Math 571: Topology II

1/17/2018 Course overview:

Chp 1: Fundamental group — Van Kampen's theorem, covering spaces.

Chp 2: Homology — long exact sequences, cellular homology, the formal viewpoint, homology with coefficients.

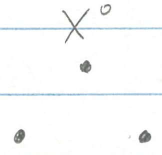
Chp 3: Cohomology - cup products, Poincaré duality.

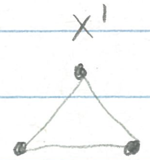
Chp 0: Cell complexes

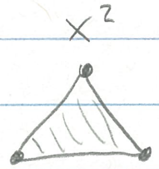
Def A CW complex X is built by

(1) Starting with a discrete set X^0

(2) Inductively forming the n-skeleton X^n from X^{n-1} by attaching open n-cells e^n_α via $\varphi_\alpha : S^{n-1} \to X^{n-1}$.





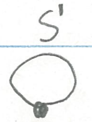


That is, $X^n = (X^{n-1} \amalg_\alpha D^n_\alpha) / (x \sim \varphi_\alpha(x) \text{ for } x \in \partial D^n_\alpha)$ for each D^n_α a closed n-disk.

(3) Set $X = \bigcup_n X^n$. X is given the Weak topology: $A \subseteq X$ is open iff $A \cap X^n$ is open $\forall n$.

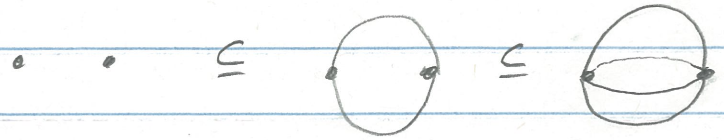
A consequence is Closure-finiteness: the closure of each cell intersects only finitely many other cells.

Ex 0.3 The sphere S^n $(n \geq 1)$ has a CW complex structure with a 0-cell e^0 and a single n-cell e^n attached via $S^{n-1} \to e^0$.



An alternate CW structure is two 0-cells, two 1-cells, two 2-cells, ..., two n-cells.

S^0 $\quad$ S^1 $\quad$ S^2

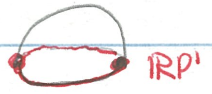


This allows one to define $S^\infty = \bigcup_n S^n$, which turns out to be contractible.

Ex 0.4 Real projective n-space

$\mathbb{R}P^n = \{\text{all lines through origin in } \mathbb{R}^{n+1}\}$
$= (\mathbb{R}^{n+1} \setminus \{\vec{0}\}) / (v \sim \lambda v \text{ for } 0 \neq \lambda \in \mathbb{R}) = S^n / (v \sim -v)$

$\mathbb{R}P^0 = e^0$ $\quad$ $\mathbb{R}P^1 = e^0 \cup e^1$ $\quad$ $\mathbb{R}P^2 = e^0 \cup e^1 \cup e^2$



It follows by induction that $\mathbb{R}P^n$ has a CW structure $e^0 \cup e^1 \cup \dots \cup e^n$ with one i-cell e^i for each $0 \leq i \leq n$

(RREF from Chris Peterson class)

1/19/2018

Ex 0.6 Complex projective n-space

$\mathbb{C}P^n = \{\text{all complex lines through origin in } \mathbb{C}^{n+1}\}$
$= (\mathbb{C}^{n+1} \setminus \{\vec{0}\}) / (v \sim \lambda v \text{ for } 0 \neq \lambda \in \mathbb{C})$
$= S^{2n+1} / (v \sim \lambda v \text{ for } |\lambda| = 1)$.

$\mathbb{C}P^0 = \{\cdot\} = S^1 / S^1$ $\quad$ $\mathbb{C}P^1 = S^3 / (\text{each point identified with a circle})$
$= S^2$

Recall the Hopf fibration

$$\begin{tikzcd} S^1 \arrow[r] & S^3 \arrow[d] \\ & S^2 \end{tikzcd}$$

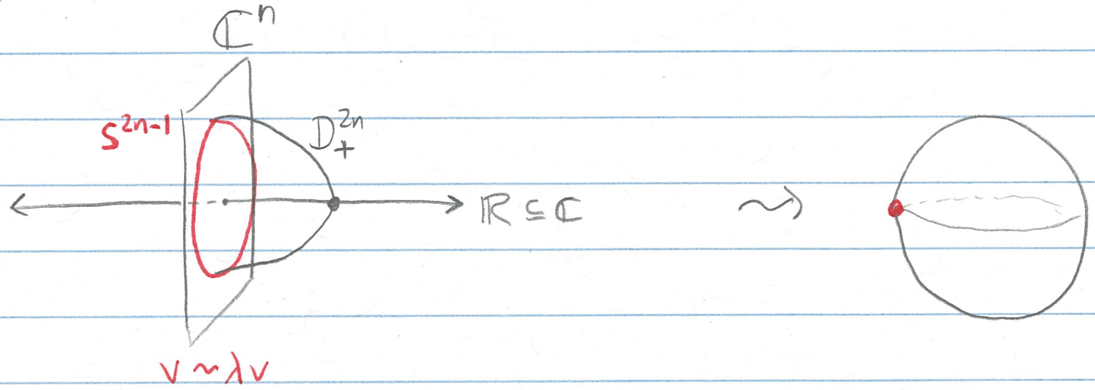
Claim $\mathbb{C}P^n = D^{2n}_+ / (v \sim \lambda v$ for $v \in \partial D^{2n}_+ = S^{2n-1}$ and $|\lambda| = 1)$ $S^{2n-1} \to \mathbb{C}P^{n-1}$

Consequence $\mathbb{C}P^n$ is obtained from $\mathbb{C}P^{n-1}$ by attaching a cell e^{2n}, so by induction we obtain a CW structure $e^0 \cup e^2 \cup e^4 \cup \dots \cup e^{2n}$ for $\mathbb{C}P^n$.

(RREF from Chris Peterson class)

Proof of Claim

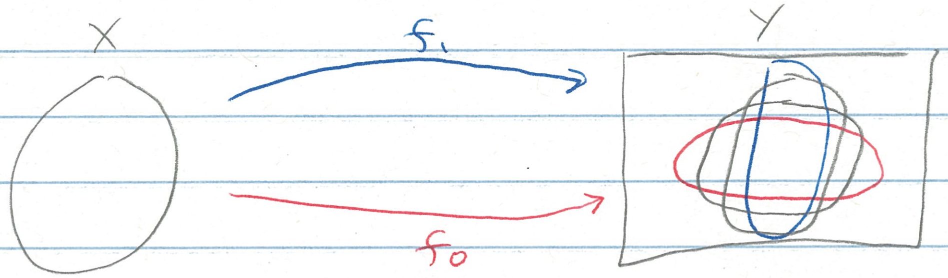
Pic $n=1$



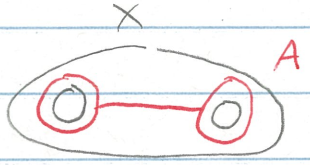
- Each vector in S^{2n+1} is equivalent under $v \sim \lambda v$ with $|\lambda| = 1$ to a vector $(w, \sqrt{1-|w|^2}) \in \mathbb{C}^n \times \mathbb{R} \subseteq \mathbb{C}^n \times \mathbb{C}$ with last coordinate real and non-negative, and $|w| \le 1$.
- These vectors form a disk D^{2n}_+ bounded by the sphere $S^{2n-1} = \{(w,0) \mid |w| = 1\}$.
- When the last coordinate is zero, we have the remaining identifications $v \sim \lambda v$ for $v \in S^{2n-1}$ (i.e. $\mathbb{C}P^{n-1}$).

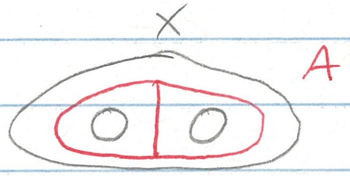
Pages 2–3

Def A *homotopy* is a family of maps $f_t : X \to Y$, $t \in I$ such that the associated map $F : X \times I \to Y$ given by $F(x,t) = f_t(x)$ is continuous.
We write $f_0 \simeq f_1$

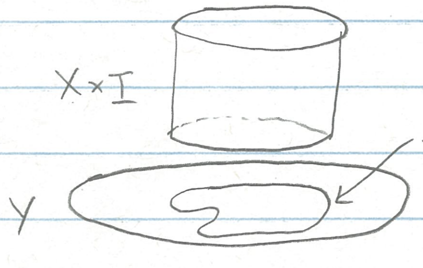


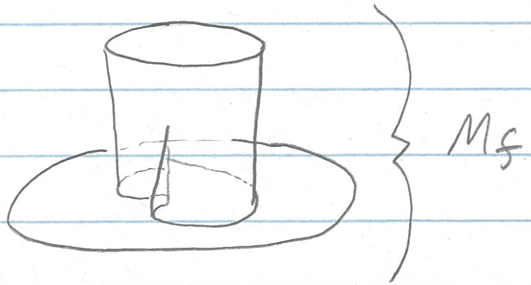
Def For $A \subseteq X$, a *deformation retraction* is a homotopy $f_t : X \to X$ such that $f_0 = \mathbb{1}_X$ $(= id_X)$, $f_1(X) = A$, and $f_t|_A = \mathbb{1}_A$ $\forall t$.





Ex Given $f : X \to Y$, the *mapping cylinder* M_f is the quotient space
$$\big((X \times I) \amalg Y\big) \big/ \big((x,1) \sim f(x) \text{ for } x \in X\big)$$





$X = S^1$, $Y = D^2 = \{z \in \mathbb{R}^2 \mid \|z\| \leq 1\}$

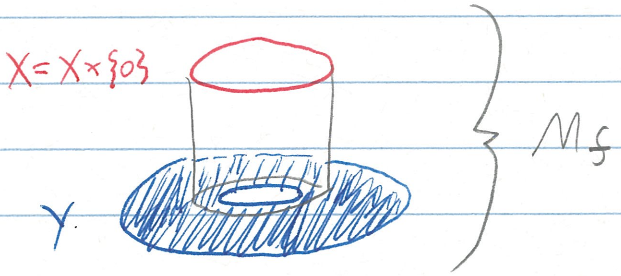
M_f deformation retracts onto Y by sliding each point (x,t) along the segment $\{x\} \times I \subseteq M_f$ to $f(x) \in Y$.

