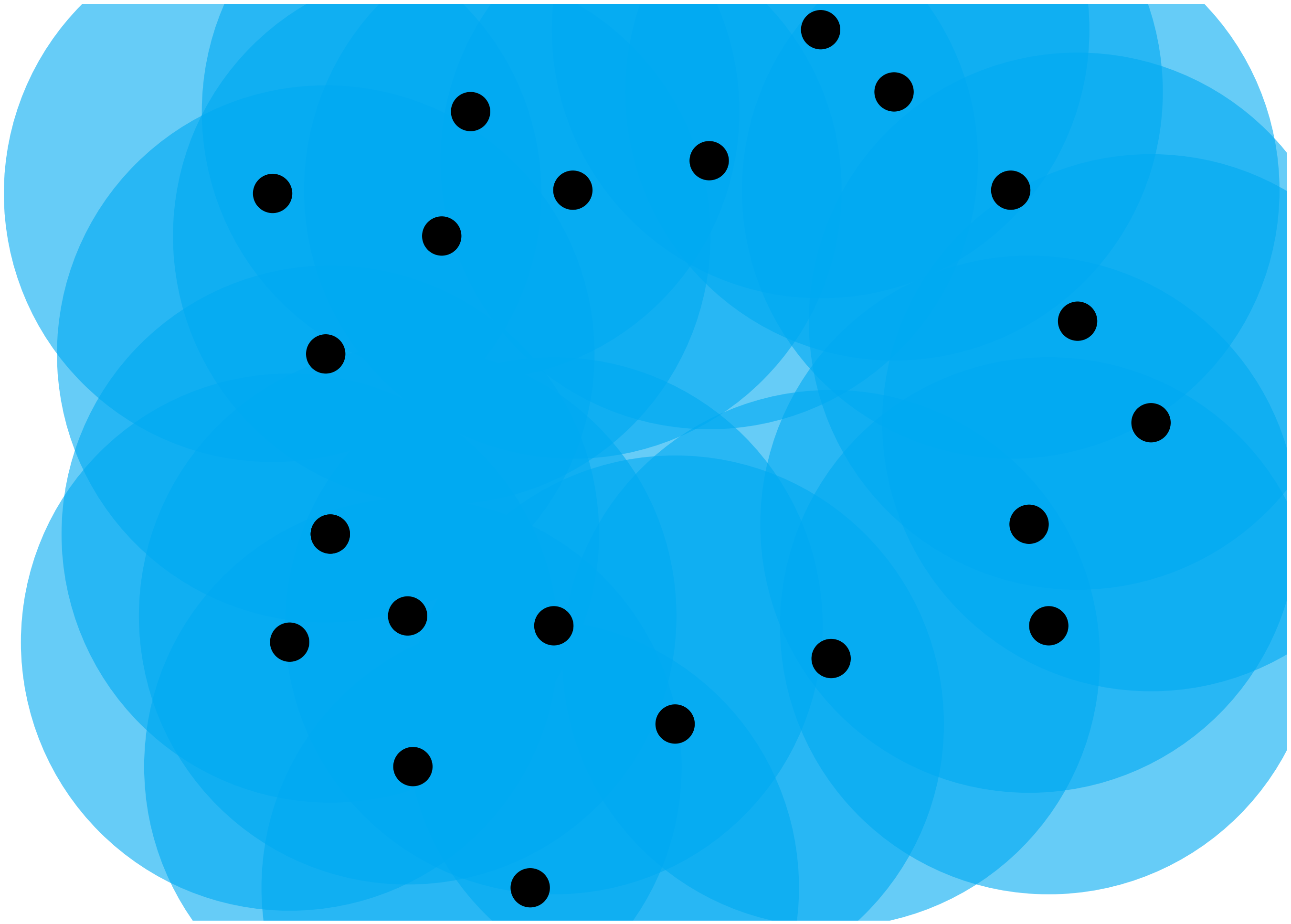


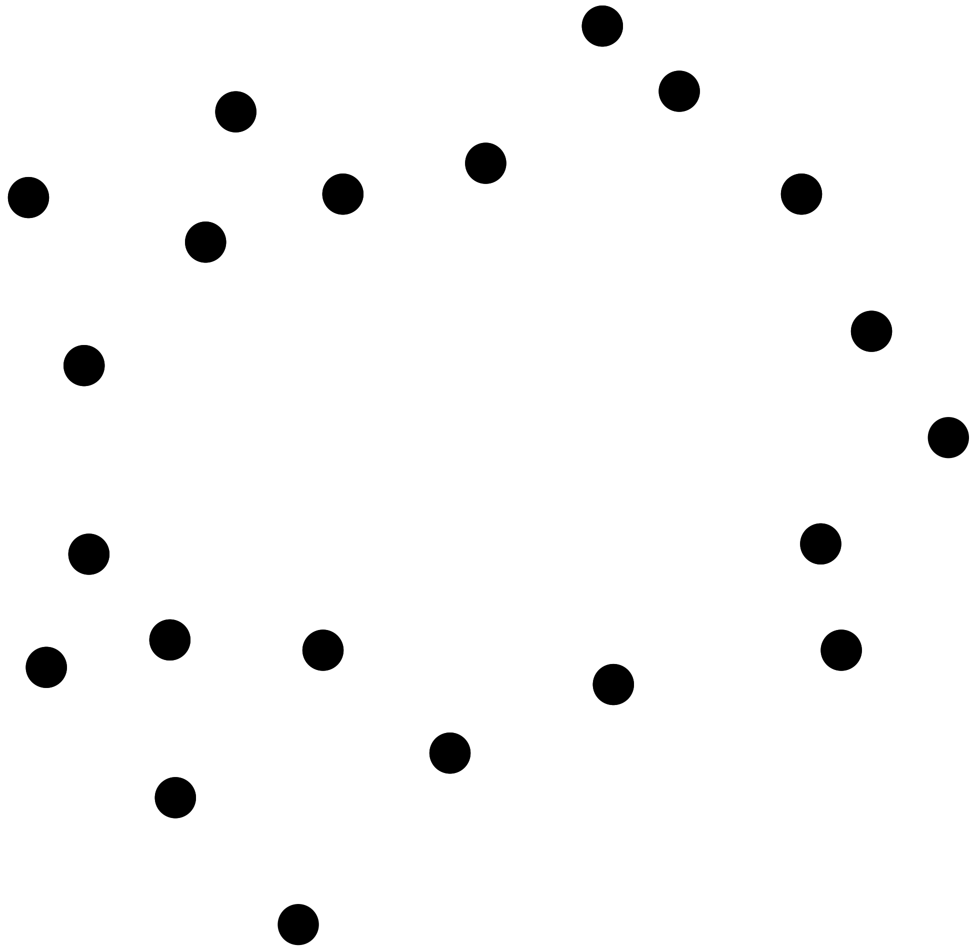








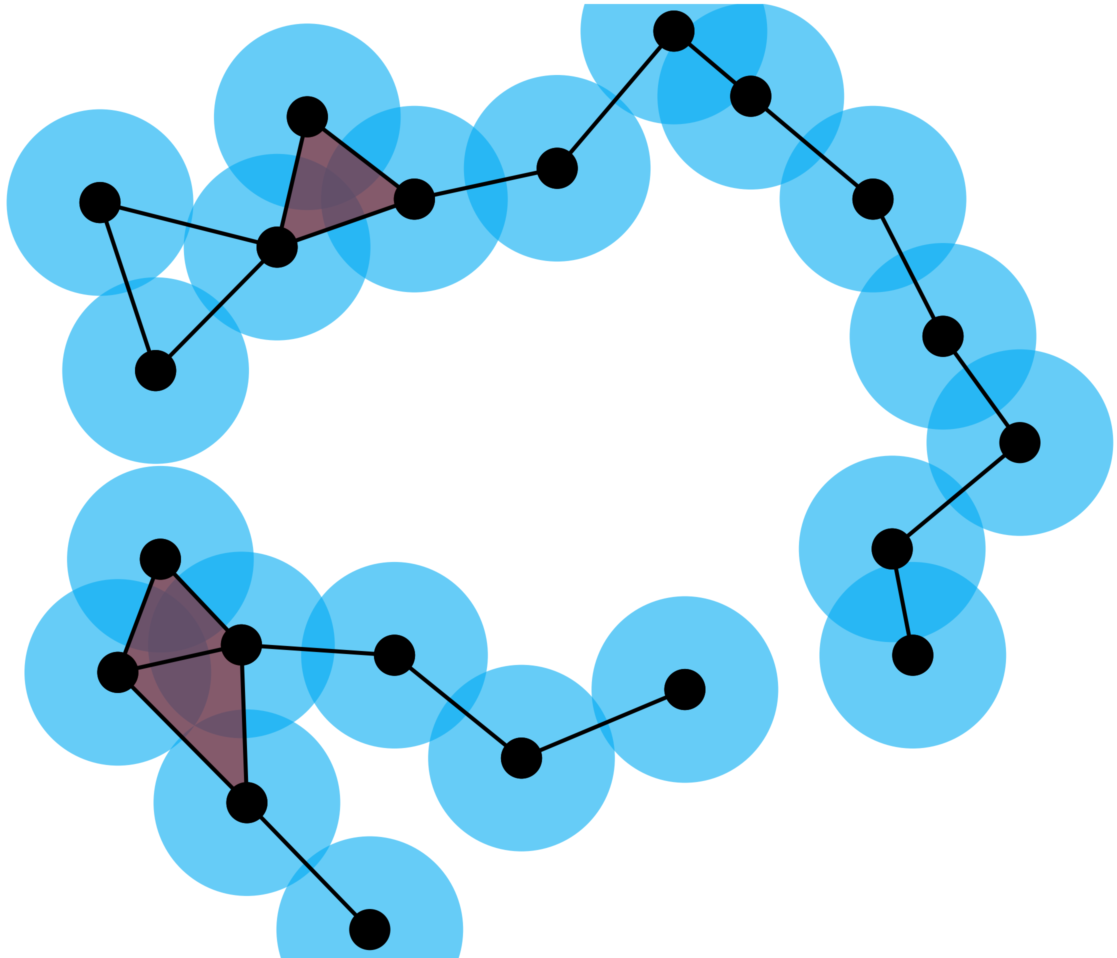




Definition

For metric spaces $X \subseteq Y$ and scale $r \geq 0$, the *Čech simplicial complex* $\check{\mathrm{C}}(X; r)$ has

- vertex set X
- finite simplex $[x_0, x_1, \ldots, x_k]$ when $\cap_{i=0}^{k} B_Y(x_i, \frac{r}{2}) \neq \emptyset$.



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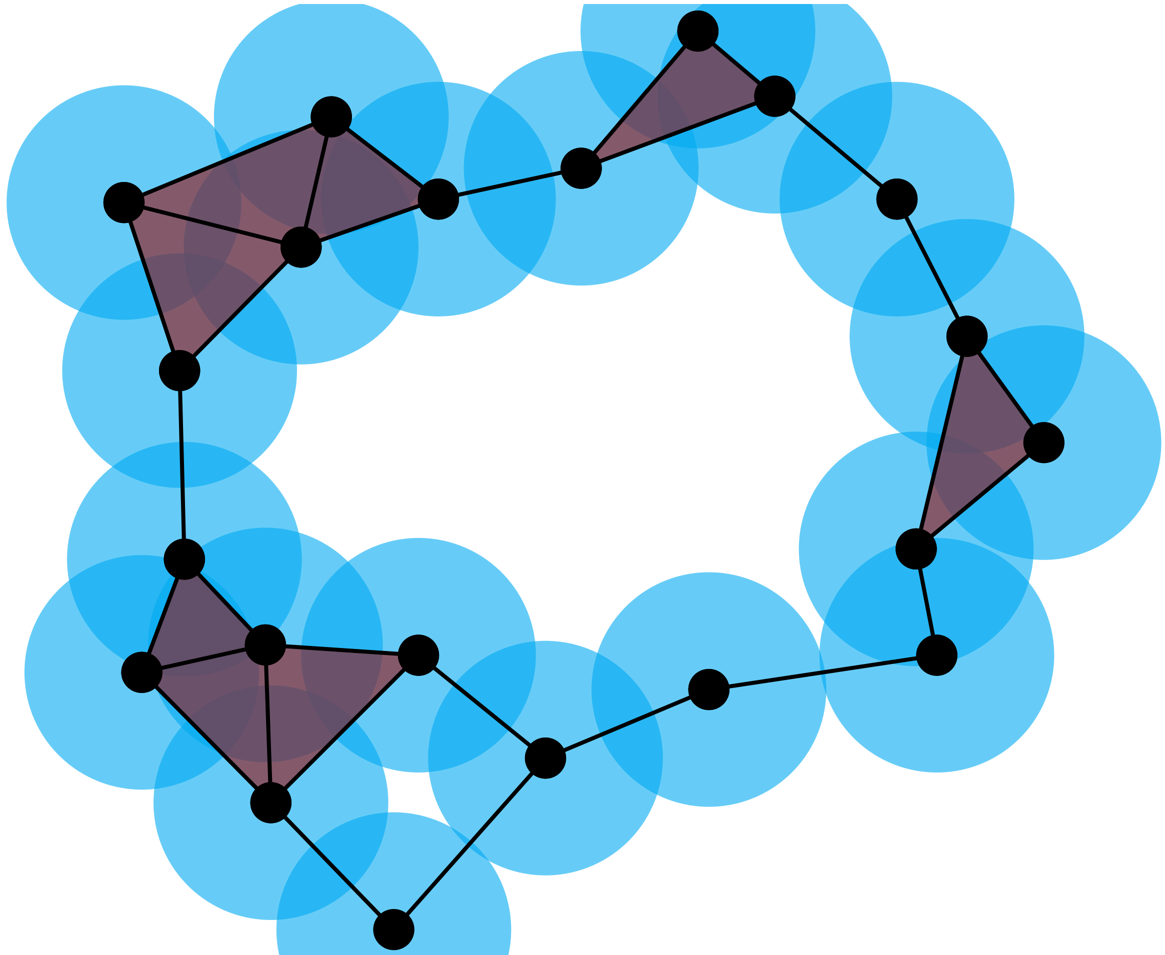

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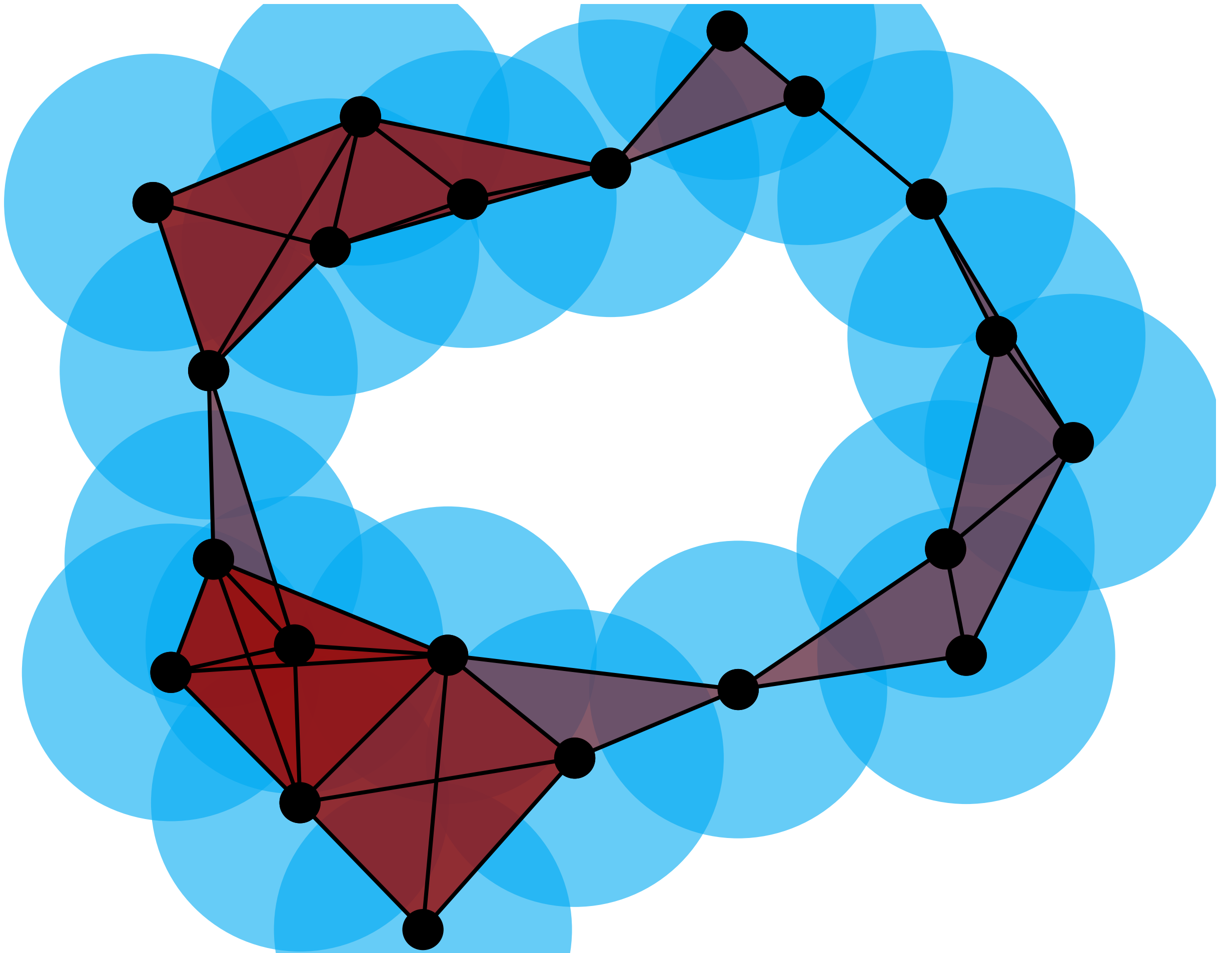