Recall Two spaces X and Y are homotopy equivalent $(X \simeq Y)$ if $\exists$ $f: X \to Y$ and $g: Y \to X$ such that $gf \simeq 1_X$ and $fg \simeq 1_Y$.

Fact Two spaces X and Y are homotopy equivalent iff $\exists$ a space Z containing both X and Y as deformation retracts.

Pf Let $f: X \to Y$ be a homotopy equivalence.
Let $Z = M_f$.
We always have that M_f deformation retracts onto Y.
One can use the fact that f is a homotopy equivalence to also show M_f deformation retracts onto $X \times \{0\} = X$.

Pic $X = S^1$
Y = annulus



1/22/18 Today our goal is to prove

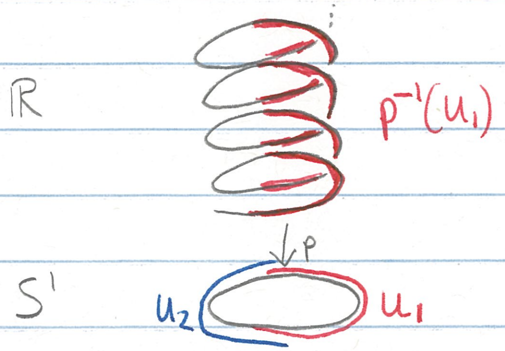
Thm 1.7 $\pi_1(S^1) \cong \mathbb{Z}$.

I think it's "dishonest" to not mention covering spaces when doing this.

Def (pg 56) A covering space of a space X is a space $\tilde{X}$ together with a map $p: \tilde{X} \to X$ such that

- there exists an open cover $\{U_\alpha\}$ of X such that for each α, $p^{-1}(U_\alpha)$ is a disjoint union of open sets in $\tilde{X}$, each of which is mapped by p homeomorphically onto U_α.

Ex $p: \mathbb{R} \to S^1$ given by $p(t) = (\cos 2\pi t, \sin 2\pi t)$ is a covering space of S^1.

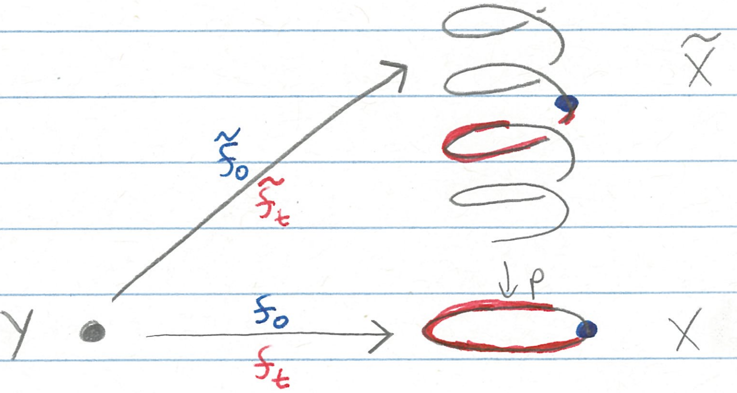


Def (pg 60) A lift of a map $f: Y \to X$ is a map $\tilde{f}: Y \to \tilde{X}$ such that $p\tilde{f} = f$

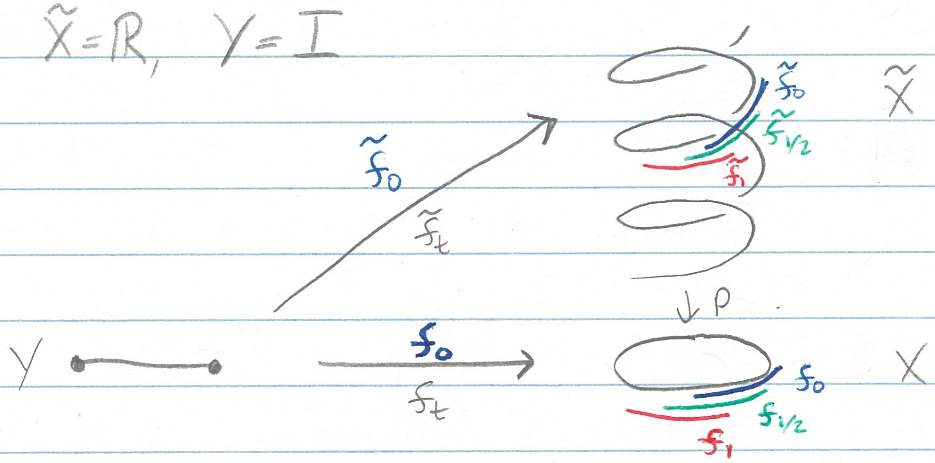
$$\begin{tikzcd}
 & \tilde{X} \arrow[d, "p"] \\
Y \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] \arrow[r, phantom, shift left=4, "\circlearrowright"] & X
\end{tikzcd}$$

Prop 1.30 (Homotopy lifting property) Given a covering space $p: \tilde{X} \to X$, a homotopy $f_t : Y \to X$, and a map $\tilde{f}_0 : Y \to \tilde{X}$ lifting f_0, there exists a unique homotopy $\tilde{f}_t : Y \to \tilde{X}$ of $\tilde{f}_0$ that lifts f_t.

Ex $X = S^1$, $\tilde{X} = \mathbb{R}$, $Y = \{0\}$ is a point.



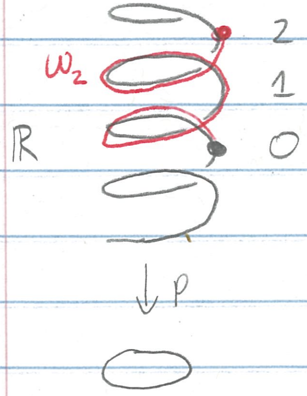
Ex $X = S^1$, $\tilde{X} = \mathbb{R}$, $Y = I$



Pf Sketch Note unique lifts exist over any open set U_α by the homeomorphism property (•). It turns out you can piece together these local lifts (this takes a page to prove in Hatcher).

Thm 1.7 The map $\Phi : \mathbb{Z} \to \pi_1(S^1, (1,0))$ via $n \longmapsto [\omega_n]$, where $\omega_n(s) = (\cos 2\pi ns, \sin 2\pi ns)$, is an isomorphism.

Pf



Note $\tilde{\omega}_n : I \to \mathbb{R}$ via $\tilde{\omega}_n(s) = ns$ is a lift of $\omega_n : I \to S^1$ (since $p\tilde{\omega}_n = \omega_n$).

Note $\Phi(n)$ is equal to $[p\tilde{f}]$ for $\tilde{f}$ any path in $\mathbb{R}$ from 0 to n (since $\tilde{f} \simeq \omega_n$ by a linear homotopy).

Φ is a homomorphism Let $\tau_m : \mathbb{R} \to \mathbb{R}$ be the translation $\tau_m(x) = x + m$.
Since $\tilde{\omega}_m \cdot (\tau_m \tilde{\omega}_n)$ is a path in $\mathbb{R}$ from 0 to $m+n$, we have

$$\begin{aligned} \Phi(m+n) &= [p(\tilde{\omega}_m \cdot (\tau_m \tilde{\omega}_n))] \\ &= [\omega_m \cdot \omega_n] \\ &= \Phi(m) \cdot \Phi(n). \end{aligned}$$

Φ is surjective Let $f : I \to S^1$ be any loop basal at $(1,0)$. By Prop 1.30 (with $Y = \{0\}$) there is a (unique) lift $\tilde{f} : I \to \mathbb{R}$ with $\tilde{f}(0) = 0$. Necessarily $\tilde{f}(1) = n$ for some $n \in \mathbb{Z}$, giving $\Phi(n) = [p\tilde{f}] = [f]$.

Φ is injective Suppose $\Phi(m) = \Phi(n)$, i.e. $\overset{f_0}{\overset{\shortparallel}{\omega_m}} \underset{f_t}{\simeq} \overset{f_1}{\overset{/\!/}{\omega_n}}$.
By Prop 1.30 (with $Y = I$) this lifts to a homotopy $\tilde{f}_t$ of paths starting at 0. By uniqueness in the paragraph above, $\tilde{f}_0 = \tilde{\omega}_m$ and $\tilde{f}_1 = \tilde{\omega}_n$. Hence $m = \tilde{f}_0(1) = \tilde{f}_1(1) = n$ since $\tilde{f}_t$ is a homotopy of paths.

1/24/18

Section 1.2: Van Kampen's Theorem

Free products of groups

Def 1 Let G_α be a collection of groups. Their free product $*_\alpha G_\alpha$ consists of all reduced words $g_1 g_2 \dots g_m$, where $g_i \in G_{\alpha_i}$ (reduced means $\alpha_i \neq \alpha_{i+1}$, and $g_i \neq id_{G_{\alpha_i}}$ $\forall i$). The group operation is juxtaposition, $(g_1 \dots g_m)(h_1 \dots h_n) = g_1 \dots g_m h_1 \dots h_n$, followed by reducing.

Ex $G_1 = \mathbb{Z} = \langle a \rangle$, $G_2 = \mathbb{Z} = \langle b \rangle$, $G_3 = \mathbb{Z} = \langle c \rangle$

In $G_1 * G_2 * G_3$ we have

$(ab^{-2}a^7bc^2b)(b^{-1}c^{-2}b^{-1}a^4c) = ab^{-2}a^{11}c$.

Also $(cb^{-6}ca)^{-1} = a^{-1}c^{-1}b^6c^{-1}$.

The identity is .

← The empty word

Ex $G_1 = \mathbb{Z}/2\mathbb{Z} = \langle a \mid a^2 = 1 \rangle$ $G_2 = \langle b \mid b^2 = 1 \rangle$

Elements of $G_1 * G_2$ are

$a, ab, aba, abab, \dots$

$b, ba, bab, baba, \dots$

It turns out that

$$G_1 * G_2 \cong \mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z} \cong \langle ab, a \mid a^2 = 1, a(ab)a^{-1} = (ab)^{-1} \rangle$$
$$= \langle r, s \mid s^2 = 1, srs^{-1} = r^{-1} \rangle$$

is the "infinite dihedral group".