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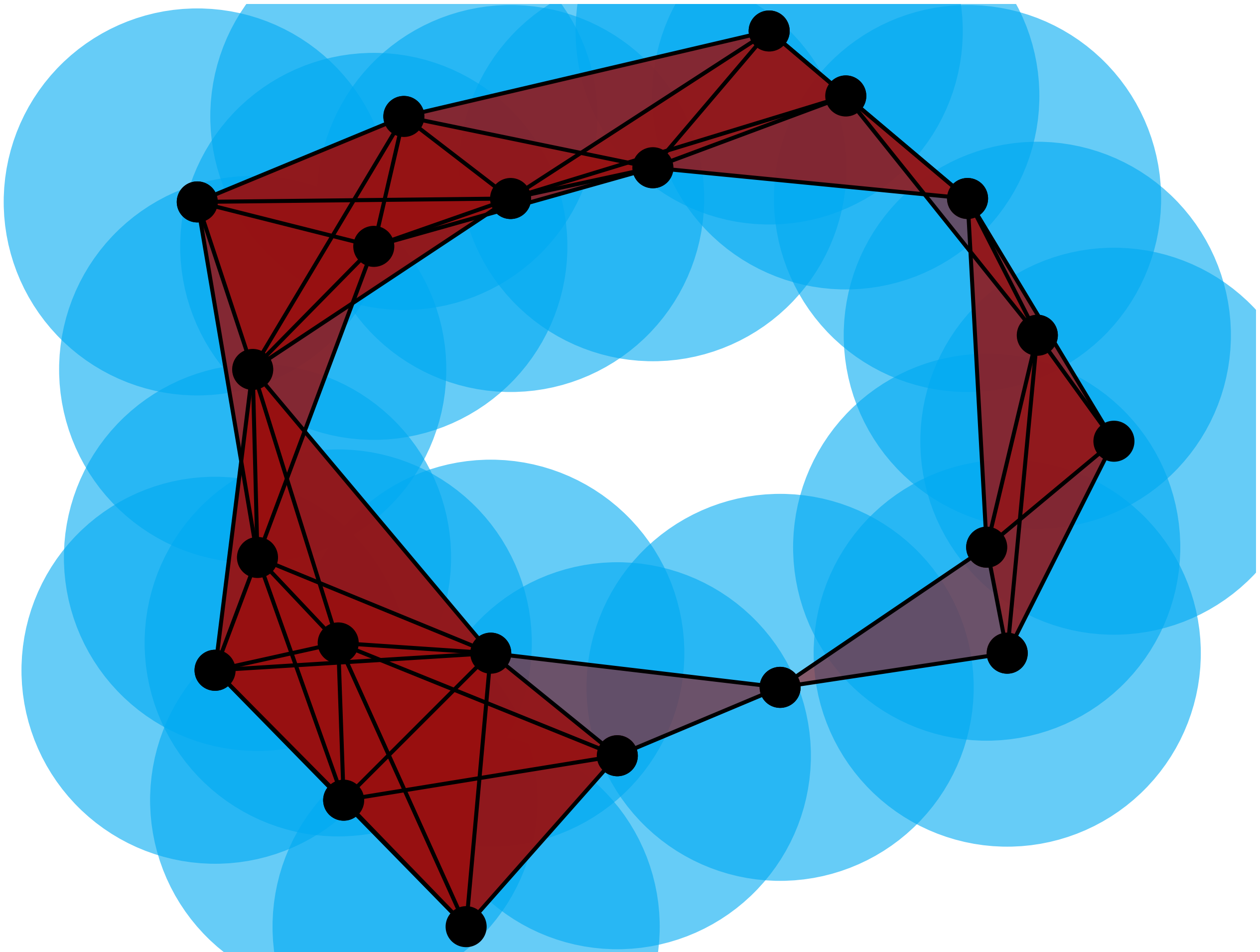

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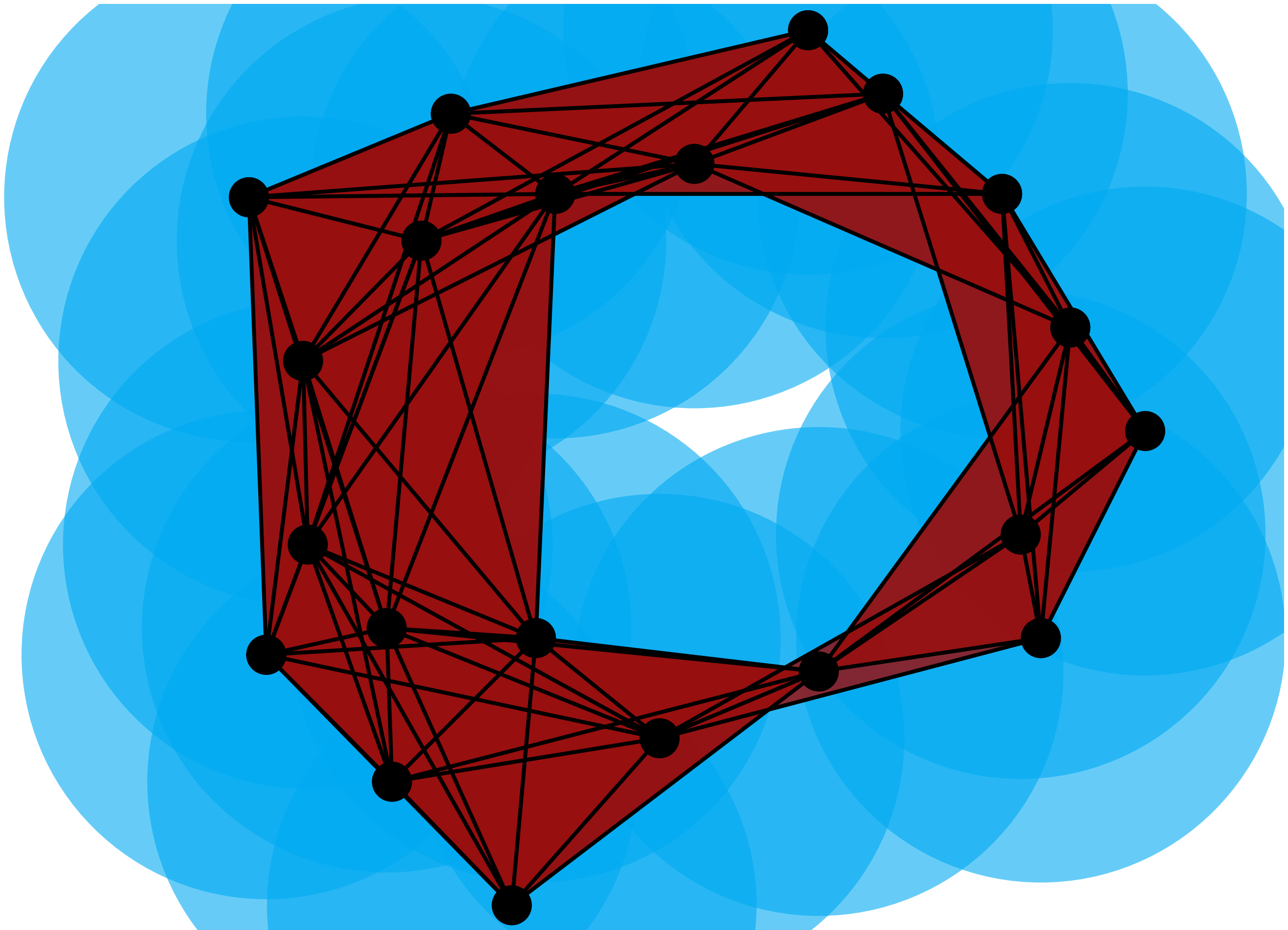

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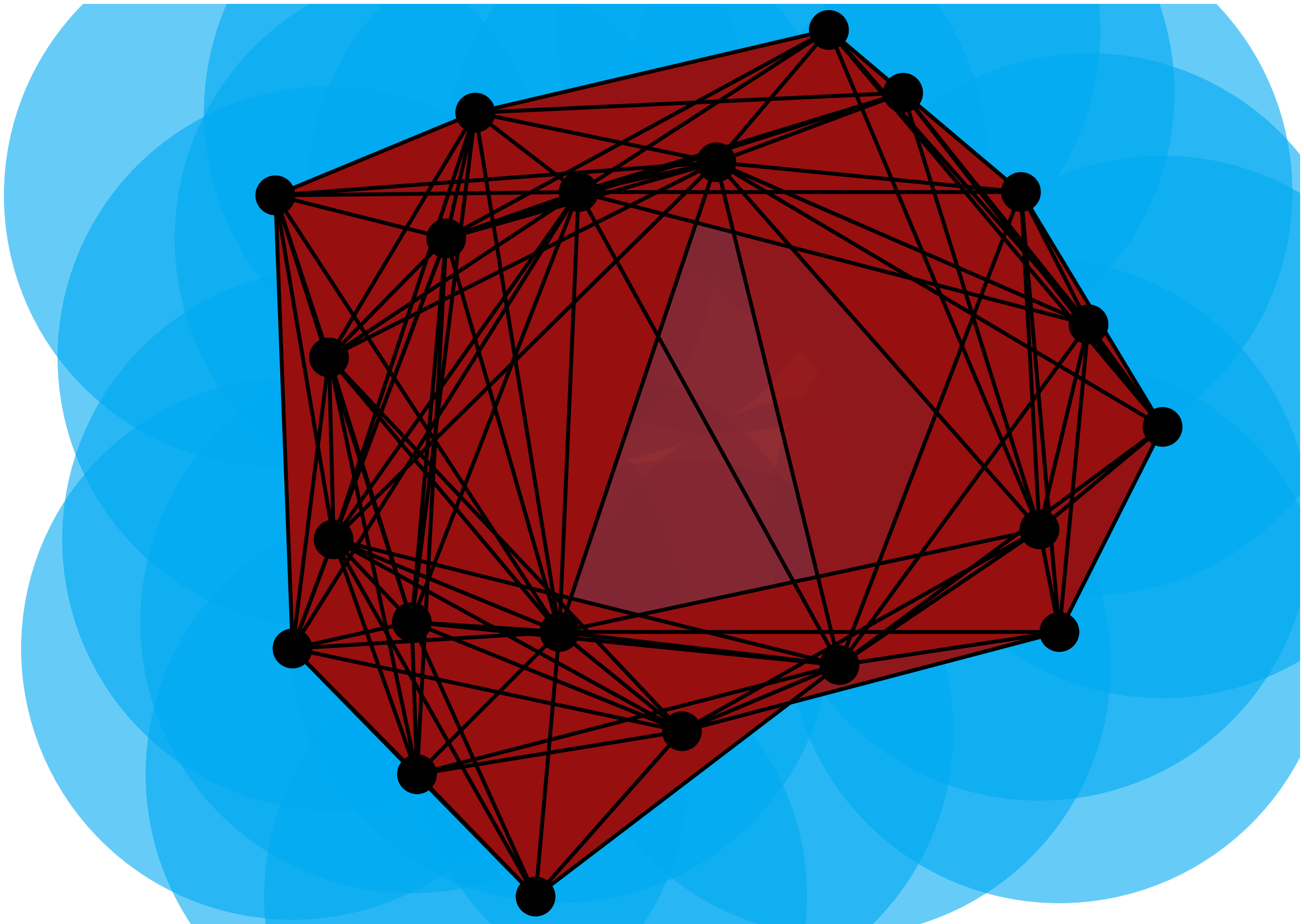

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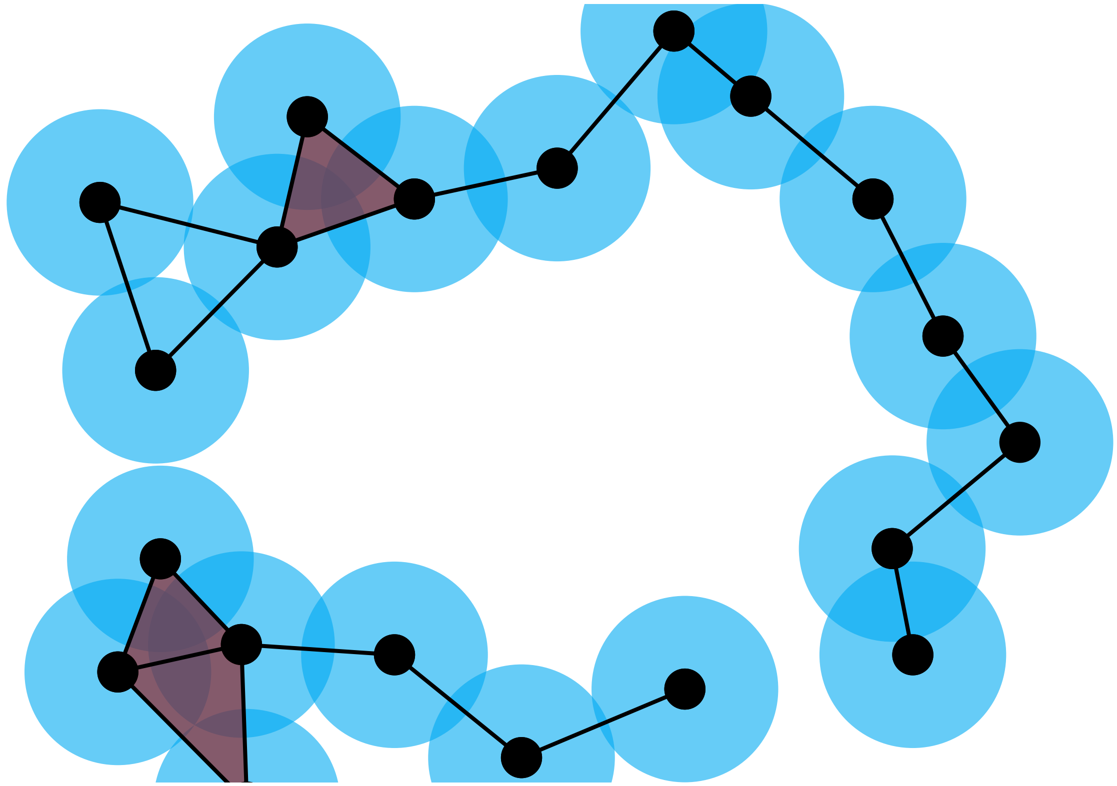

Nerve Lemma. $\check{C}(X;r) \simeq \text{union of balls}$

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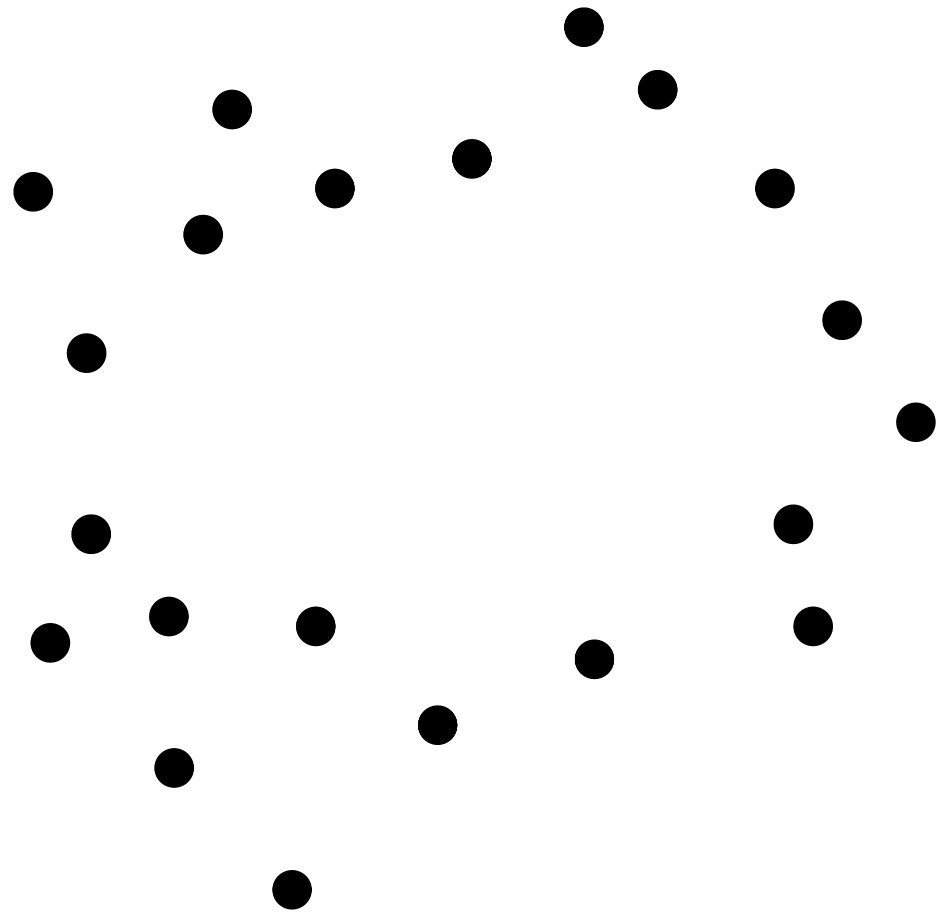

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For metric space X and scale $r \geq 0$, the *Vietoris–Rips simplicial complex* $\mathrm{VR}(X; r)$ has

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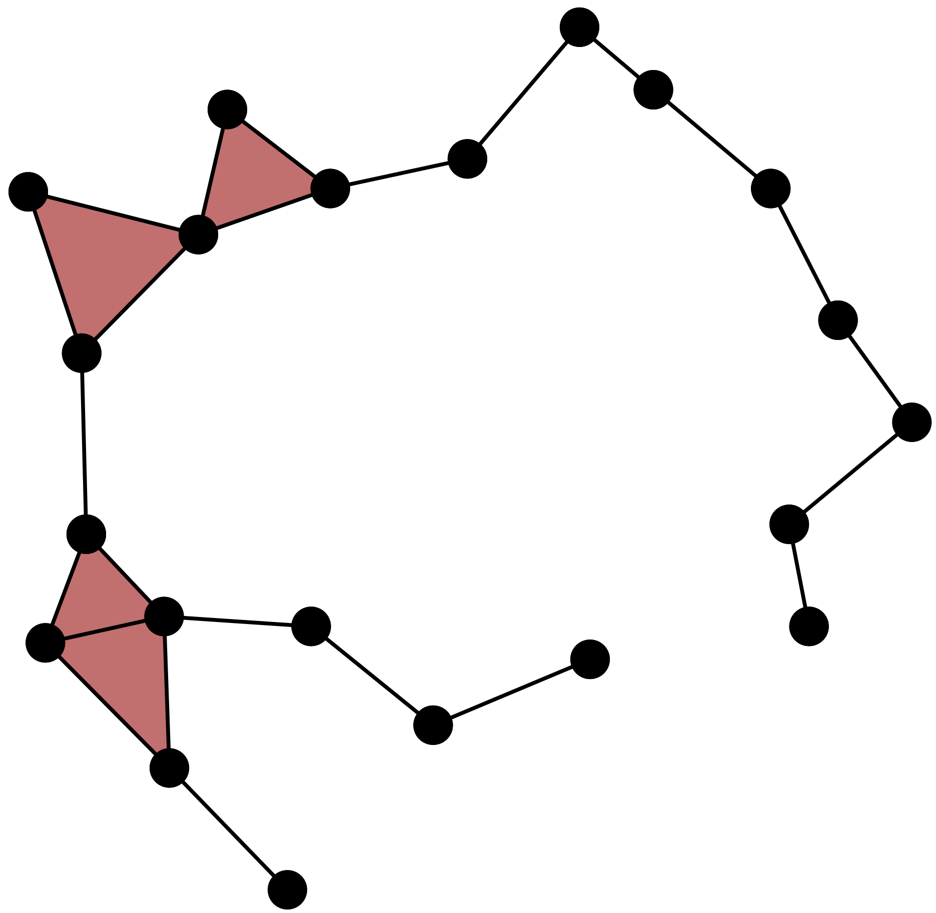

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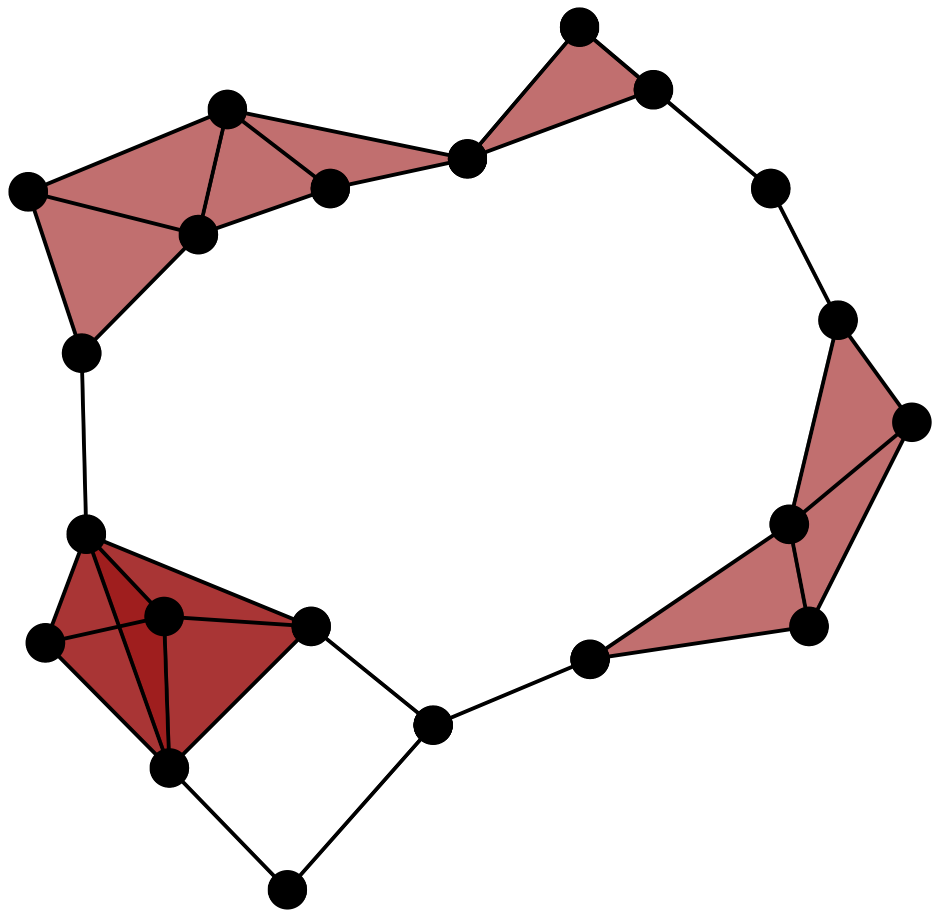

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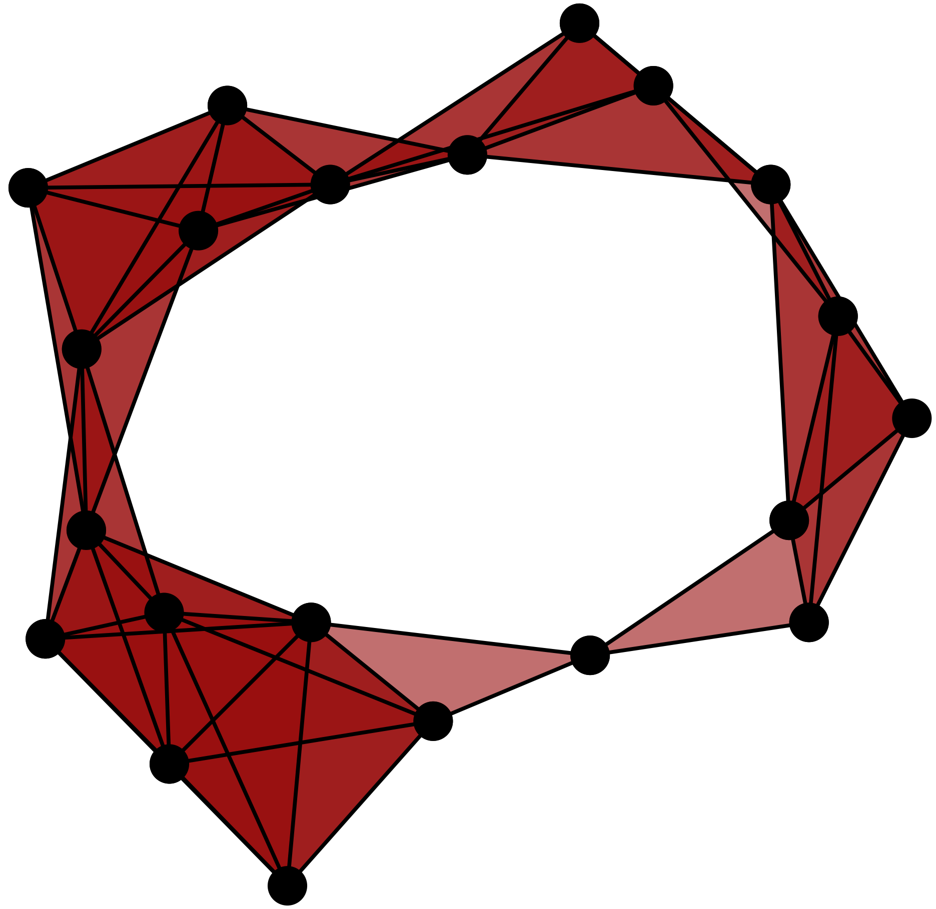

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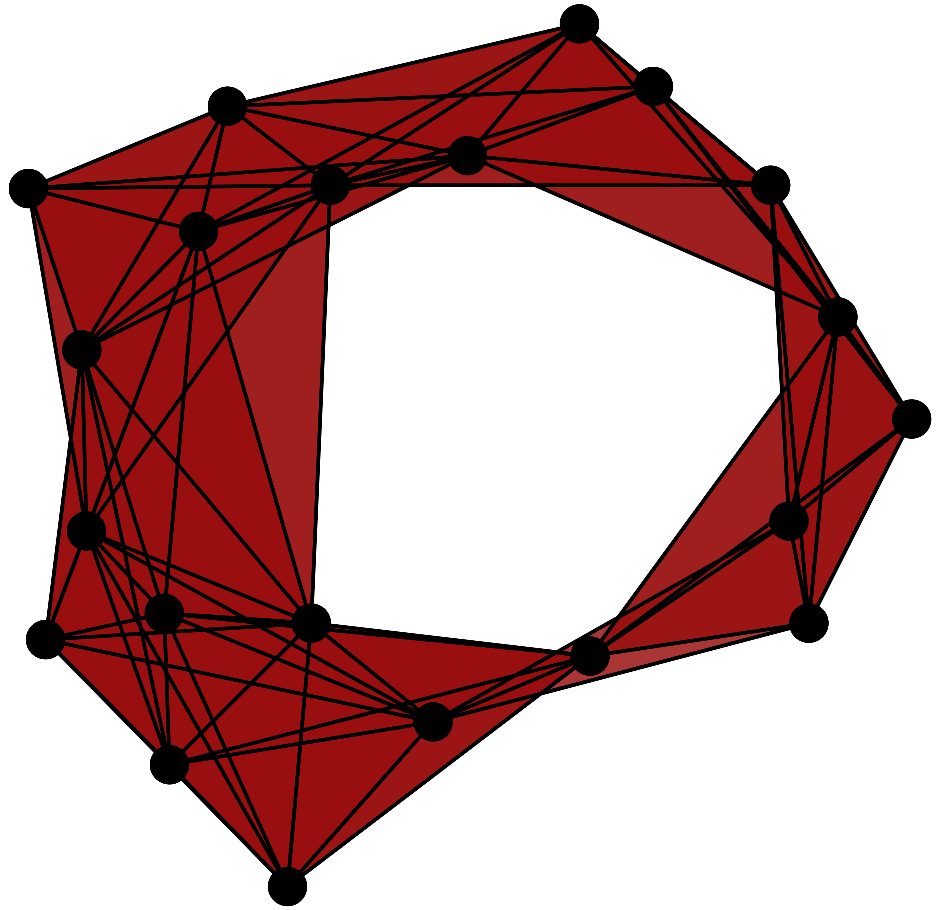

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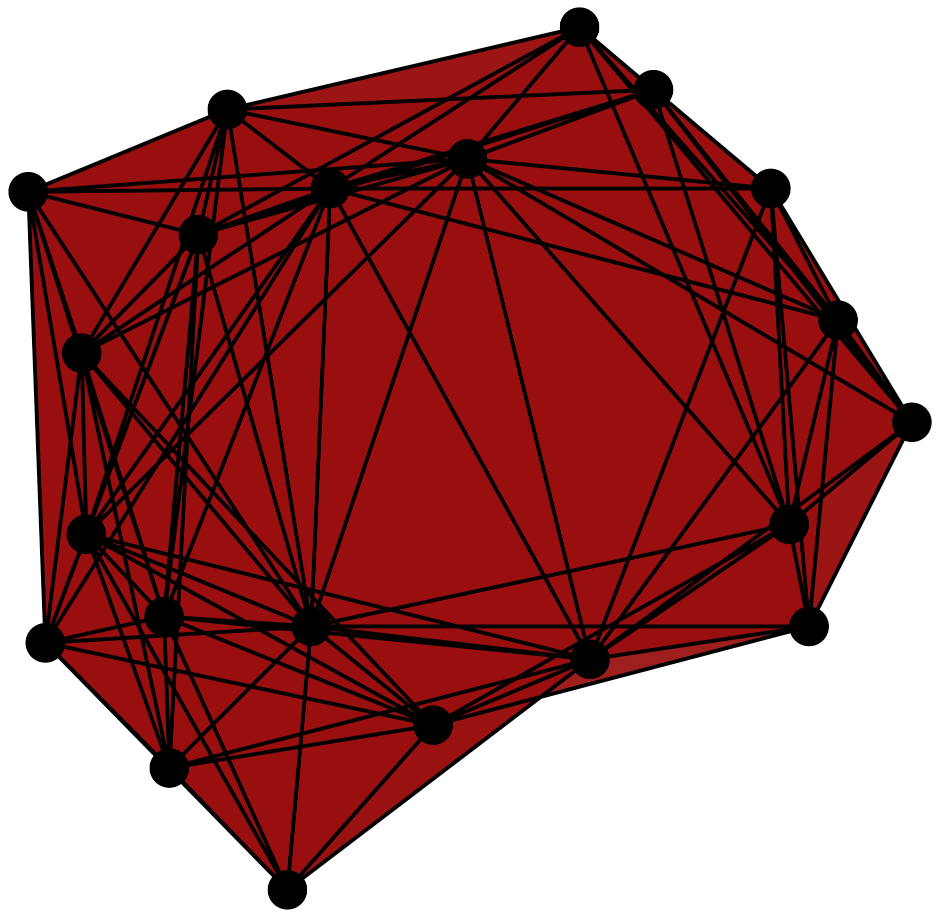

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For metric space X and scale $r \geq 0$, the *Vietoris–Rips simplicial complex* $\mathrm{VR}(X; r)$ has