Rmk Proving $*_\alpha G_\alpha$ is associative takes work.

Def 2 The free product is the coproduct in the category of groups.

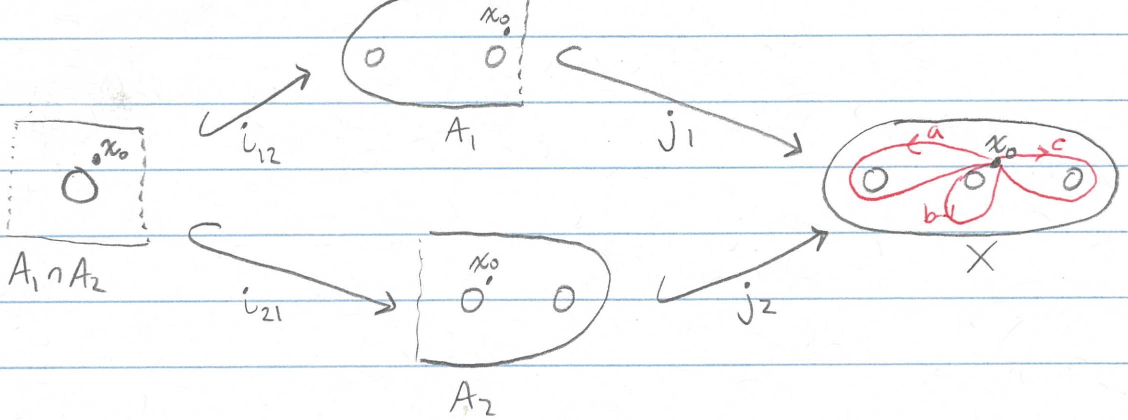
$$\begin{tikzcd} & H & \\ G_1 \arrow[ur, "f_1"] \arrow[r, "\iota_1"'] & G_1 * G_2 \arrow[u, "\exists! f"'] & G_2 \arrow[ul, "f_2"'] \arrow[l, "\iota_2"] \end{tikzcd}$$

$$\begin{tikzcd} & H \\ G_\alpha \arrow[ur, "f_\alpha"] \arrow[r, "\iota_\alpha"'] \arrow[ur, phantom, shift right=3ex, "\circlearrowright"] & *_\alpha G_\alpha \arrow[u, dashed, "\exists! f"'] \end{tikzcd}$$

Rmk One often proves that two free products are not isomorphic by observing that their abelianizations are not isomorphic.

Ex $\mathbb{Z} * \mathbb{Z} * \mathbb{Z} \not\cong \mathbb{Z} * \mathbb{Z}$ since $ab(\mathbb{Z} * \mathbb{Z} * \mathbb{Z}) \cong \mathbb{Z}^3 \not\cong \mathbb{Z}^2 \cong ab(\mathbb{Z} * \mathbb{Z})$.

The van Kampen Theorem



$$\begin{tikzcd}
& \pi_1(X) & \\
\pi_1(A_1) \arrow[ur, "j_1"] \arrow[r] & \pi_1(A_1) * \pi_1(A_2) \arrow[u, "\exists ! \Phi"] & \pi_1(A_2) \arrow[l] \arrow[ul, "j_2"']
\end{tikzcd}$$

Thm 1.20, special case of two sets Suppose X is the union of path-connected open sets A_1 and A_2, each containing the basepoint x_0, and $A_1 \cap A_2$ is path-connected.

Then $\Phi : \pi_1(A_1) * \pi_1(A_2) \longrightarrow \pi_1(X)$ is surjective, with kernel N the normal subgroup generated by all elements of the form $i_{12}(w) i_{21}(w)^{-1}$ for $w \in \pi_1(A_1 \cap A_2)$, and so $\pi_1(X) \cong (\pi_1(A_1) * \pi_1(A_2)) / N$.

How does this work in our picture?

$\pi_1(X) \cong \langle a, b, c \rangle \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$

$\pi_1(A_1) \cong \langle a, b_1 \rangle$

$\pi_1(A_2) \cong \langle b_2, c \rangle$

$\pi_1(A_1 \cap A_2) \cong \langle b \rangle$ with $i_{12}(b) = b_1$, $i_{21}(b) = b_2$

Note $(\pi_1(A_1) * \pi_1(A_2))/N$

$$\begin{aligned}
&\cong \langle a, b_1, b_2, c \rangle / N \\
&\cong \langle a, b_1, b_2, c \mid i_{12}(w)\, i_{21}(w)^{-1} = 1 \text{ for } w \in \pi_1(A_1 \cap A_2) \rangle \\
&\cong \langle a, b_1, b_2, c \mid b_1 b_2^{-1} = 1 \rangle \\
&\cong \langle a, b_1, b_2, c \mid b_1 = b_2 \rangle \\
&\cong \langle a, b, c \rangle \\
&\cong \pi_1(X).
\end{aligned}$$

Can be relaxed some

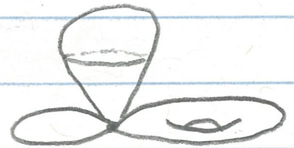
Thm 1.20, special case of wedge sums

If X_α is a collection of pointed CW complexes and $\vee_\alpha X_\alpha$ is their wedge sum, then $\Phi : *_\alpha\, \pi_1(X_\alpha) \longrightarrow \pi_1(\vee_\alpha X_\alpha)$ is an isomorphism.

Ex $X_1 =$ 

$X_2 =$ 

$X_3 =$ 

$X_1 \vee X_2 \vee X_3 =$ 

$$\begin{aligned}
\pi_1(X_1 \vee X_2 \vee X_3) &\cong \pi_1(X_1) * \pi_1(X_2) * \pi_1(X_3) \\
&\cong \mathbb{Z} * (\mathbb{Z} \times \mathbb{Z}) * (\text{trivial group}) \\
&\cong \mathbb{Z} * (\mathbb{Z} \times \mathbb{Z}).
\end{aligned}$$

Rmk For "nice spaces," the fundamental group takes the coproduct of pointed spaces to the coproduct of fundamental groups.

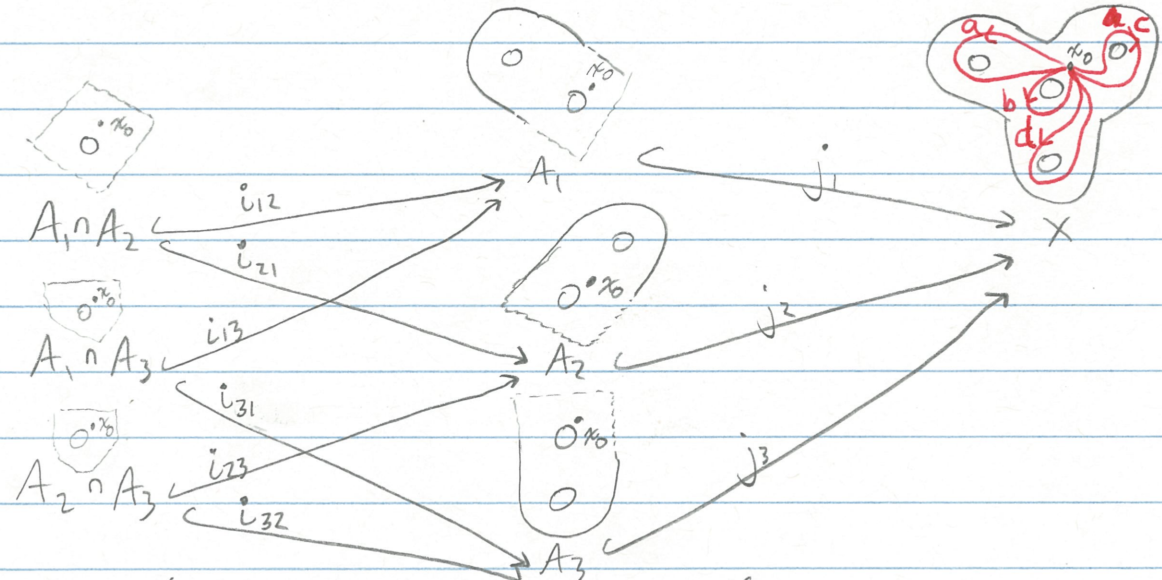
$$\begin{tikzcd}
 & Y \\
X_\alpha \arrow[r] \arrow[ur] & \vee_\alpha X_\alpha \arrow[u, dashed, "\exists!"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & H \\
G_\alpha \arrow[r] \arrow[ur] & *_\alpha G_\alpha \arrow[u, dashed, "\exists!"']
\end{tikzcd}$$

1/26/18 Thm 1.20 (van Kampen's) If X is the union of path-connected open sets A_α each containing the basepoint x_0, and if each $A_\alpha \cap A_\beta$ is path-connected, then $\underset{\uparrow}{\Phi} : *_\alpha \pi_1(A_\alpha) \to \pi_1(X)$ is surjective.

Defined by concatenating loops

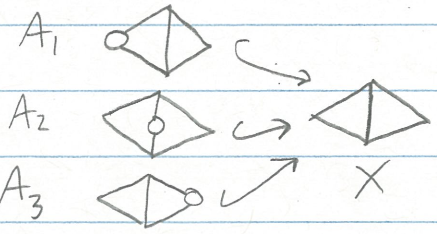
If in addition each $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected, then the kernel of Φ is the normal subgroup N generated by all elements $i_{\alpha\beta}(w)\, i_{\beta\alpha}(w)^{-1}$ for $w \in \pi_1(A_\alpha \cap A_\beta)$, and so $\pi_1(X) \cong *_\alpha \pi_1(A_\alpha)/N$.

Ex



$$\begin{aligned}
\pi_1(X) &\cong (\pi_1(A_1) * \pi_1(A_2) * \pi_1(A_3))/N \\
&\cong (\langle a, b_1\rangle * \langle b_2, c\rangle * \langle b_3, d\rangle)/N \\
&\cong \langle a, b_1, b_2, b_3, c, d\rangle / N \\
&\cong \langle a, b_1, b_2, b_3, c, d \mid b_1 b_2^{-1},\ b_1 b_3^{-1},\ b_2 b_3^{-1}\rangle \\
&\cong \langle a, b, c, d\rangle.
\end{aligned}$$

Rmk The theorem can fail if some $A_\alpha \cap A_\beta \cap A_\gamma$ is not path-connected:



$\pi_1(X) \cong \mathbb{Z} * \mathbb{Z}$ but

$(\pi_1(A_1) * \pi_1(A_2) * \pi_1(A_3))/N$

$\cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$ since

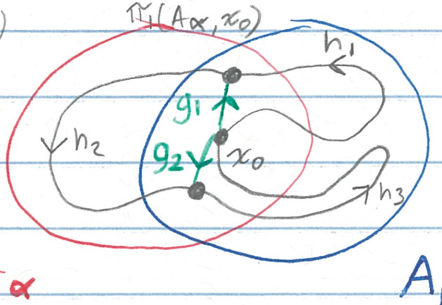
N is trivial (because each $A_\alpha \cap A_\beta$ is contractible).

Proof

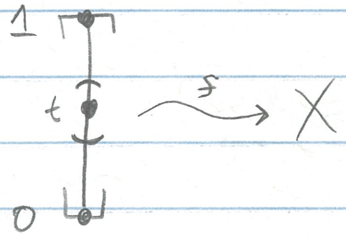
A factorization of $[f]\in\pi_1(X,x_0)$ is a product $[f]=[f_1]\cdots[f_k]$ with each $[f_i]\in\pi_1(A_\alpha,x_0)$ for some α.

Φ is surjective since each $[f]$ has a factorization

$$[f]=[h_1\cdot h_2\cdot\cdots h_k]$$
$$=\underset{\underset{\pi_1(A_\beta,x_0)}{\rotatebox{-90}{$\in$}}}{[h_1\cdot\bar{g}_1]}\underset{\underset{\pi_1(A_\alpha,x_0)}{\rotatebox{-90}{$\in$}}}{[g_1\cdot h_2\cdot\bar{g}_2]}[g_2\cdot h_3\cdot\bar{g}_3]\cdots[g_{k-1}h_k].$$



Finding $f=h_1\cdots h_k$ with $\mathrm{im}(h_i)\in A_{\alpha_i}$ for some α_i relies on the compactness of I:



Each $t\in I$ is in a small open interval mapped into some A_α; apply compactness to get a finite cover.

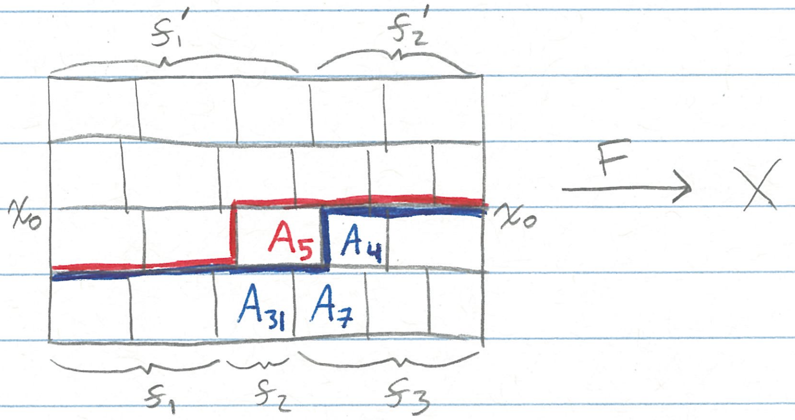
Choosing g_i with $\mathrm{im}(g_i)\in A_{\alpha_i}\cap A_{\alpha_{i+1}}$ relies on the path-connectedness of each such intersection.

It's clear $N \subseteq \ker(\Phi)$ since

$$\begin{aligned} \Phi\left(i_{\alpha\beta}(\omega)\, i_{\beta\alpha}(\omega)^{-1}\right) &= \Phi\left(i_{\alpha\beta}(\omega)\right) \cdot \Phi\left(i_{\beta\alpha}(\omega)\right)^{-1} \\ &= j_\alpha\left(i_{\alpha\beta}(\omega)\right) \cdot j_\beta\left(i_{\beta\alpha}(\omega)\right)^{-1} \\ &= \text{identity} \end{aligned}$$

(since $j_\alpha \circ i_{\alpha\beta} = j_\beta \circ i_{\beta\alpha}$ as maps of spaces, and hence also as maps of groups.)

It remains to show $\ker(\Phi) \subseteq N$, i.e. if $[f_1] \cdots [f_k] = [f_1'] \cdots [f_\ell']$ (two factorizations have the same image under Φ) then one can be obtained from the other by regarding $[f_i] \in \pi_1(A_\alpha \cap A_\beta)$ as lying in $\pi_1(A_\alpha)$ or $\pi_1(A_\beta)$ and reducing (i.e. by applying relations in N).

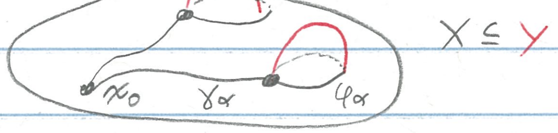


- Let $F: I \times I \to X$ be a homotopy from $f_1 \cdots f_k$ to $f_1' \cdots f_\ell'$.
- By compactness of $I \times I$ we can subdivide into many small squares, each with image in some A_α.
- At each vertex (3-fold intersection of squares), choose a path to x_0 in $A_\alpha \cap A_\beta \cap A_\gamma$ (using path-connectedness).
- Obtain the red factorization from the blue one by regarding the blue loops in A_{31}, A_7, A_4 instead as lying in A_5, and then performing a homotopy across the "A_5" square.
- Continue until we obtain $[f_1'] \cdots [f_\ell']$ from $[f_1] \cdots [f_k]$ by applying relations in N.

1/29/18

Section 1.2: Applications (of van Kampen's theorem) to Cell Complexes.

Let X be path-connected, and form Y by gluing 2-cells e^2_α to X along attaching maps $\varphi_\alpha : S^1 \to X$. Fix $x_0 \in X$, and for each α fix a path γ_α in X from x_0 to a basepoint of γ_α.

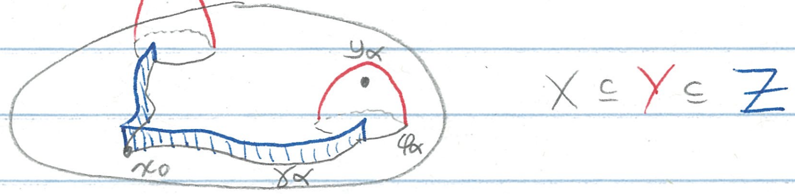


The normal subgroup N of $\pi_1(X, x_0)$ generated by all loops $\gamma_\alpha \varphi_\alpha \overline{\gamma_\alpha}$ lies in the kernel of $\pi_1(X) \to \pi_1(Y)$.

<u>Prop 1.26</u> **(a)** The inclusion $X \hookrightarrow Y$ induces a surjection $\pi_1(X) \to \pi_1(Y)$ with kernel N, and thus $\pi_1(Y) \cong \pi_1(X)/N$.

(b) Attaching n-cells for $n > 2$ does not affect π_1.

<u>Proof</u> (a)



Let $A = Z - \bigcup_\alpha \{y_\alpha\}$, which deformation retracts to X.

Let $B = Z - X$, which is contractible.

Note $A \cap B \simeq \bigvee_\alpha S^1$.

Apply van Kampen's to the cover $\{A, B\}$ of Z to get

$$\pi_1(Y) \cong \pi_1(Z) \cong (\pi_1(A) * \pi_1(B)) / \langle i_{AB}(w)\, i_{BA}(w)^{-1} \text{ for } w \in \pi_1(A \cap B) \rangle$$

$$\cong \pi_1(X) / \langle i_{AB}(w) \text{ for } w \in \pi_1(A \cap B) \rangle \quad \text{since } B \simeq *$$

$$\cong \pi_1(X)/N \quad \text{since } \pi_1(A \cap B) \xrightarrow{i_{AB}} \pi_1(A)$$

has image N.

(b) The same construction works, except now with $A \cap B \simeq \vee_\alpha S^{n-1}$, giving

$\pi_1(X$ with n-cells added for $n > 2)$

$\cong \pi_1(X) / \langle i_{AB}(w)$ for $w \in \pi_1(A \cap B) \rangle$ since $B \simeq *$

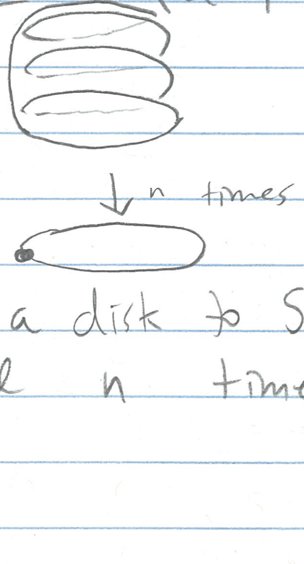
$\cong \pi_1(X)$ since $\pi_1(A \cap B)$ is trivial.

Cor 1.28 For every group G there is a 2-dimensional CW complex X_G with $\pi_1(X_G) \cong G$.

Pf Any group G has a presentation $G = \langle g_\alpha / r_\beta \rangle$ (since any group G is a quotient of a free group F, say $F \xrightarrow{h} G$, so one can take the g_α's to be the generators of F, and the r_β's to be the generators of the kernel of h.)

Construct X_G from $\vee_\alpha S^1$ by attaching 2-cells e^2_β via maps specified by the words r_β.

Ex 1.29 $\mathbb{Z}/n\mathbb{Z} \cong \langle a \mid a^n = 1 \rangle$



Attach a disk to S^1 by wrapping its boundary around n times.

Thm Any connected closed surface is homeomorphic to a sphere, a connected sum of g tori for $g \geq 1$, or a connected sum of k $\mathbb{RP}^2$'s for $k \geq 1$

Pic $T \# T$ (torus connected sum torus)

Ex Klein bottle $\cong \mathbb{RP}^2 \# \mathbb{RP}^2$

Fact $T \# \mathbb{RP}^2 \cong (\text{Klein bottle}) \# \mathbb{RP}^2 \cong \mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2$

Corollary 1.27 and the following paragraph

None of the surfaces in the above classification theorem are homeomorphic to each other, since their fundamental groups have non-isomorphic abelianizations.