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Vietoris-Rips is a controlled approximation:

$$\check{C}(X;r) \subseteq \mathrm{VR}(X;r) \subseteq \check{C}(X;2r)$$

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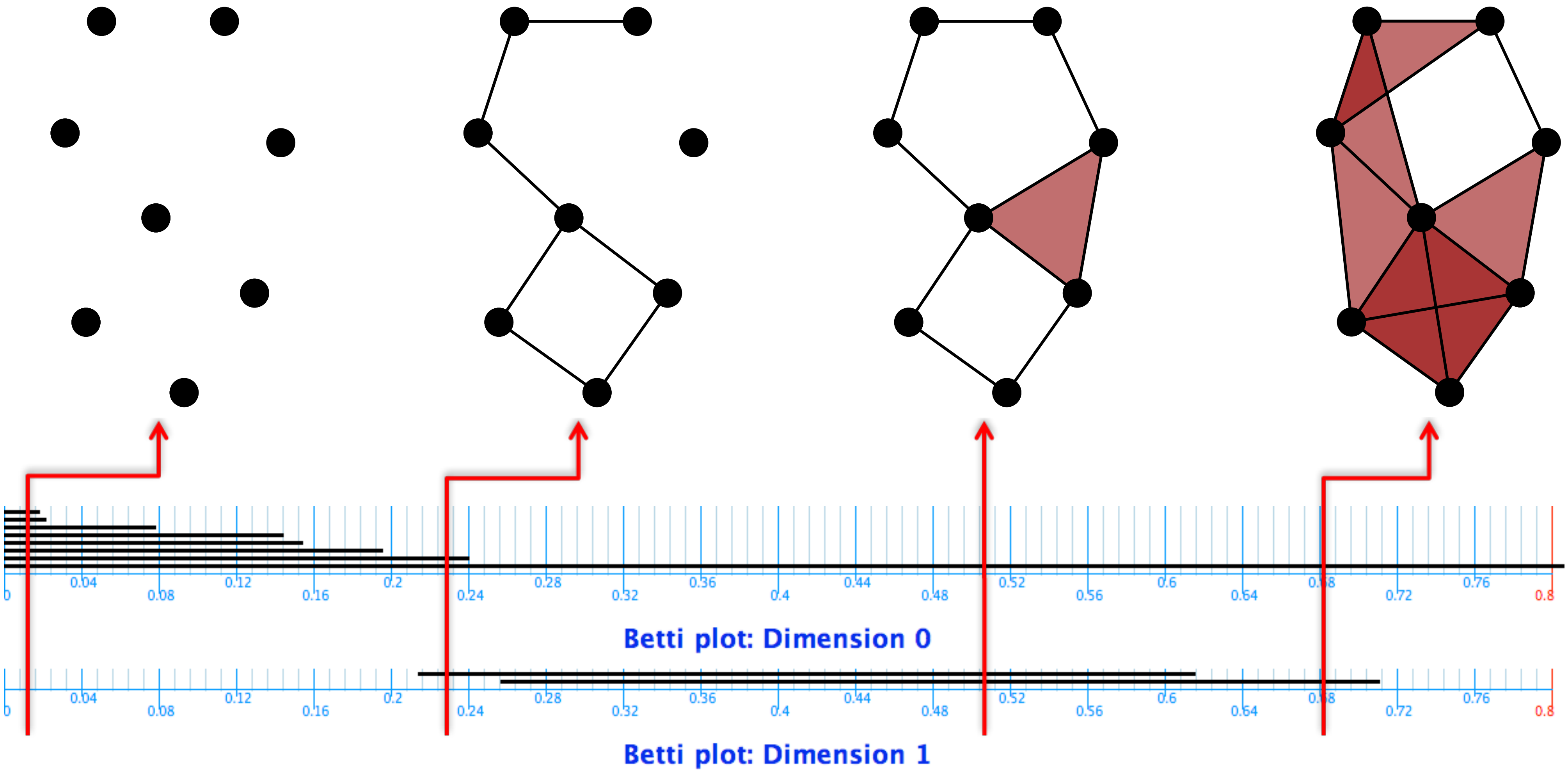
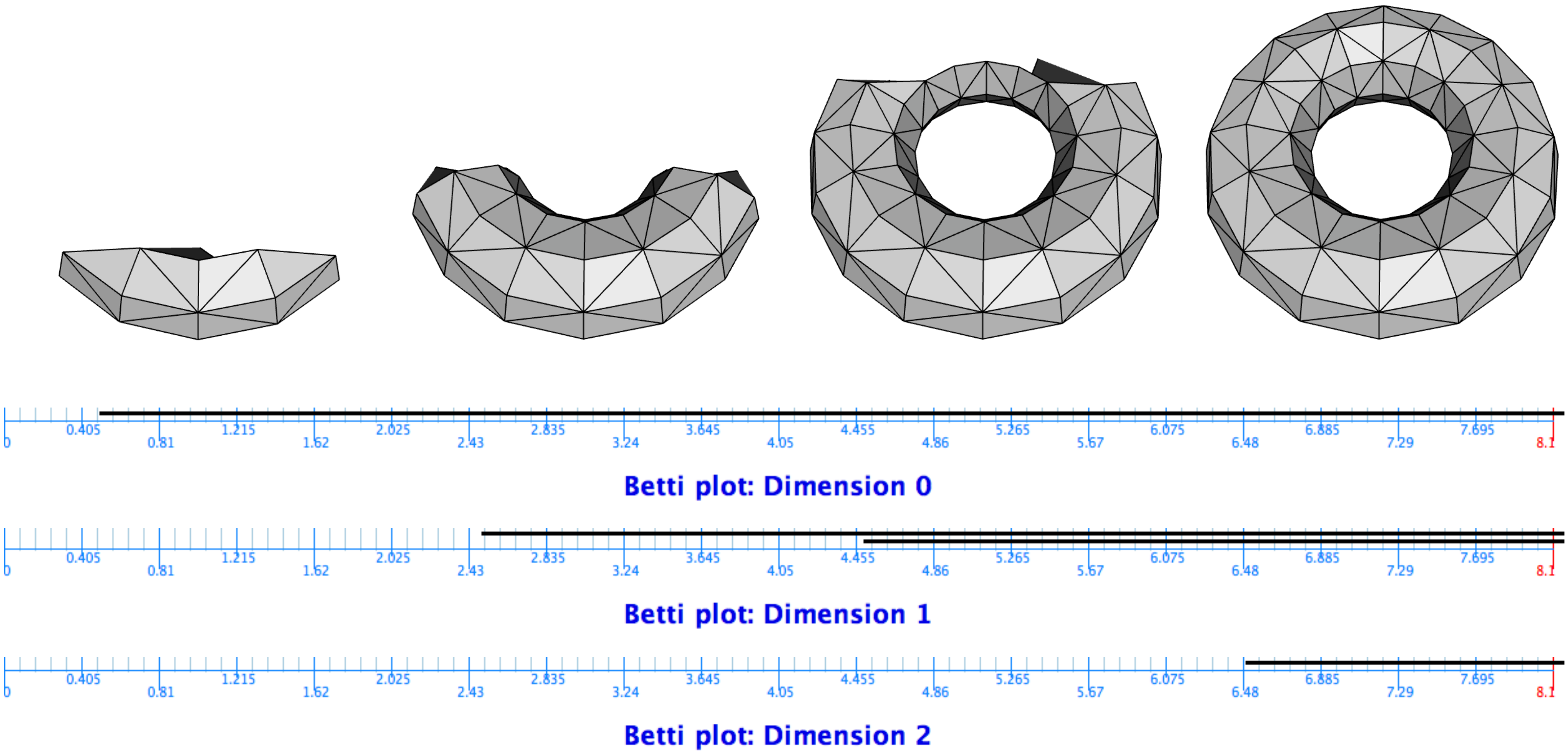
Persistent homology



- Input: filtered topological space. Output: barcode.
- Significant features persist.
- Cubic computation time in the number of simplices.

Persistent homology



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Persistent homology

- Definition. A *persistence module* over diagram

$$\bullet \to \bullet \to \ldots \to \bullet \to \bullet$$

has a vector space at each vertex and a linear map at each edge.

- Example.

$$\bullet \to \bullet \to \bullet \to \bullet \to \bullet$$

$$V_1 \to V_2 \to V_3 \to V_4 \to V_5$$

Zigzag Persistence by G. Carlsson and V. de Silva

Zigzag Persistent homology

- <u>Definition.</u> A *zigzag persistence module* over diagram

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- <u>Example.</u>

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$$\bigoplus$$

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- <u>Theorem (Gabriel).</u> Indecomposables classified by intervals.

$$0 \longleftrightarrow \ldots \longleftrightarrow 0 \longleftrightarrow \overline{k \overset{id}{\longleftrightarrow} \ldots \overset{id}{\longleftrightarrow} k} \longleftrightarrow 0 \longleftrightarrow \ldots \longleftrightarrow 0$$

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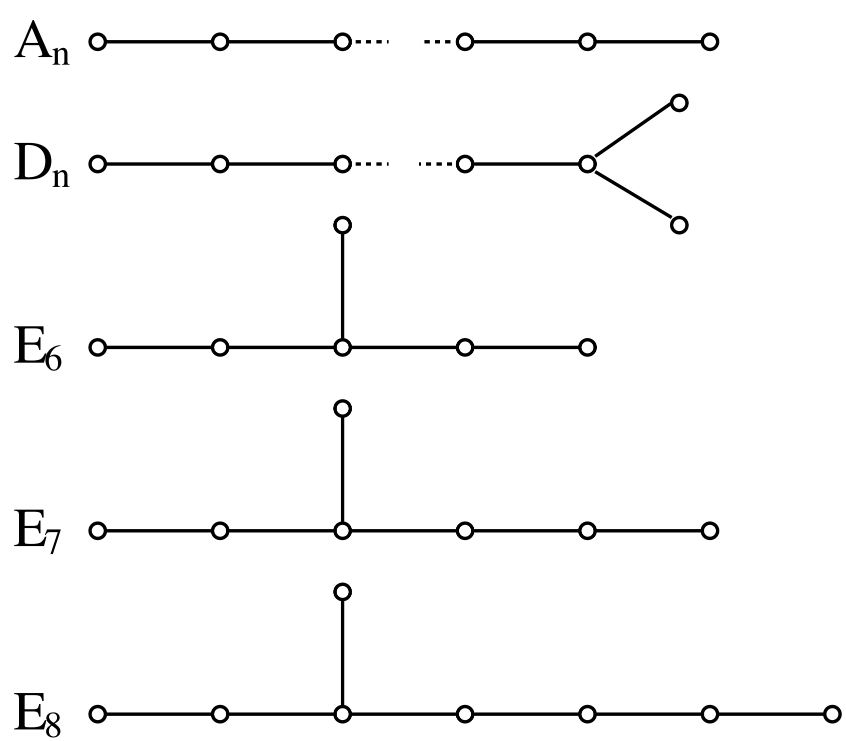
Zigzag Persistent homology

- <u>Theorem (Gabriel).</u>

  A diagram has a finite number of indecomposables $\Longleftrightarrow$ it's a union of certain Dynkin diagrams.



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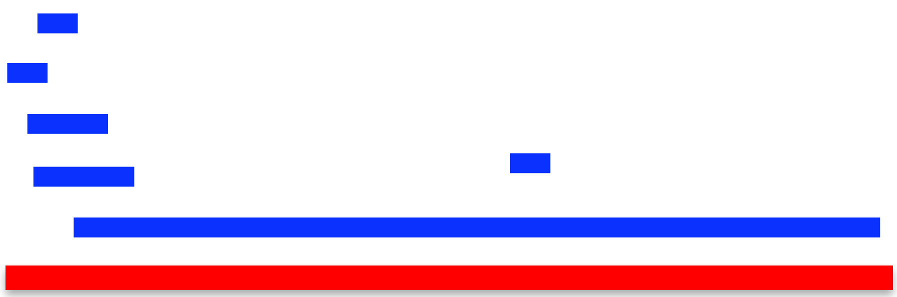
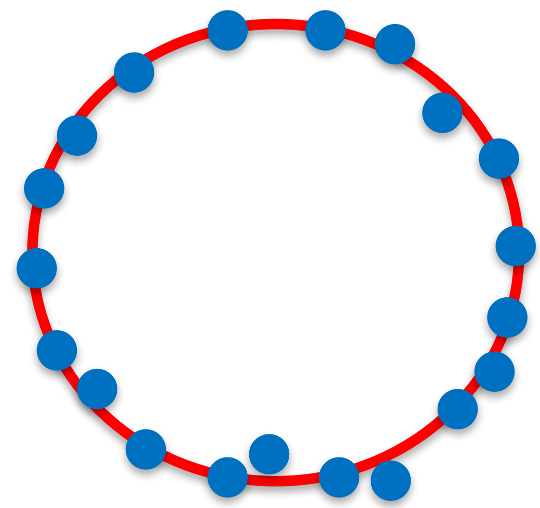
Persistent homology

- <u>Stability Theorem.</u>

  If X and Y are metric spaces, then

$$d_B(\mathrm{PH}(\mathrm{VR}(X)), \mathrm{PH}(\mathrm{VR}(Y))) \leq 2 d_{GH}(X, Y)$$





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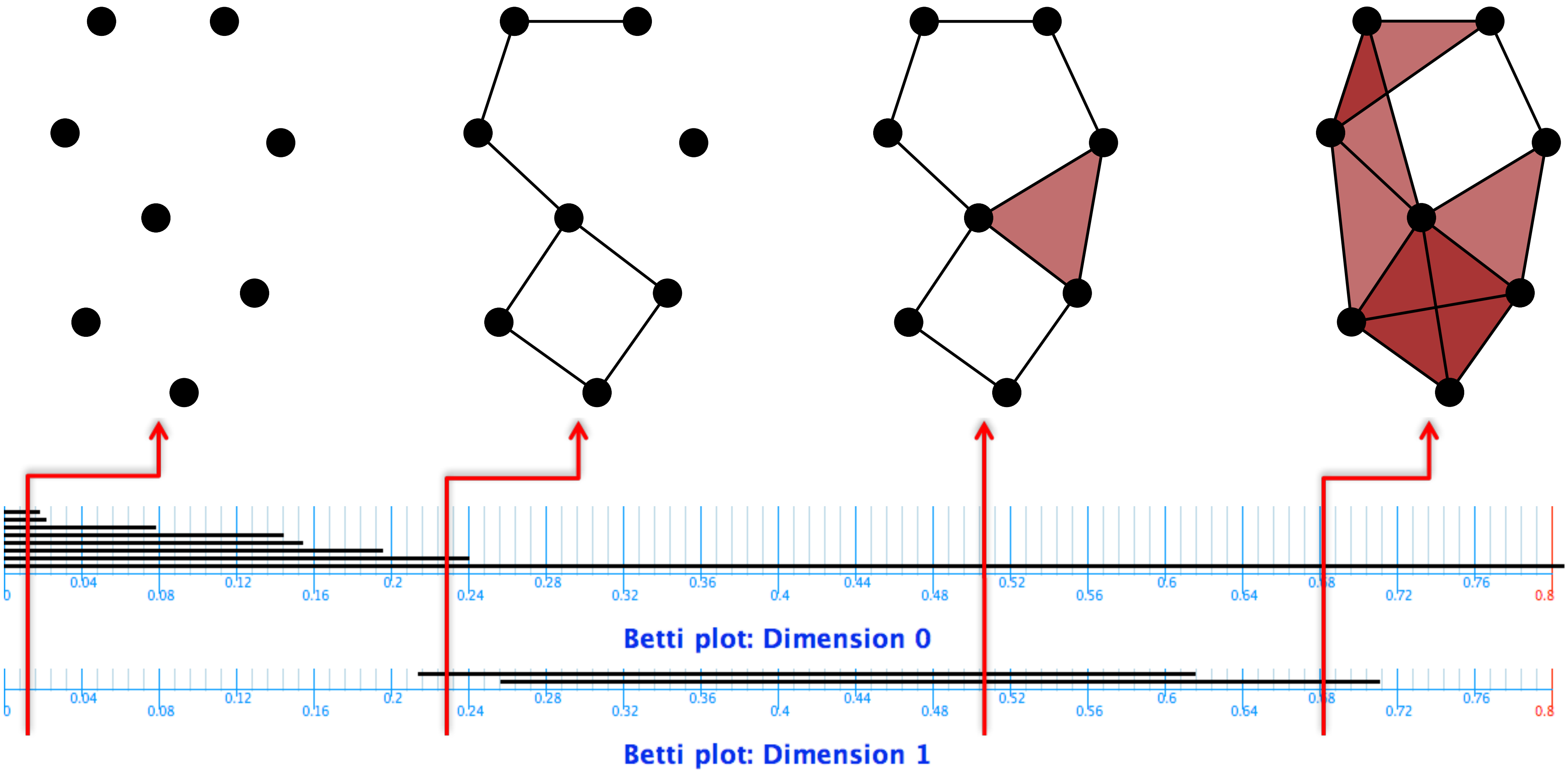

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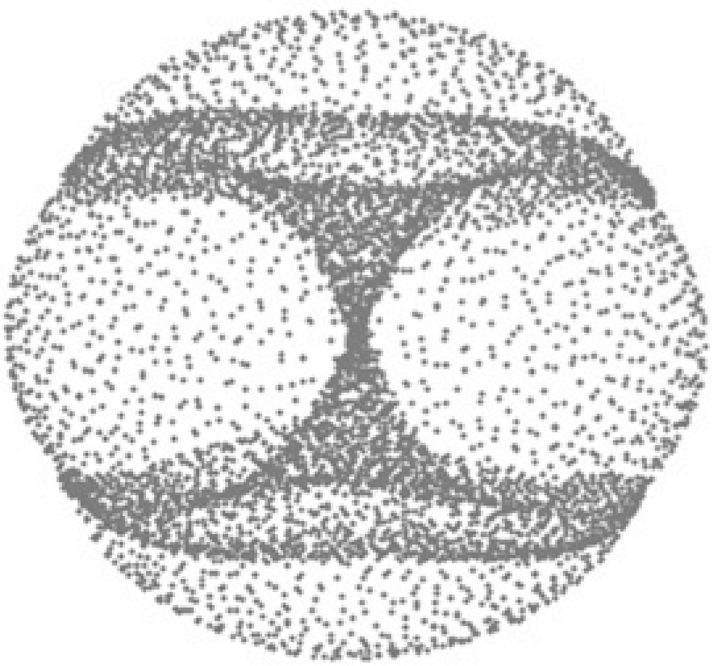
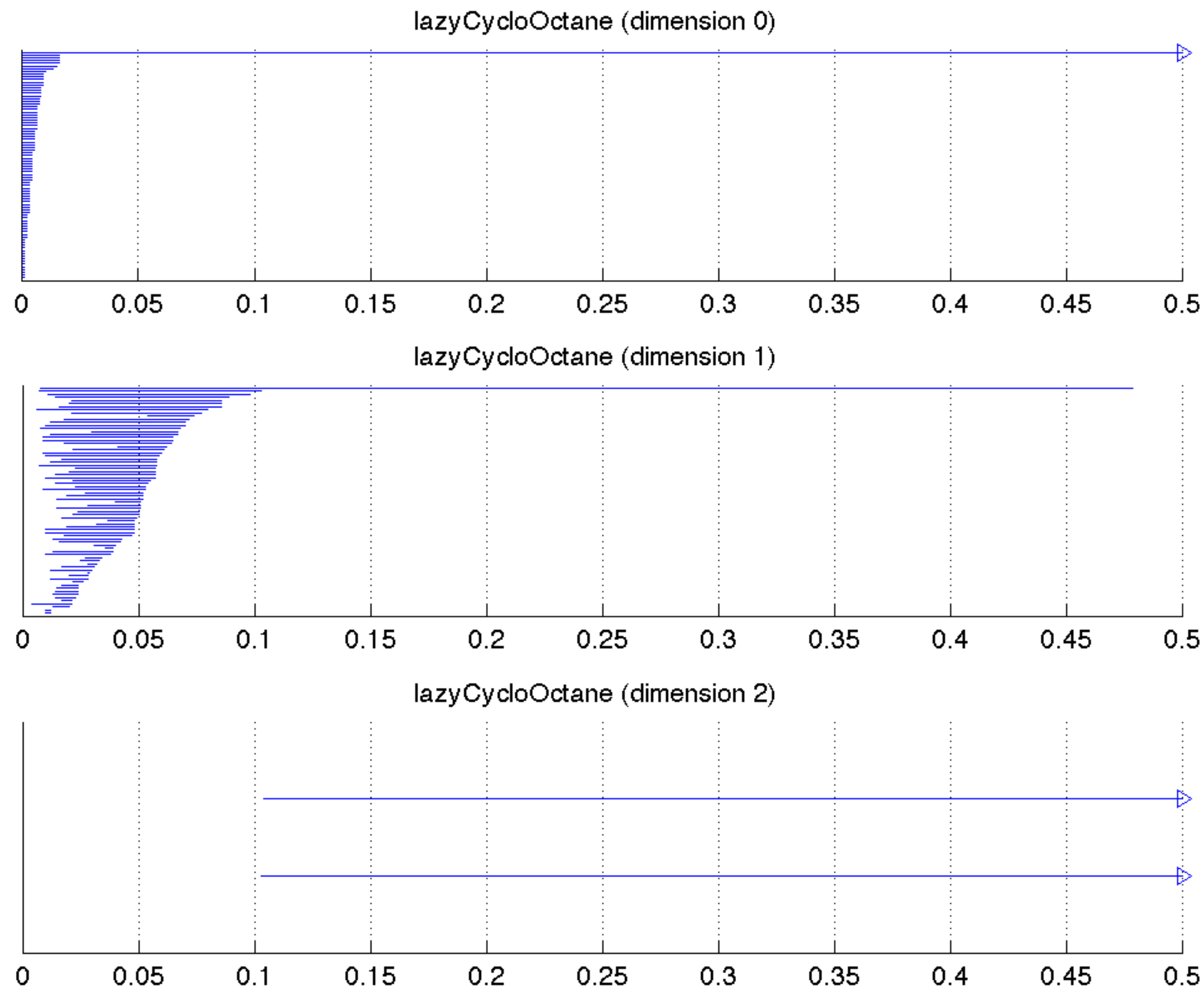
Persistent homology applied to data

Example: Cyclo-Octane (C_8H_{16}) data

1,031,644 points in 72-dimensional space





Non-Manifold Surface Reconstruction from High Dimensional Point Cloud Data by Shawn Martin and Jean-Paul Watson, 2010.

Persistent homology applied to data

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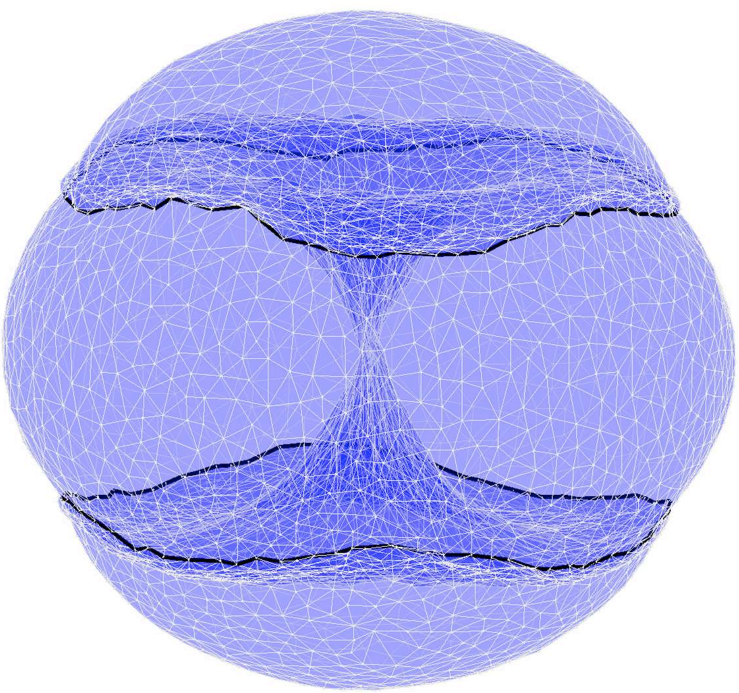

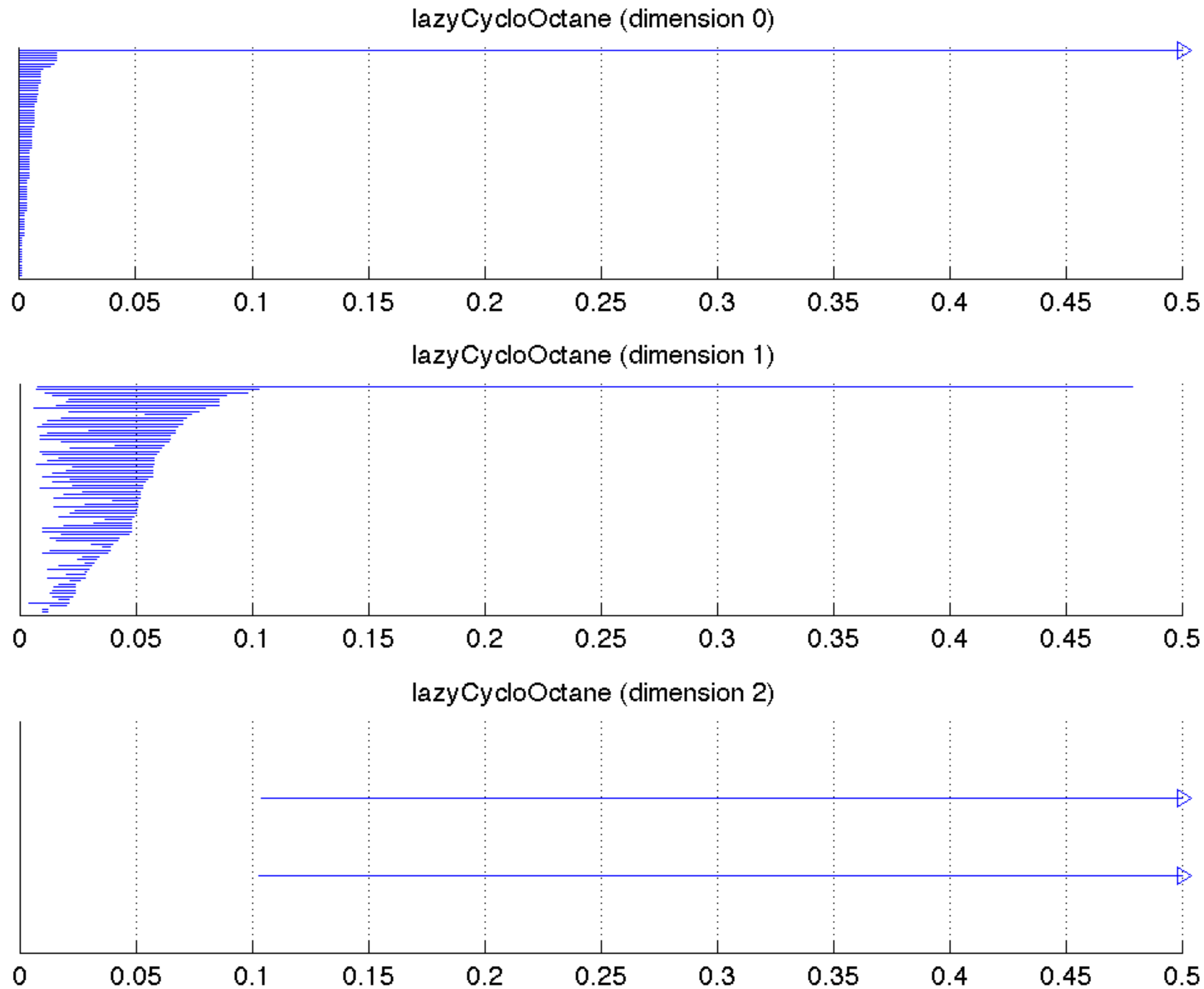

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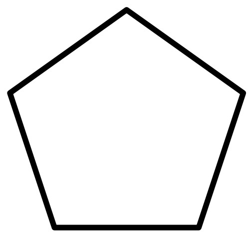
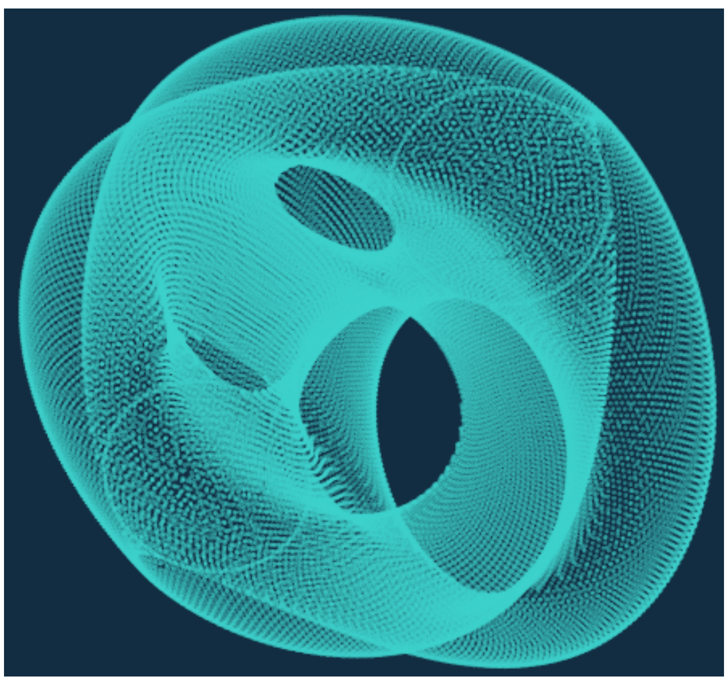
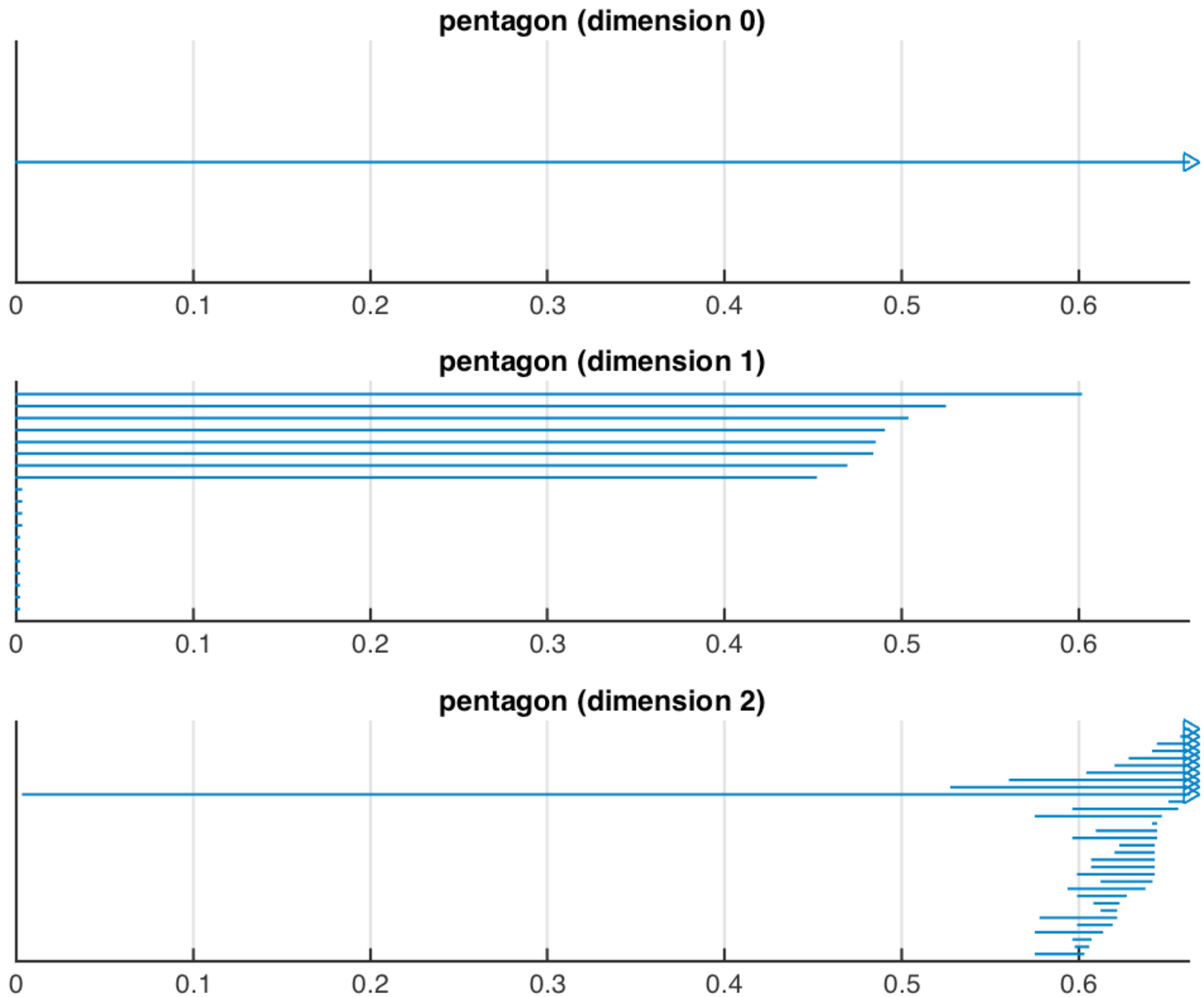
Persistent homology applied to data

Example: Equilateral pentagons in the plane



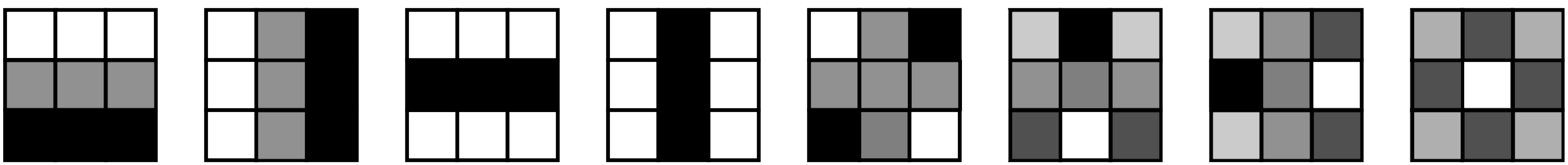




Persistent homology applied to data

Example: 3x3 high-contrast patches from images

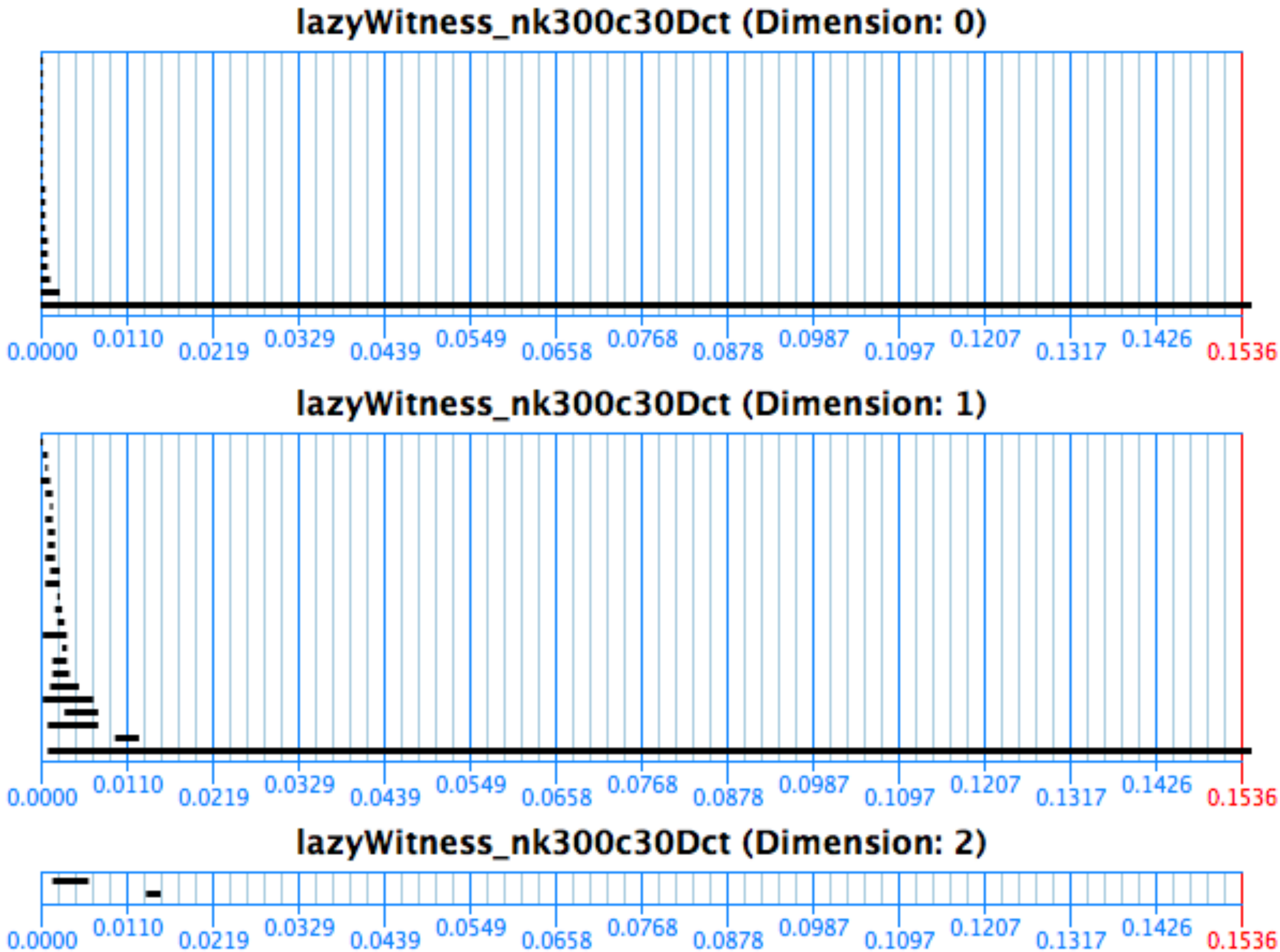
Points in 9-dimensional space



On the Local Behavior of Spaces of Natural Images by Gunnar Carlsson, Tigran Ishkhanov, Vin de Silva, and Afra Zomorodian, 2008.