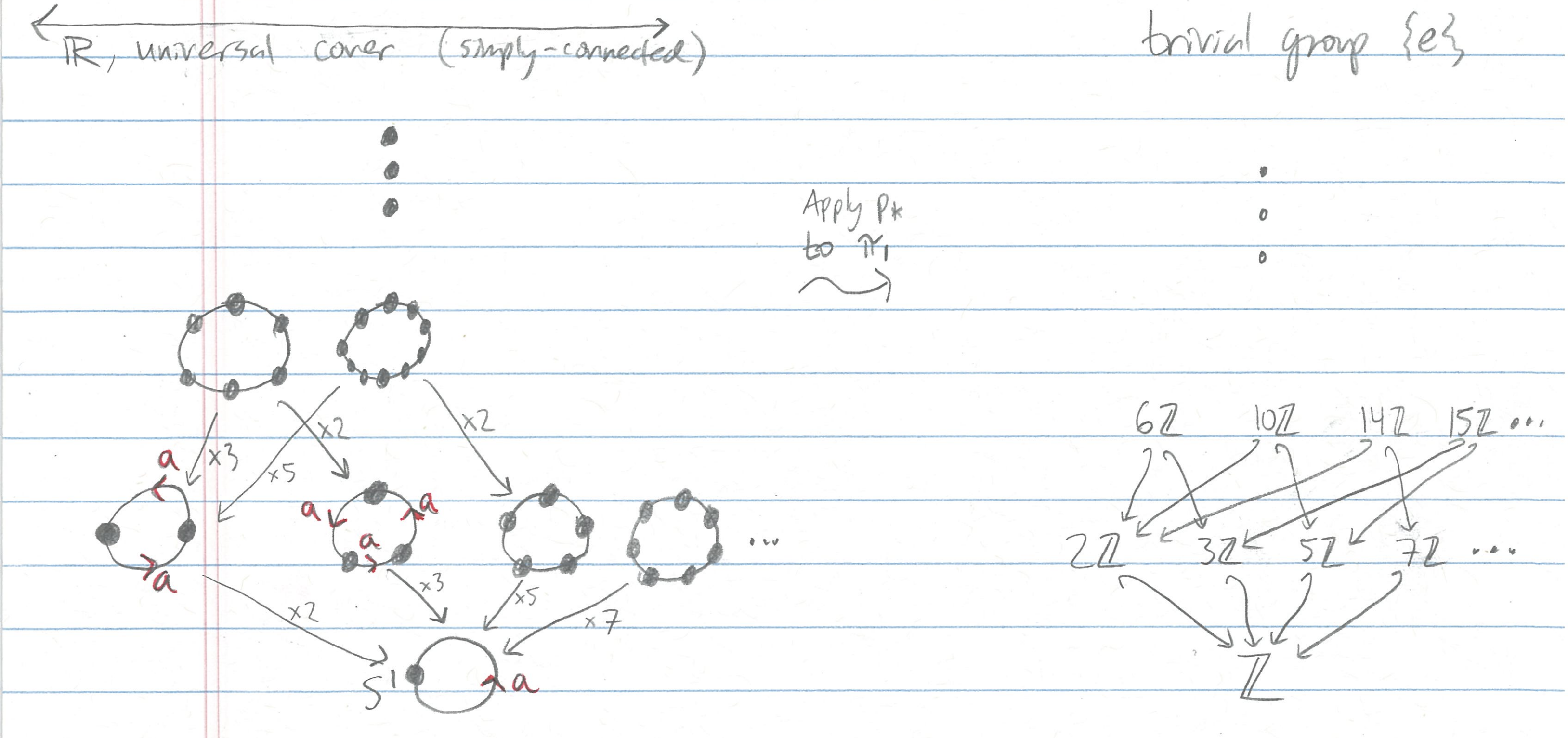
1/31/18 Section 1.3: Covering Spaces

Def (pg 56) A covering space of a space X is a space $\tilde{X}$ together with a map $p: \tilde{X} \to X$ such that

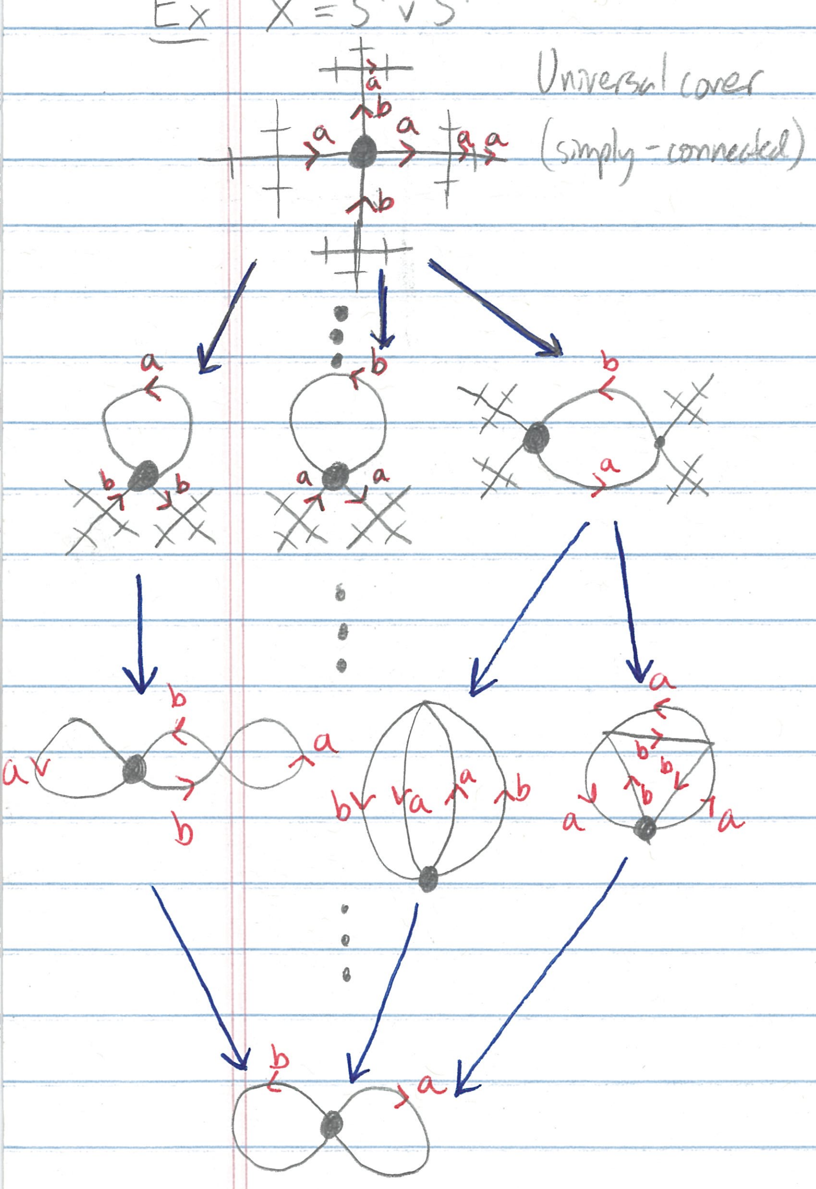
- there exists an open cover $\{U_\alpha\}$ of X such that for each α, $p^{-1}(U_\alpha)$ is a disjoint union of open sets in $\tilde{X}$, each of which is mapped by p homeomorphically onto U_α.

Main Theorem (1.38) Let X be path-connected, locally path-connected, and semilocally simply connected (i.e., "nice enough"). Then there is a bijection between the set of basepoint-preserving isomorphism classes of path-connected covering spaces $p: (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ and the set of subgroups of $\pi_1(X, x_0)$, obtained by associating the covering space $(\tilde{X}, \tilde{x}_0)$ to the subgroup $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.

Ex $X = S^1$



Ex $X = S^1 \vee S^1$



Apply p_* to π_1 $\rightsquigarrow$

$$\begin{tikzcd}
 & \text{trivial group } \{e\} \arrow[dl] \arrow[d] \arrow[dr] & \\
\langle a \rangle \arrow[dd] & \langle b \rangle & \langle ab \rangle \cdots \arrow[ddl] \arrow[dd] \\
 & \vdots & \\
\langle a, b^2, bab^{-1} \rangle \arrow[dr] & \langle a^2, b^2, ab \rangle \arrow[d] & \langle a^3, b^3, ab, ba \rangle \cdots \arrow[dl] \\
 & \langle a, b \rangle &
\end{tikzcd}$$

Recall <u>Prop 1.30 (Homotopy lifting property)</u> Given a covering space $p: \tilde{X} \to X$, a homotopy $f_t: Y \to X$, and a map $\tilde{f}_0: Y \to \tilde{X}$ lifting f_0, $\exists!$ homotopy $\tilde{f}_t: Y \to \tilde{X}$ of $\tilde{f}_0$ that lifts f_t.

$$\begin{tikzcd}
 & \tilde{X} \arrow[d, "p"] \\
Y \arrow[ur, "\tilde{f}_0", "\Rightarrow \exists! \tilde{f}_t"'] \arrow[r, "f_t"'] & X
\end{tikzcd}$$

Prop 1.31 If $p:(\tilde{X},\tilde{x}_0)\to(X,x_0)$ is a covering space, then $p_*:\pi_1(\tilde{X},\tilde{x}_0)\to\pi_1(X,x_0)$ is injective, with image consisting of all based loops in X whose based lifts to $\tilde{X}$ are loops.

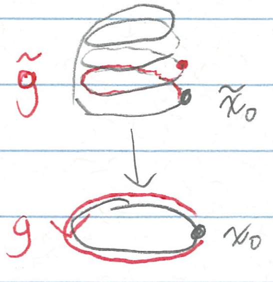
Pf An element of $\ker(p_*)$ is represented by a loop $\tilde{f}_0: I\to\tilde{X}$ with a homotopy $f_t: I\to X$ to the trivial loop f_1. By Prop 1.30 (with $Y=I$), we get a lifted homotopy $\tilde{f}_t$ starting at $\tilde{f}_0$ and ending at the trivial loop $\tilde{f}_1$ in $\tilde{X}$.
Hence $[\tilde{f}_0]$ is the identity in $\pi_1(\tilde{X},\tilde{x}_0)$ and p_* is injective.