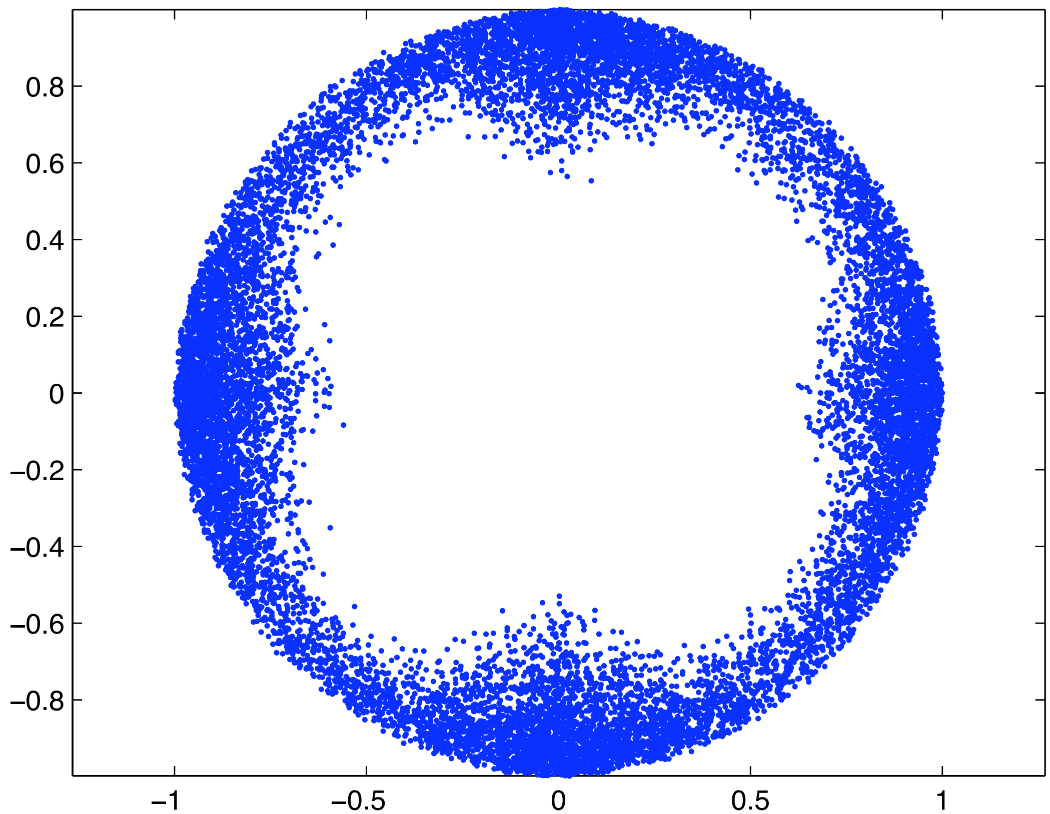
Persistent homology applied to data

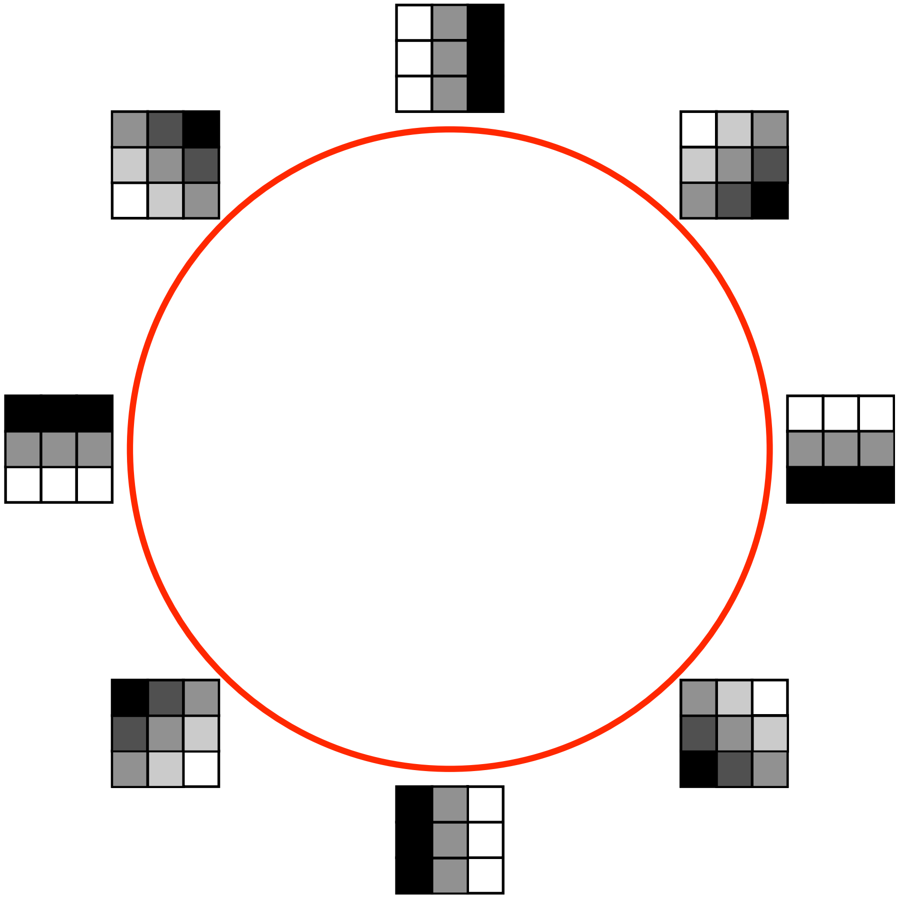
1. Densest patches according to a global estimate



Persistent homology applied to data

1. Densest patches according to a global estimate

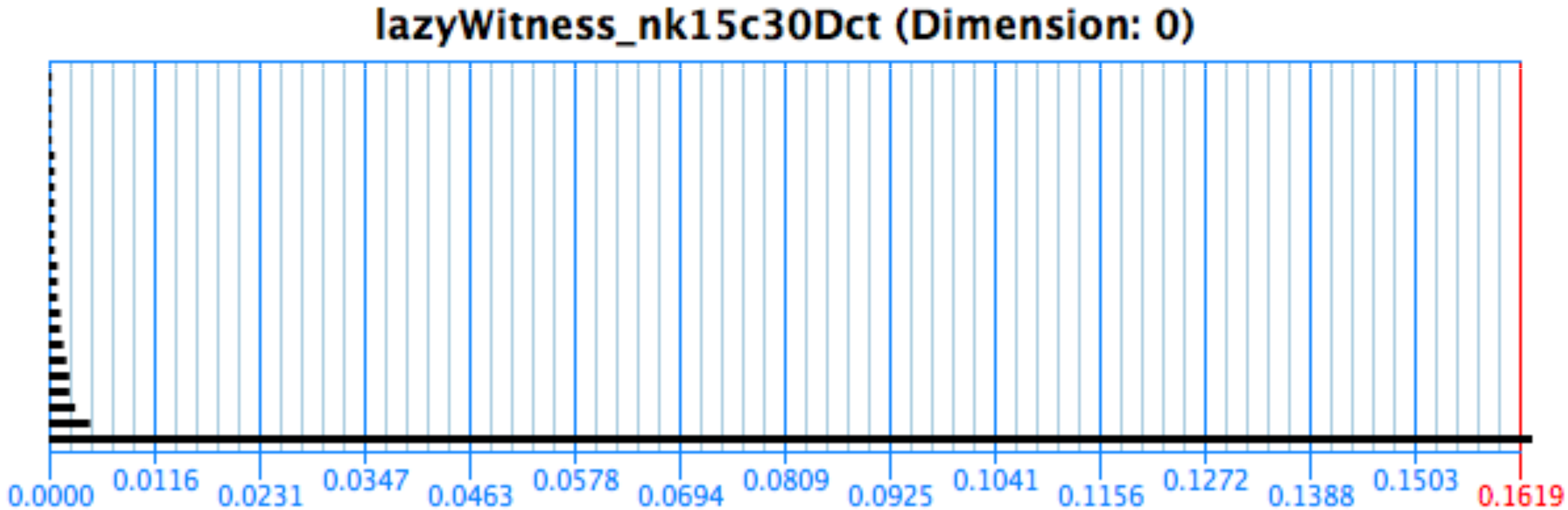


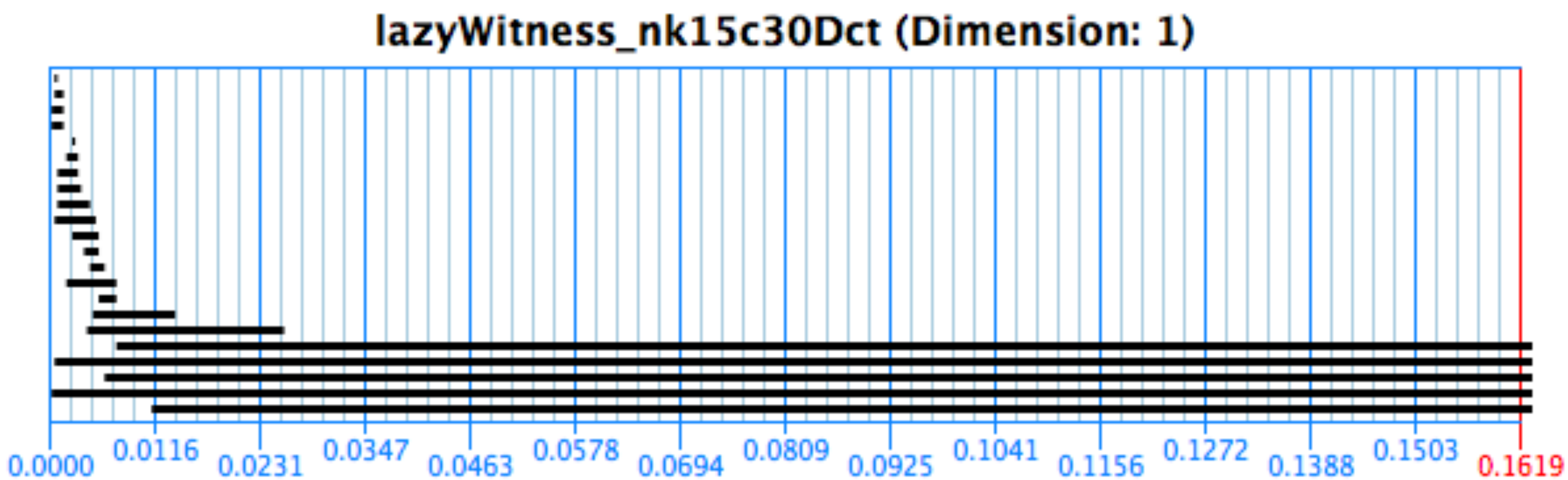


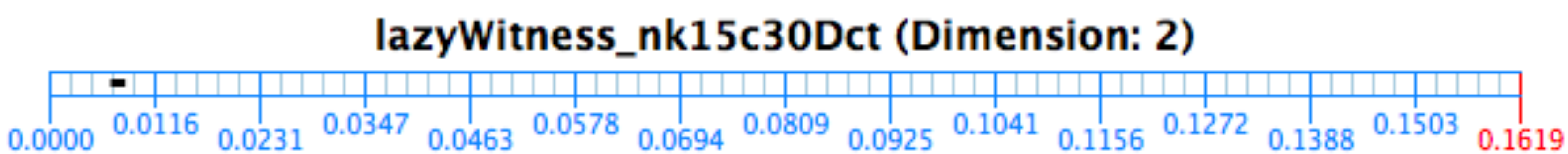
Interpretation: nature prefers linearity