To describe $\operatorname{im}(p_*)$, note loops lifting to loops are in the image. Conversely, a loop f_1 in X representing an element of $\operatorname{im}(p_*)$ is homotopic to a loop f_0 (in X) lifting to a loop $\tilde{f}_0$ (in $\tilde{X}$), and hence itself lifts to a loop $\tilde{f}_1$ in $\tilde{X}$ by Prop 1.30 (with $Y=I$).

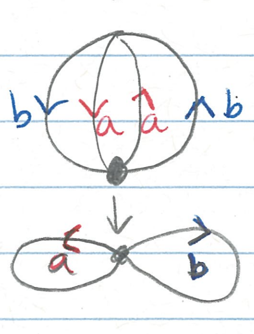
2/2/18

Prop 1.32 The number of sheets of a covering space $p:(\tilde{X},\tilde{x}_0)\to(X,x_0)$ with X and $\tilde{X}$ path-connected equals the index of $p_*(\pi_1(\tilde{X},\tilde{x}_0))$ in $\pi_1(X,x_0)$.

Ex



$4\mathbb{Z}$ has index 4 in $\mathbb{Z}$



$\langle a^2, b^2, ab\rangle$ has index 2 in $\langle a,b\rangle$.

Pf Let $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$

Define $\Phi : \{\text{cosets of } H \text{ in } \pi_1(X, x_0)\} \to p^{-1}(x_0)$ ← # of sheets

by $\qquad H[g] \longmapsto \tilde{g}(1)$

where g is a loop in X based at x_0,

and $\tilde{g}$ is its lift starting at $\tilde{x}_0$.

Φ is well-defined since elements in H lift to loops in $\tilde{X}$!

Φ is surjective since $\tilde{X}$ is path-connected.

(and since any path between $\tilde{x}_0$ and a point in $p^{-1}(x_0)$ projects to a loop in X)

Φ is injective since

$$\begin{aligned} \Phi(H[g_1]) = \Phi(H[g_2]) &\Rightarrow \tilde{g}_1 \cdot \overline{\tilde{g}_2} \text{ is a loop in } \tilde{X} \text{ based at } \tilde{x}_0 \\ &\Rightarrow [g_1][g_2]^{-1} \in H \\ &\Rightarrow [g_1] \in H[g_2] \\ &\Rightarrow H[g_1] = H[g_2] \; \blacksquare \end{aligned}$$

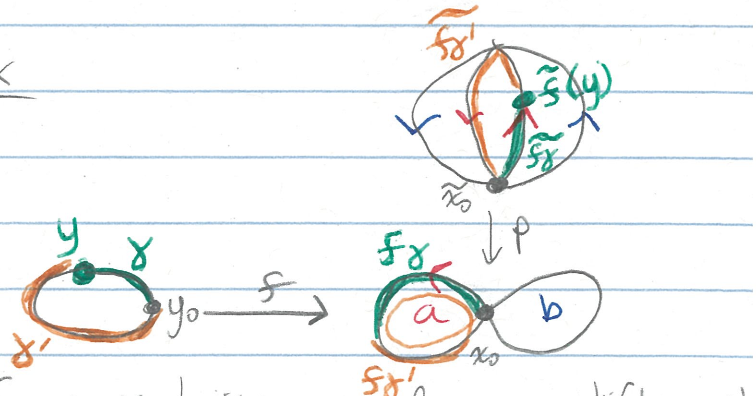
Prop 1.33 Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering space, and $f : (Y, y_0) \to (X, x_0)$ with Y path-connected and locally path-connected. Then a lift $\tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ exists $\iff f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.

Ex 7 shows this is necessary

$\tilde{f}$ exists $\iff$ $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(X, x_0))$

$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\ (Y, y_0) \arrow[ur, dashed] \arrow[r, "f"] & (X, x_0) \end{tikzcd}$$

Ex



If f wraps twice around a, a lift exists.

Not so if f wraps once around a.

Pf ($\Rightarrow$) Clear since $f_* = p_* \tilde{f}_*$.

($\Leftarrow$) Define $\tilde{f}: Y \to \tilde{X}$ by $\tilde{f}(y) = \widetilde{f\gamma}(1)$, where γ is any path in Y from y_0 to y.

(This relies on Prop 1.30, the homotopy lifting property)

To see $\tilde{f}$ is well-defined, note that if γ' is another path from y_0 to y, then

$[(f\gamma') \cdot (\overline{f\gamma})] \in f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$,

so $(f\gamma') \cdot (\overline{f\gamma})$ lifts to a loop in $\tilde{X}$ by Prop 1.31,

so $\widetilde{f\gamma'}(1) = \widetilde{f\gamma}(1)$ $(= \tilde{f}(y))$.

For why $\tilde{f}$ is continuous, see Hatcher.

Prop 1.34 Given a covering space $p: \tilde{X} \to X$ and a map $f: Y \to X$, if two lifts $\tilde{f}_1, \tilde{f}_2: Y \to \tilde{X}$ of f agree at one point of Y and Y is connected, then $\tilde{f}_1 = \tilde{f}_2$.

Pf Point-set topology. You can show the set of all points where $\tilde{f}_1 = \tilde{f}_2$, and also where $\tilde{f}_1 \neq \tilde{f}_2$, are both open.

2/5/18 For all of class today, let X be path-connected, locally path-connected, and semilocally simply-connected (each $x \in X$ has a neighborhood U such that $\pi_1(U,x) \to \pi_1(X,x)$ is trivial.)

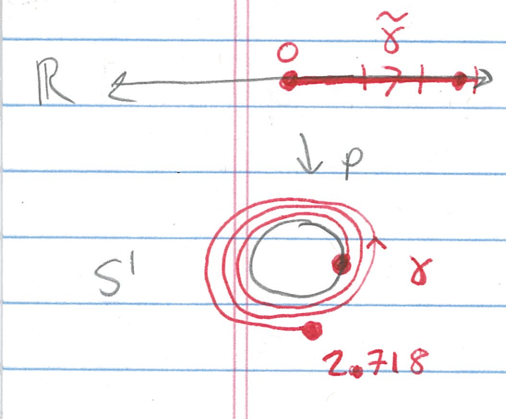
How do we find a covering space $p: (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ with $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = \text{trivial group} \leq \pi_1(X, x_0)$? Recall p_* is injective, so $\tilde{X}$ must be simply-connected.

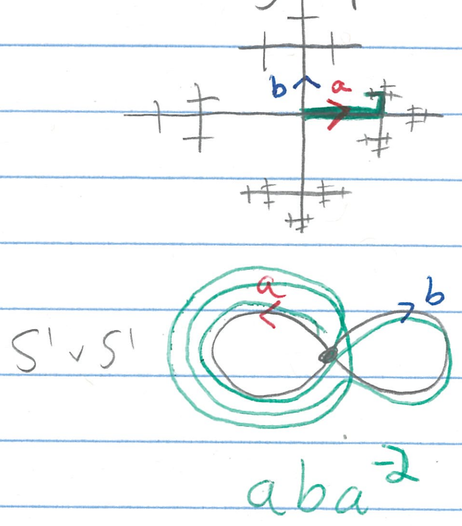
path-connected and trivial π_1

Def (pg 64) The universal cover $\tilde{X}$ of X is defined by $\tilde{X} = \{[\gamma] \mid \gamma \text{ is a path in } X \text{ starting at } x_0\}$.

Homotopy class of paths; these homotopies fix $\gamma(0)$ and $\gamma(1)$

The function $p: \tilde{X} \to X$ is defined by $p[\gamma] = \gamma(1)$.



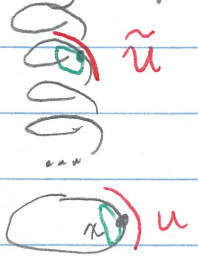


Rmk p is surjective since X is path-connected.

Rmk The basepoint $\tilde{x}_0 \in \tilde{X}$ is the homotopy class of the constant path: $\tilde{x}_0 = [x_0]$.

Rmk Hatcher defines a topology on $\tilde{X}$ and shows that p is a covering space map. His description of a basis for $\tilde{X}$ relies on local path-connectivity and semilocal simple-connectivity of X.

To see the necessity of X being semilocally simply-connected, suppose $p: \tilde{X} \to X$ is a covering space with $\tilde{X}$ simply-connected.



A loop in a local covering set U lifts to a loop in $\tilde{U}$, which is nullhomotopic in $\tilde{X}$, and hence projects to a nullhomotopic loop in X.

$\tilde{X}$ is path-connected For $[\gamma] \in \tilde{X}$, let $\gamma_t : I \to X$ via

$$\gamma_t(s) = \begin{cases} \gamma(s) & 0 \leq s \leq t \\ \gamma(t) & t \leq s \leq 1. \end{cases}$$

Note $t \mapsto [\gamma_t]$ is a path in $\tilde{X}$ starting at the basepoint $\tilde{x}_0 = [x_0]$ and ending at $[\gamma]$.

$\tilde{X}$ is simply-connected Since p_* is injective it suffices to show $p_*(\pi_1(\tilde{X}, [x_0])) = 0$ (in $\pi_1(X, x_0)$).

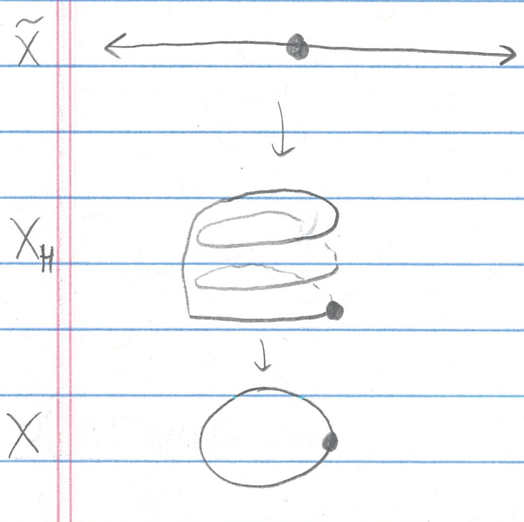
The elements of $\operatorname{im}(p_*)$ are loops in X lifting to loops in $\tilde{X}$.