Persistent homology applied to data

2. Densest patches according to an intermediate estimate

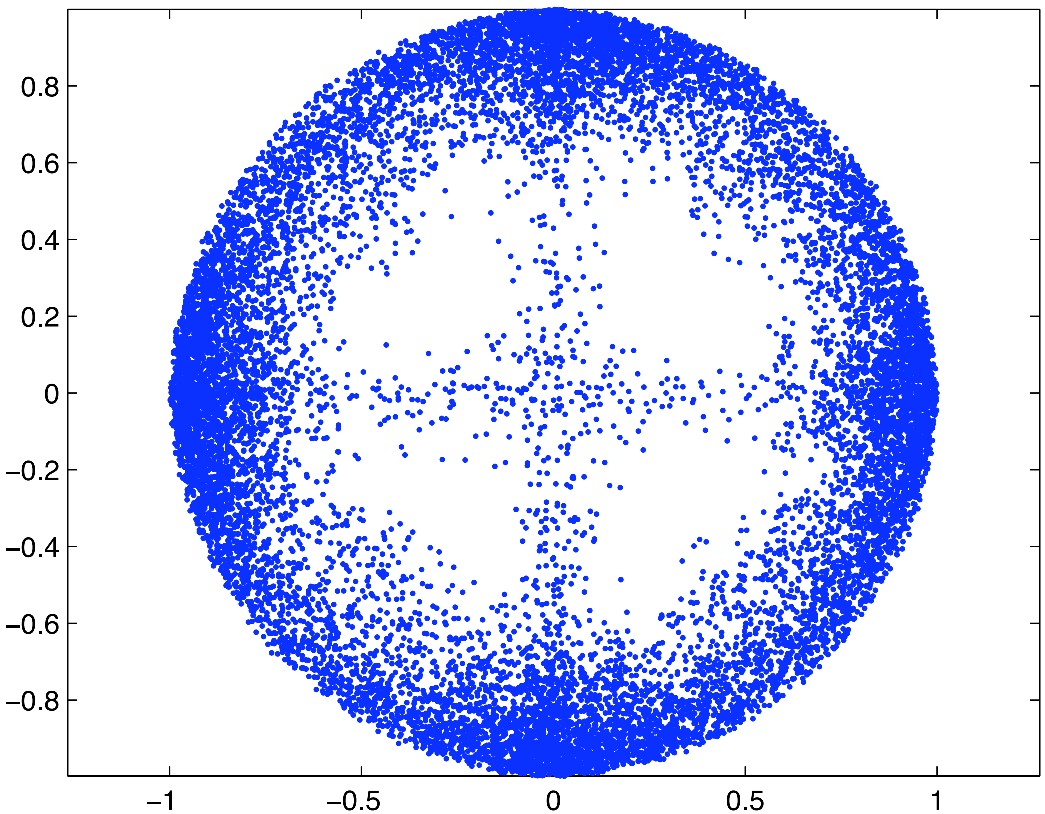


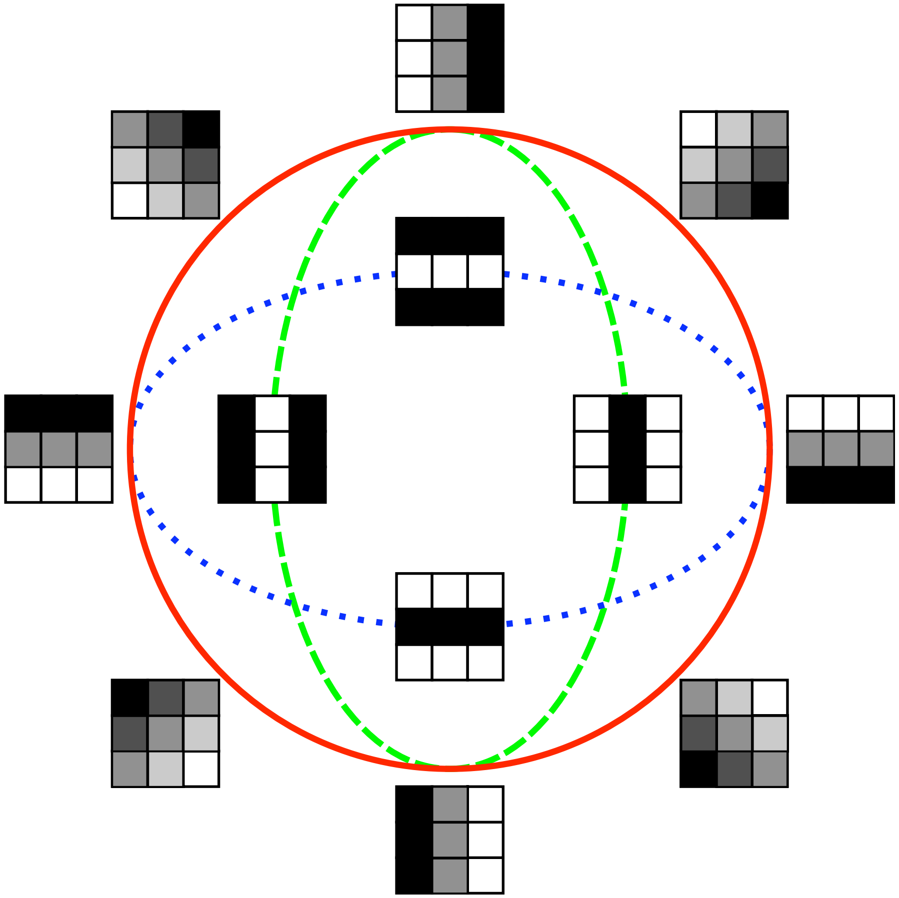




Persistent homology applied to data

2. Densest patches according to an intermediate estimate

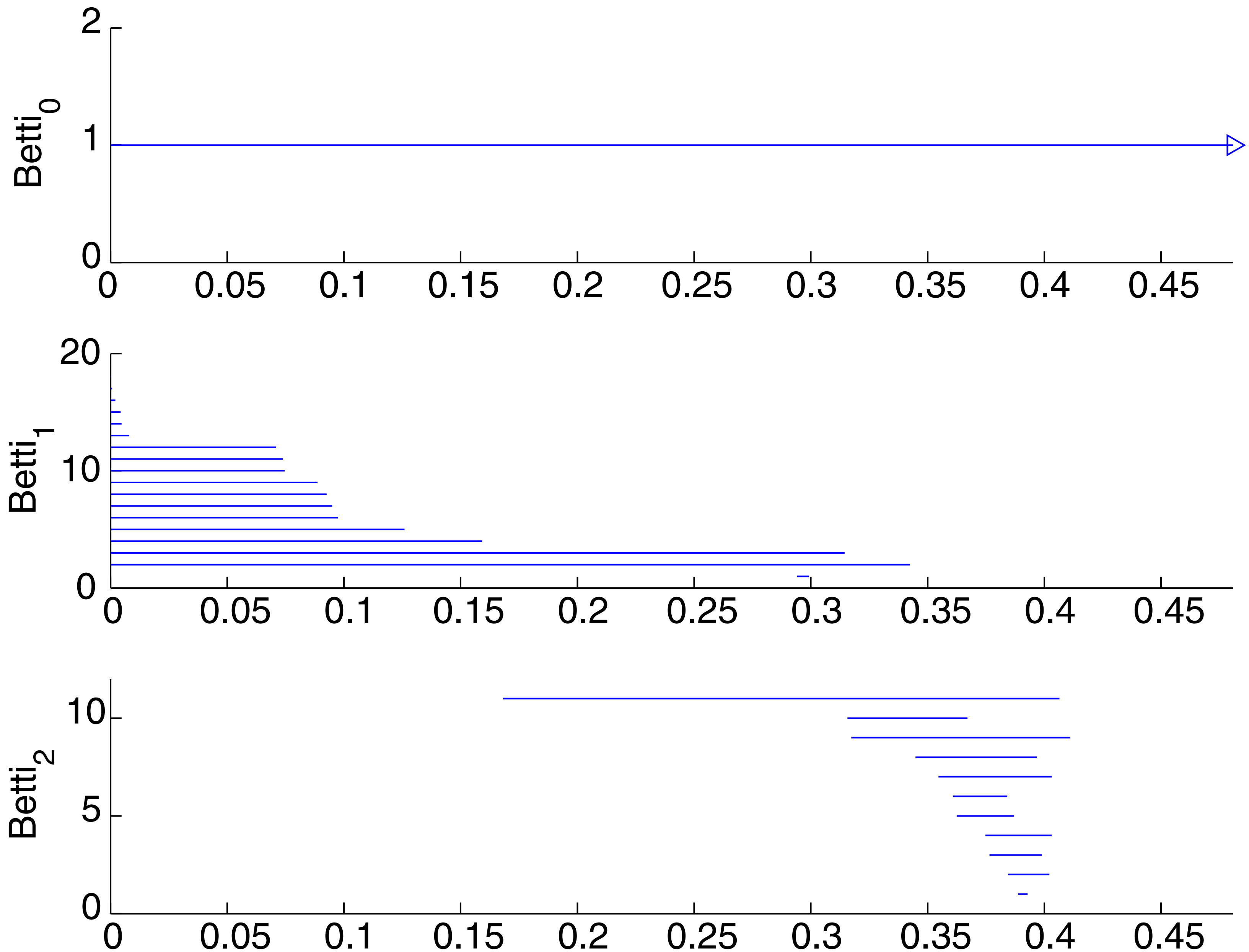




Interpretation: nature prefers horizontal and vertical directions

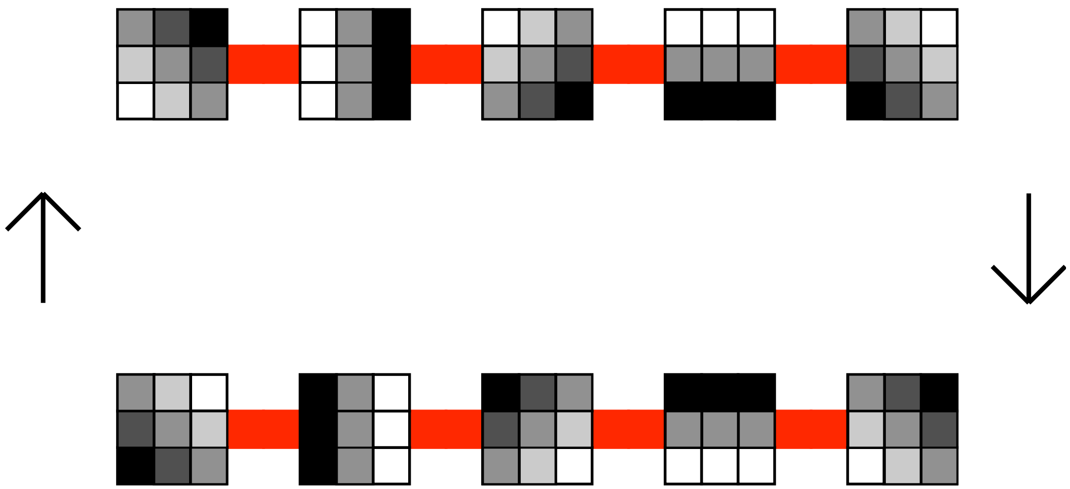
Persistent homology applied to data

3. Densest patches according to a local estimate



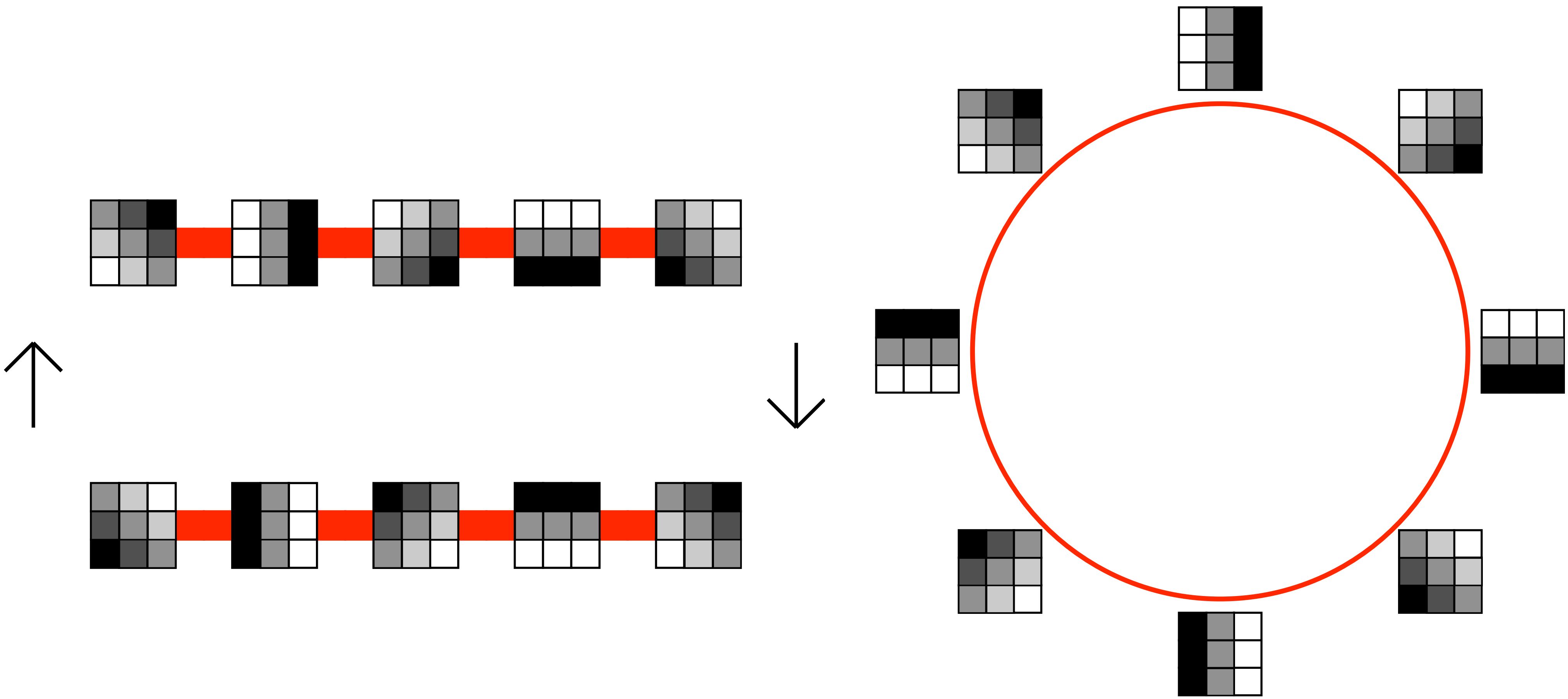
Persistent homology applied to data

3. Densest patches according to a local estimate



Persistent homology applied to data

3. Densest patches according to a local estimate



Persistent homology applied to data

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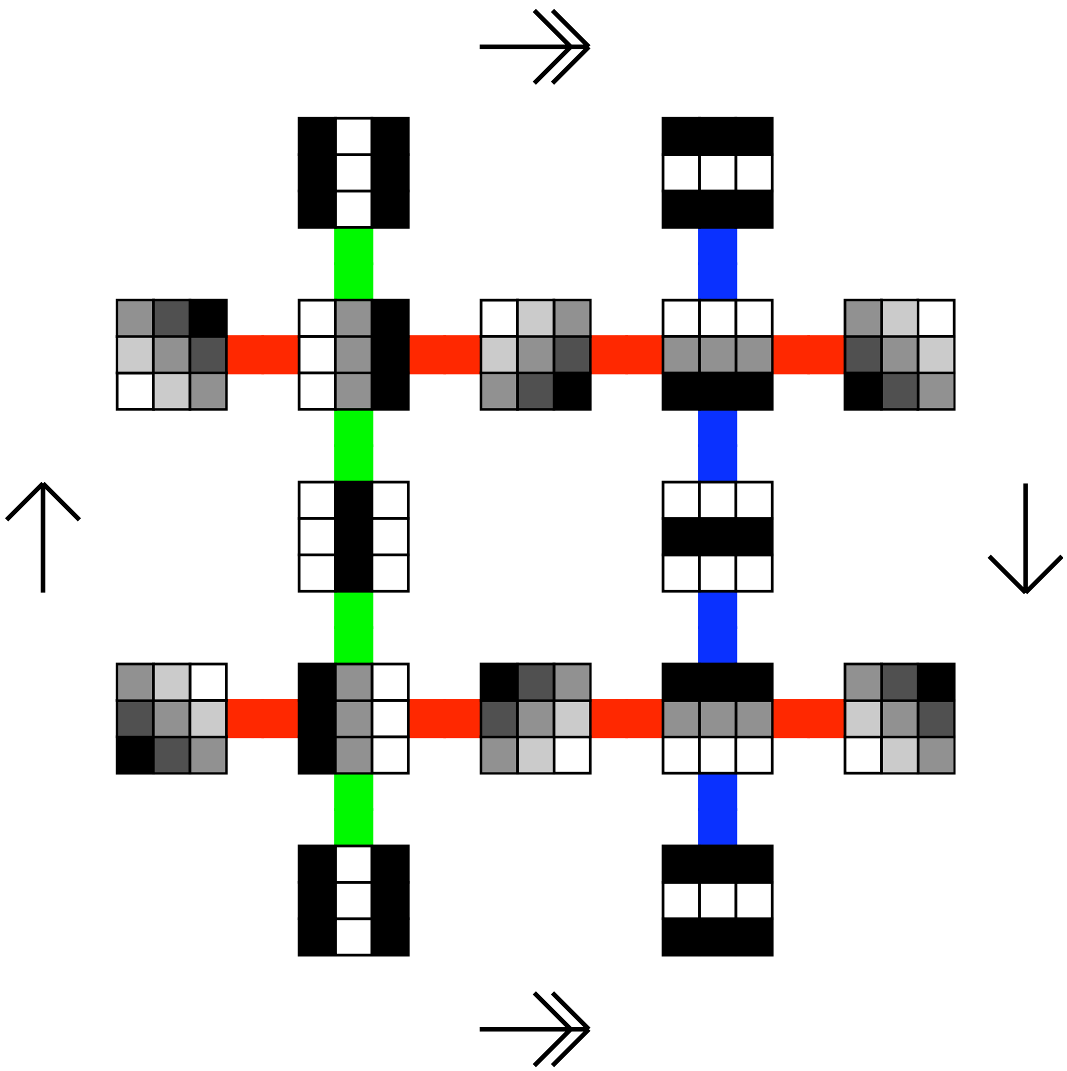

Persistent homology applied to data

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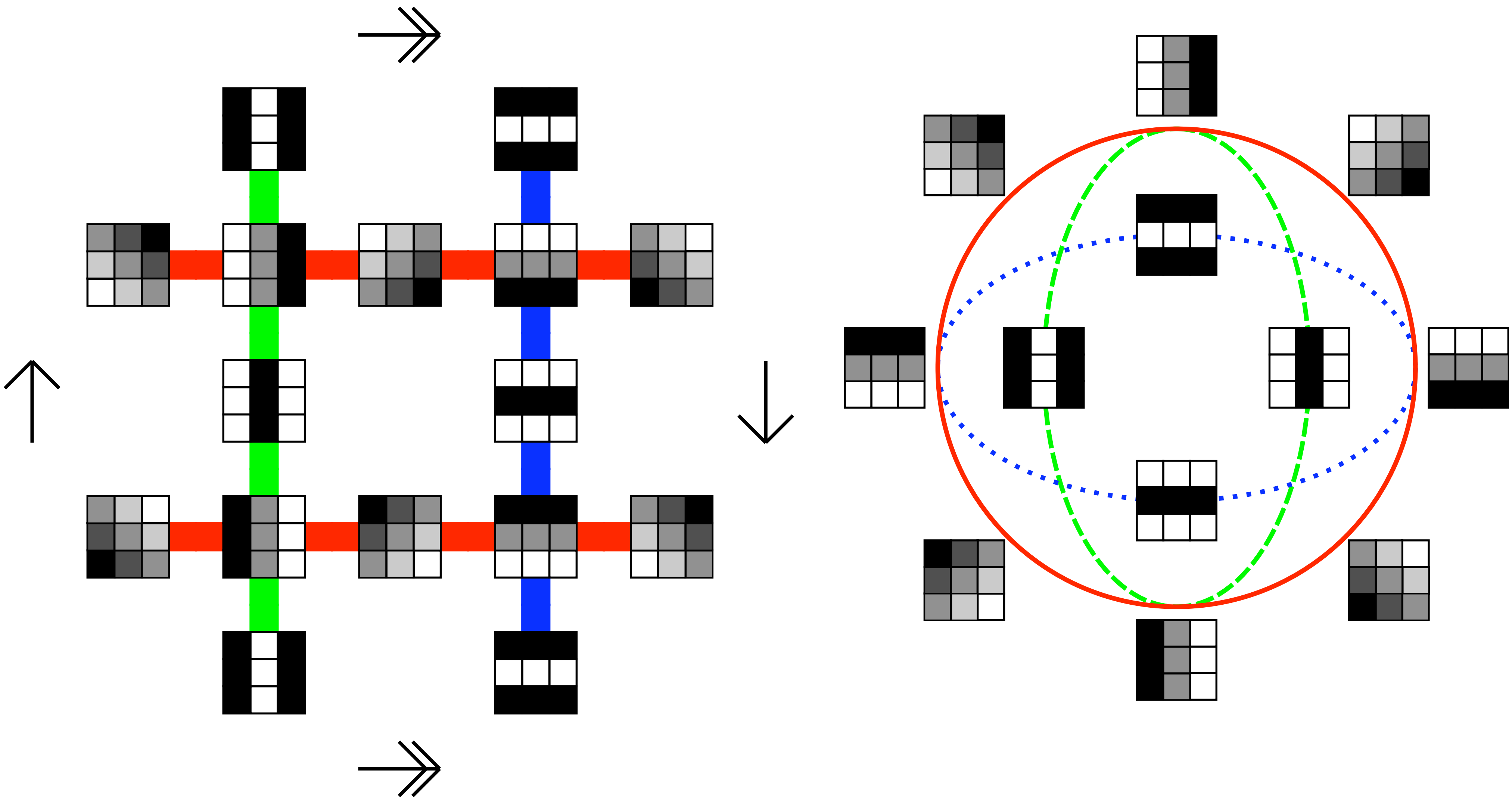