A loop γ in X lifts to $t \mapsto [\gamma_t]$ in $\tilde{X}$, which is a loop iff $[X_0] = [\gamma_1] = [\gamma]$, iff γ is trivial in $\pi_1(X, x_0)$.

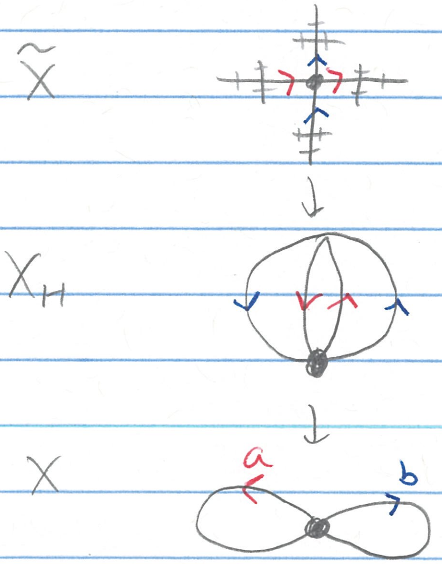
More generally,

Prop 1.36 X is path-connected, locally path-connected, and semilocally simply-connected. Then for every subgroup $H \in \pi_1(X, x_0)$ there is a covering space $p: (X_H, \tilde{x}_0) \to (X, x_0)$ such that $p_*(\pi_1(X_H, \tilde{x}_0)) = H$.

Ex



$H = 3\mathbb{Z} \subseteq \mathbb{Z}$



$H = \langle a^2, b^2, ab \rangle \subseteq \langle a, b \rangle$

Pf Let $\tilde{X}$ be the universal cover.
For $[\gamma], [\gamma'] \in \tilde{X}$, let $[\gamma] \sim [\gamma']$ mean $\gamma(1) = \gamma'(1)$ and $[\gamma \cdot \bar{\gamma}'] \in H$.
This is an equivalence relation since H is a subgroup:
- reflexive since $\mathrm{id} \in H$
- symmetric since H contains inverses
  $([\gamma \cdot \bar{\gamma}'] \in H \Rightarrow [\gamma' \cdot \bar{\gamma}] = [\gamma \cdot \bar{\gamma}']^{-1} \in H)$
- transitive since H is closed under multiplication
  $([\gamma \cdot \bar{\gamma}'] \in H, [\gamma' \cdot \bar{\gamma}''] \in H \Rightarrow [\gamma \cdot \bar{\gamma}''] = [\gamma \cdot \bar{\gamma}'][\gamma' \cdot \bar{\gamma}''] \in H)$

Let X_H be the quotient space $X_H = \tilde{X} / \sim$.

Again one can check $p: X_H \to X$ induced by $[\gamma] \mapsto \gamma(1)$ is a covering space.

We have $p_*(\pi_1(X_H, \tilde{x}_0)) = H$ since a loop γ in X lifts to a path in $\tilde{X}$ ending at $[\gamma]$, with $\underbrace{[\gamma] \sim [x_0]}_{\text{i.e. this is a loop in } X_H} \iff [\gamma] \in H$.

2/6/2018

Main Theorem (1.38) Let X be path-connected, locally path-connected, and semilocally simply-connected. Then there is a bijection

$$\left\{\begin{array}{c}\text{based isomorphism classes} \\ \text{of covering spaces} \\ p: (\tilde{X}, \tilde{x}_0) \to (X, x_0)\end{array}\right\} \xrightarrow{\Phi} \left\{ \text{subgroups of } \pi_1(X, x_0) \right\}$$

$$(X, x_0) \xmapsto{\Phi} p_*(\pi_1(\tilde{X}, x_0))$$

Last time we showed surjectivity; this time well-definedness & injectivity.

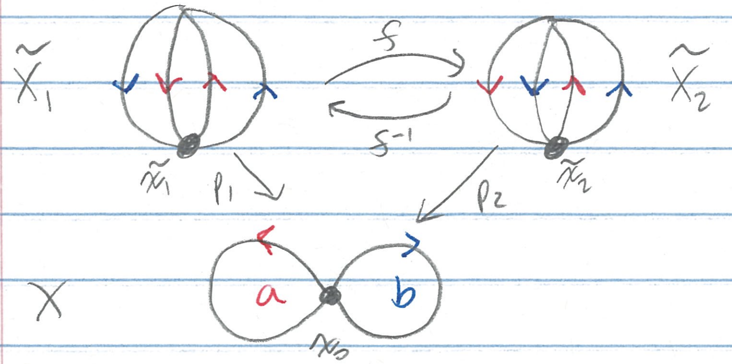
Def A based isomorphism of covering spaces $p_1: (\tilde{X}_1, \tilde{x}_1) \to (X, x_0)$ and $p_2: (\tilde{X}_2, \tilde{x}_2) \to (X, x_0)$ is a homeomorphism $f: (\tilde{X}_1, \tilde{x}_1) \to (\tilde{X}_2, \tilde{x}_2)$ such that $p_2 f = p_1$.

Picture

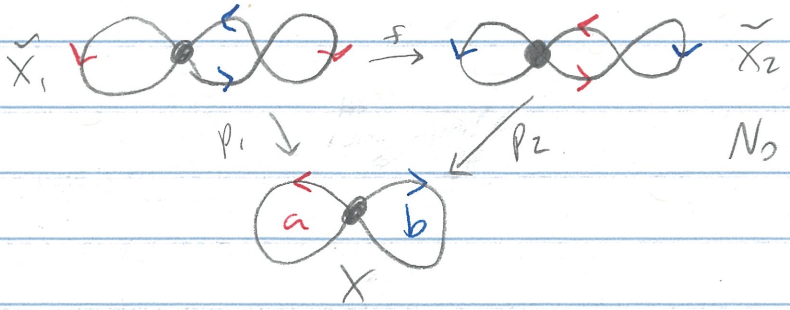
$$\begin{tikzcd} \tilde{X}_1 \arrow[rr, shift left, "f"] \arrow[dr, "p_1"'] & & \tilde{X}_2 \arrow[ll, shift left, "f^{-1}"] \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

Rmk This implies $p_1 f^{-1} = p_2$.

Ex



Non-Ex It's possible for $\tilde{X}_1$ and $\tilde{X}_2$ to be homeomorphic but not isomorphic:



No commutativity!

$p_1 \neq p_2 f$

Proof of Main Theorem (1.38)

Φ is well defined since $\tilde{X}_1$ isomorphic to $\tilde{X}_2$

$\Rightarrow p_2 f = p_1$ and $p_1 f^{-1} = p_2$

$\Rightarrow \operatorname{im}(p_{1*}) \subseteq \operatorname{im}(p_{2*})$ and $\operatorname{im}(p_{2*}) \subseteq \operatorname{im}(p_{1*})$.

So $p_{1*}(\pi_1(\tilde{X}_1, \tilde{x}_1)) = p_{2*}(\pi_1(\tilde{X}_2, \tilde{x}_2))$.

Φ is surjective by last time

Φ is injective. Suppose $p_{1*}(\pi_1(\tilde{X}_1, \tilde{x}_1)) = p_{2*}(\pi_1(\tilde{X}_2, \tilde{x}_2))$.

$$\begin{tikzcd}
 & \tilde{X}_2 \arrow[d, "p_2"] \\
\tilde{X}_1 \arrow[ur, dashed, "\exists ! f"] \arrow[r, "p_1"'] \arrow[ur, phantom, shift right=3, "\circlearrowright"] & X
\end{tikzcd}$$

By the lifting criterion (1.33), $\subseteq$ gives a lift $f : \tilde{X}_1 \to \tilde{X}_2$ with $p_2 f = p_1$.

$$\begin{tikzcd} & \tilde{X}_1 \arrow[d, "p_1"] \\ \tilde{X}_2 \arrow[ur, dashed, "\exists! g", "\circlearrowright"'] \arrow[r, "p_2"'] & X \end{tikzcd}$$

Similarly, $\supseteq$ gives a lift $g : \tilde{X}_2 \to \tilde{X}_1$ with $p_1 g = p_2$.

To see $gf = \mathbb{1}_{\tilde{X}_1}$, note $p_1 g f = p_2 f = p_1$.

$$\begin{tikzcd} & \tilde{X}_1 \arrow[d, "p_1"] \\ \tilde{X}_1 \arrow[ur, dashed, "\exists! \mathbb{1}_{\tilde{X}_1} = gf", "\circlearrowright"'] \arrow[r, "p_1"'] & X \end{tikzcd}$$

To see $fg = \mathbb{1}_{\tilde{X}_2}$, note $p_2 f g = p_1 g = p_2$.

$$\begin{tikzcd} & \tilde{X}_2 \arrow[d, "p_2"] \\ \tilde{X}_2 \arrow[ur, dashed, "\exists! \mathbb{1}_{\tilde{X}_2} = fg", "\circlearrowright"'] \arrow[r, "p_2"'] & X \end{tikzcd}$$

Hence $g = f^{-1}$, and f is an isomorphism!

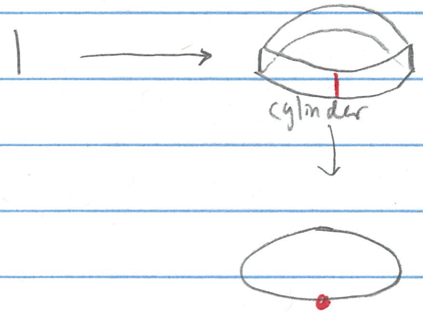
$$\begin{tikzcd} \tilde{X}_1 \arrow[rr, bend left, "f"] \arrow[dr, "p_1"'] & & \tilde{X}_2 \arrow[ll, bend left, "g = f^{-1}"] \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

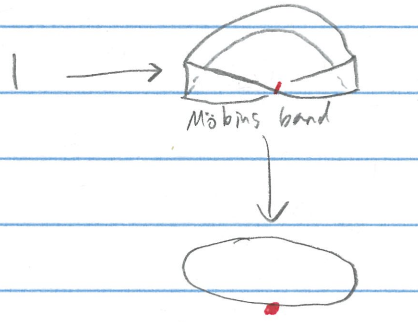
(This is analogous to our proof last semester that if limits exist in a category, then they are unique up to isomorphism.)