Persistent homology applied to data

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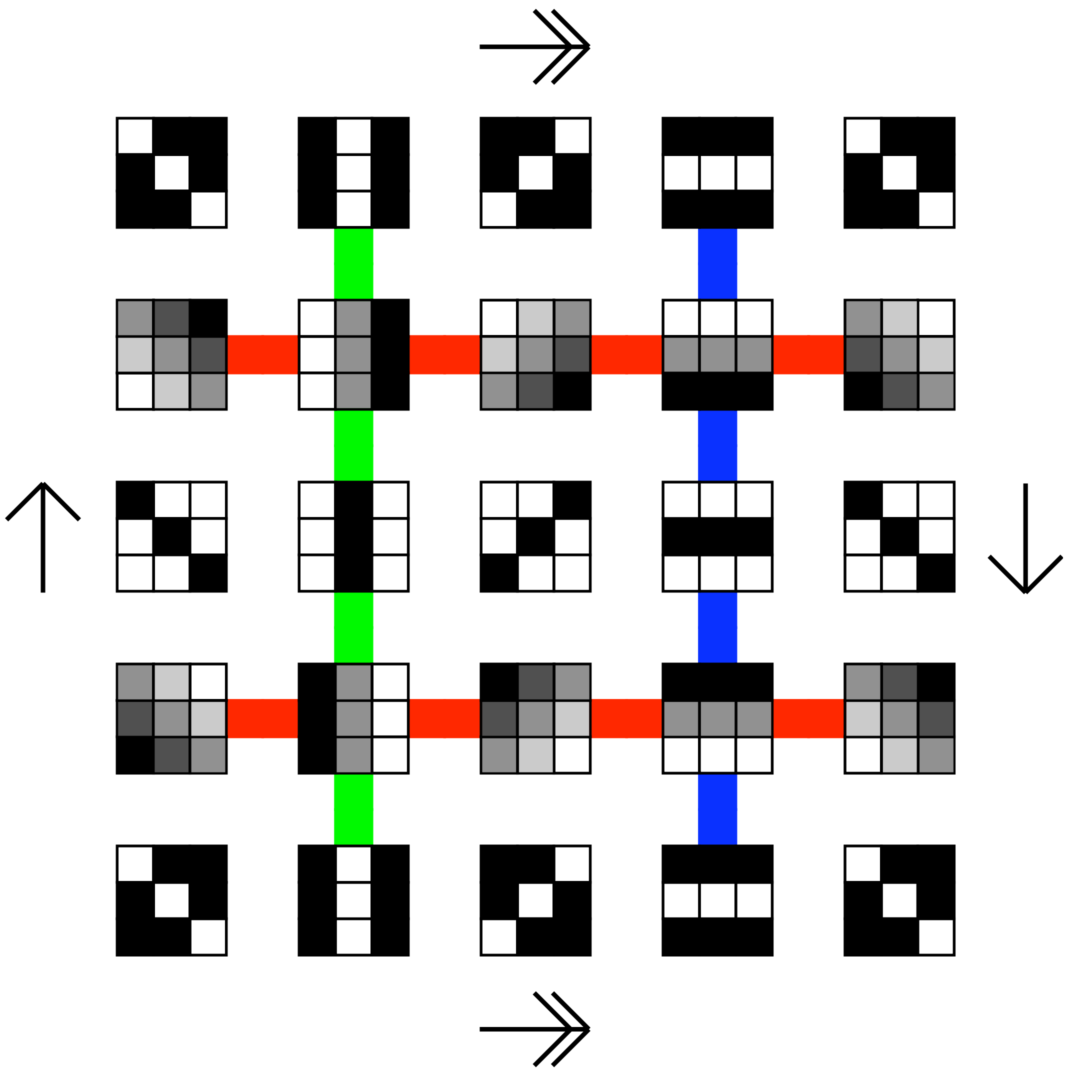
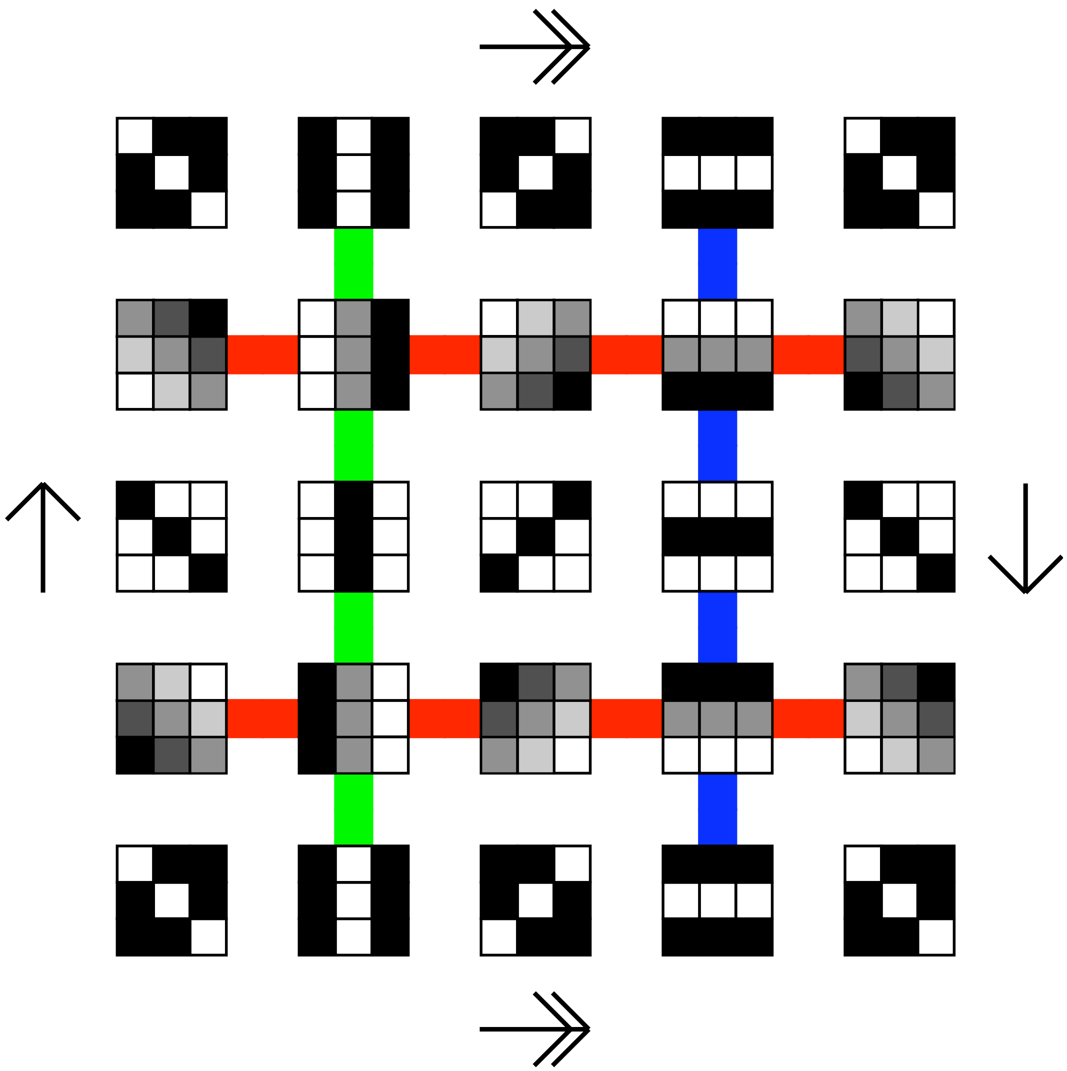
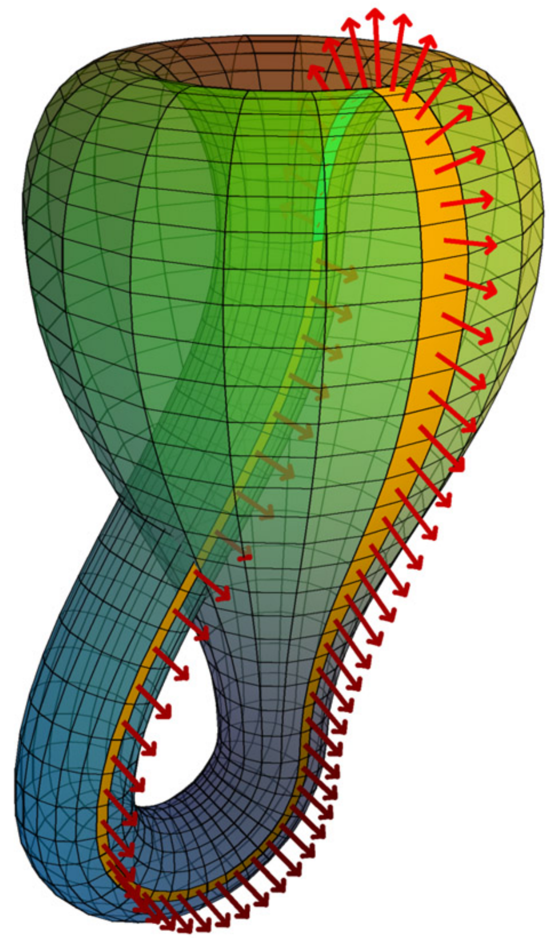


Persistent homology applied to data

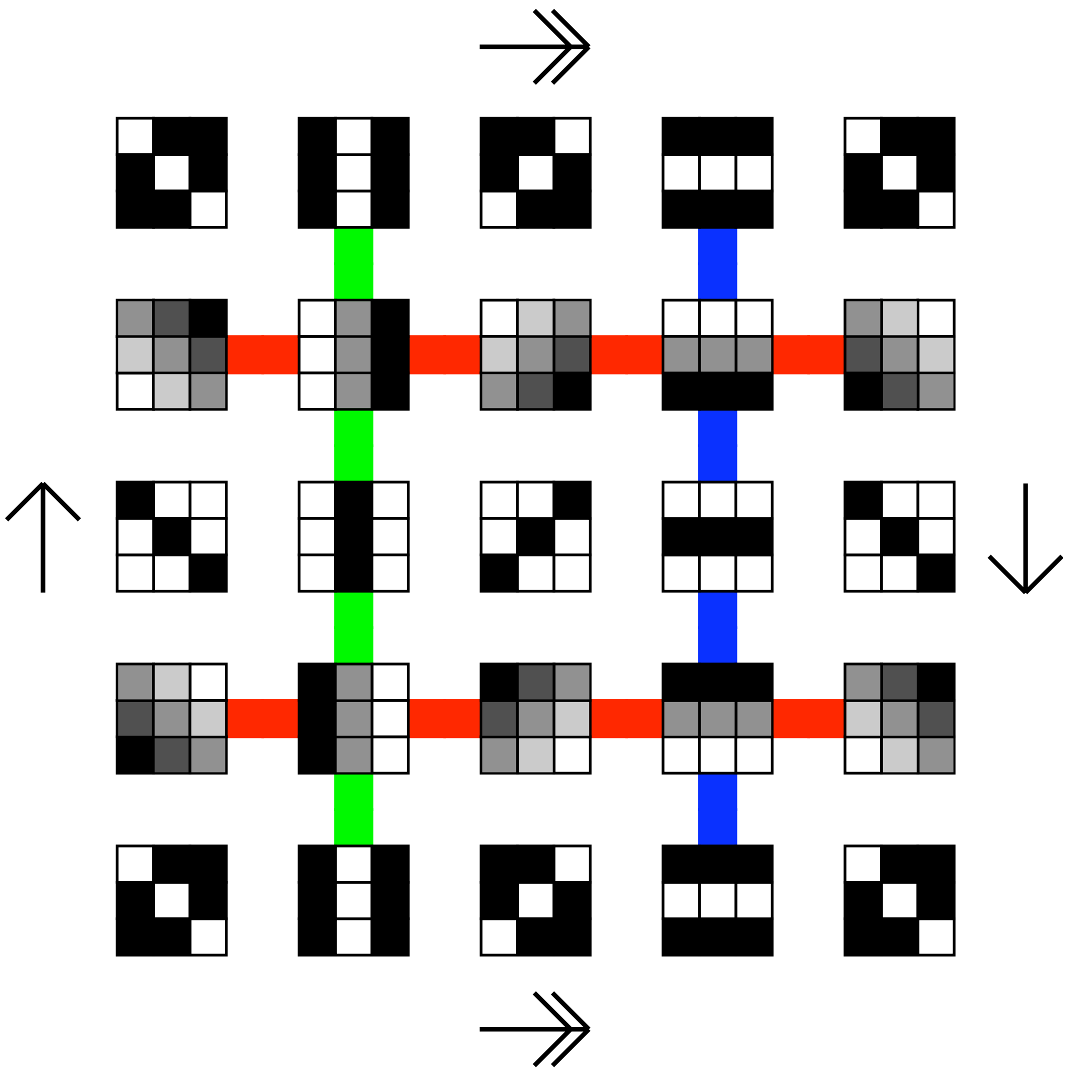
3. Densest patches according to a local estimate

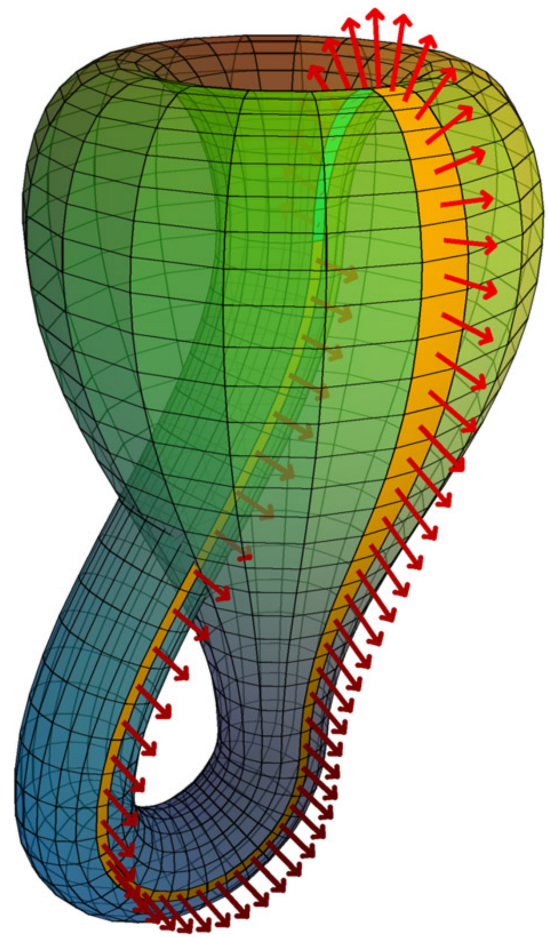




Persistent homology applied to data

3. Densest patches according to a local estimate





Interpretation: nature prefers linear and quadratic patches at all angles

References

- *An Attempt to Define the Nature of Chemical Diabetes Using a Multidimensional Analysis* by G. M. Reaven and R. G. Miller, 1979.
- *Non-Manifold Surface Reconstruction from High Dimensional Point Cloud Data* by Shawn Martin and Jean-Paul Watson, 2010.
- *On the Local Behavior of Spaces of Natural Images* by Gunnar Carlsson, Tigran Ishkhanov, Vin de Silva, and Afra Zomorodian, 2008.