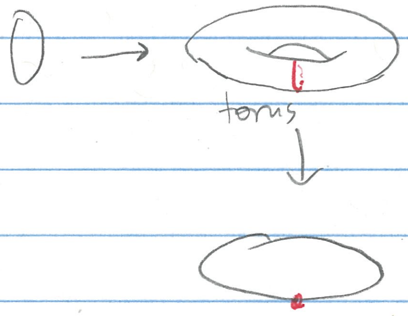
Covering spaces are the simplest examples of "fiber bundles"

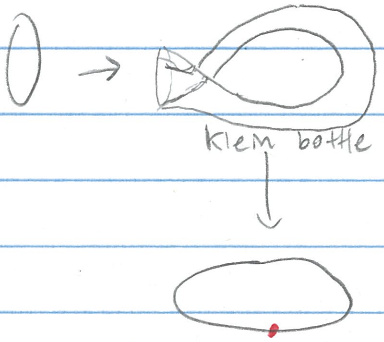
$$\begin{tikzcd} F \arrow[r] & E \arrow[d] \\ & B \end{tikzcd} \qquad \begin{tikzcd} \text{fiber} \arrow[r] & \text{total space} \arrow[d] \\ & \text{base space} \end{tikzcd}$$

Ex

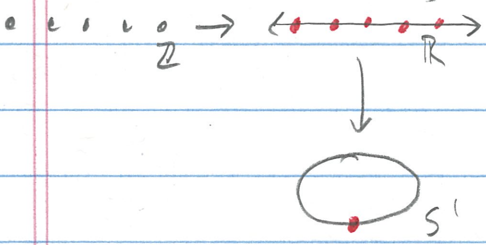


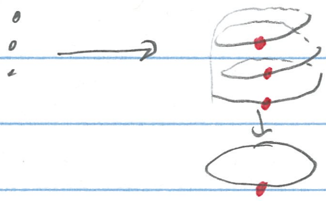






In a covering space, the fibers are discrete





Ex Hopf fibration

$$\begin{tikzcd} S^1 \arrow[r] & S^3 \arrow[d] \\ & S^2 \end{tikzcd}$$

Given a fiber bundle, there exsts a long exact sequence of homotopy groups

$$\cdots \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \cdots$$

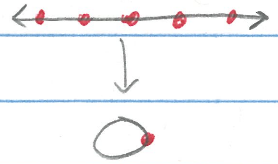
Ex

$$\to \underset{\overset{\cong}{0}}{\cancel{\pi_3}(S^1)} \to \underset{\overset{\cong}{\mathbb{Z}}}{\pi_3(S^3)} \to \underset{\Rightarrow \overset{\cong}{\mathbb{Z}}}{\pi_3(S^2)} \to \underset{\overset{=}{0}}{\cancel{\pi_2}(S^1)} \to$$

2/9/2018 Deck transformations and group actions

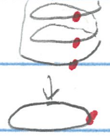
Def The <u>deck transformations</u> $G(\tilde{X})$ of a covering space $p: \tilde{X} \to X$ are the group of isomorphisms $\tilde{X} \to \tilde{X}$ (ignoring basepoints).

Ex



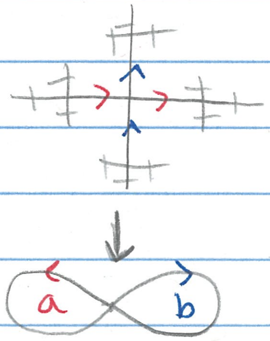
$G(\tilde{X}) \cong \mathbb{Z}$

$H = \{e\}$ normal



$G(\tilde{X}) \cong \mathbb{Z}/3\mathbb{Z}$

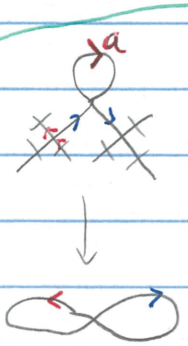
$H = 3\mathbb{Z}$ normal



$G(\tilde{X}) \cong \langle a, b \rangle$

$H = \{e\}$ normal

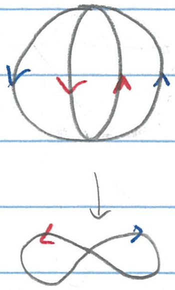
Normal



Not normal

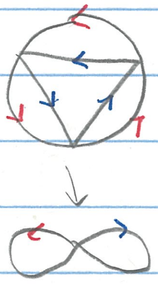
$G(\tilde{X}) \cong \{e\}$

$H = \langle a \rangle$ not normal and $N(H) = H$



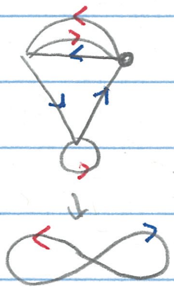
$G(\tilde{X}) \cong \mathbb{Z}/2\mathbb{Z}$

$H = \langle a^2, b^2, ab \rangle$ normal



$G(\tilde{X}) \cong \mathbb{Z}/3\mathbb{Z}$

$H = \langle a^3, b^3, ab, ba \rangle$ normal



$G(\tilde{X}) \cong \{e\}$

$H = \langle b^3, ba, ba^{-1}, b^{-1}ab \rangle$ not normal and $N(H) = H$.

Def A covering space $p: \tilde{X} \to X$ is <u>normal</u> if for each $x \in X$ and $\tilde{x}, \tilde{x}' \in \tilde{X}$, there is a deck transformation taking $\tilde{x}$ to $\tilde{x}'$.

Rmk Normal covering spaces "exhibit maximal symmetry".

Prop 1.39 Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a path-connected covering space of the path-connected, locally path-connected space X. Let $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$. Then

(a) $\tilde{X}$ is normal $\iff$ H is normal in $\pi_1(X, x_0)$.

(b) $G(\tilde{X}) \cong \pi_1(X, x_0)/H$ if $\tilde{X}$ is normal (and more generally $G(\tilde{X}) \cong N(H)/H$).

The normalizer of H in G, i.e. the largest subgroup $H \subseteq K \subseteq G$ with $H \triangleleft K$.

Rmk For the universal cover we have $G(\tilde{X}) \cong \pi_1(X)$.

Pf Sketch (a)

$$\begin{tikzcd}
(\tilde{X}, \tilde{x}_0) \arrow[rr, dashed, bend left, "f"] \arrow[dr, "p"'] & & (\tilde{X}, \tilde{x}_0') \arrow[ll, dashed, bend left, "f^{-1}"] \arrow[dl, "p"] \\
& (X, x_0) &
\end{tikzcd}$$

A deck transformation $f \in G(\tilde{X})$ exists

$\iff$ $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = p_*(\pi_1(\tilde{X}, \tilde{x}_0'))$

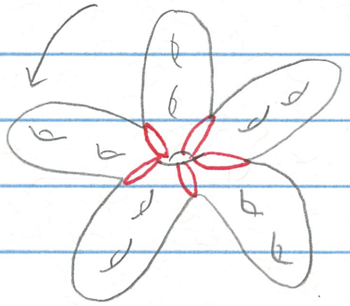
$\iff$ for a loop γ in X lifting to a path $\tilde{\gamma}$ in $\tilde{X}$ from $\tilde{x}_0$ to $\tilde{x}_0'$, we have $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = [\gamma]^{-1} \, p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \, [\gamma]$

The above is true for all $\tilde{x}_0' \in p^{-1}(x_0)$, giving (a).

Above we started with a covering space $\tilde{X}$ and produced a group $G(\tilde{X})$ acting on that space. What if we instead start with a group G acting on a space Y — is Y then a covering space?

Ex 1.41 $G \curvearrowright Y$

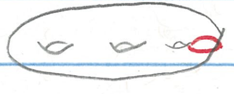
$G = \mathbb{Z}/5\mathbb{Z}$



$Y \cong M_{11}$ is the genus-11 torus

§2.2 Ex 22 For X a finite CW complex and $p: \tilde{X} \to X$ a covering space, we have $\chi(\tilde{X}) = n\chi(X)$ where $n = \#$ sheets.

"orbit space" or "quotient space"
$y \sim g \cdot y$ $\forall y \in Y$ and $g \in G$



$Y/(\mathbb{Z}/5\mathbb{Z}) \cong M_3$ is the genus-3 torus

§2.2 Ex 23 M_g is a covering space of M_h $\iff$ $g = n(h-1)+1$ for some n.

Def Group G acts on space Y properly discontinuously [(*) on Hatcher pg 72] if $\forall y \in Y$ $\exists$ an open neighborhood $y \in U$ such that $g_1(U) \cap g_2(U) \neq \emptyset \Rightarrow g_1 = g_2$ $(\forall g_1, g_2 \in G)$.

2/12/18

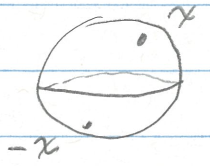
Prop 1.40 If $G \curvearrowright Y$ is properly discontinuous with Y path-connected and locally path-connected, then

(a) $p: Y \to Y/G$ $(p(y) = Gy)$ is a normal covering space.
(b) with G as its group of deck transformations
(c) and with $G \cong \pi_1(Y/G)/p_*(\pi_1(Y))$

Rmk (c) follows since

$$G \underset{\substack{\uparrow \\ (b)}}{\cong} \text{group of deck transformations} \underset{\substack{\uparrow \\ \text{Prop 1.39 (b) since the covering space is normal.}}}{\cong} \pi_1(Y/G)/p_*(\pi_1(Y))$$

Ex 1.43 $\mathbb{Z}/2\mathbb{Z} \curvearrowright S^2$ via the antipodal map.



$Y = S^2$

$\downarrow$

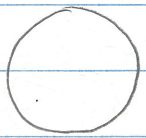
$Y/G = S^2/(\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{R}P^2$

(b) says that $\mathbb{Z}/2\mathbb{Z}$ is the group of deck transformations

(c) says $\mathbb{Z}/2\mathbb{Z} \cong \pi_1(\mathbb{R}P^2) / \underbrace{p_*(\pi_1(S^2))}_{\overset{\shortparallel}{\{e\}}} \cong \pi_1(\mathbb{R}P^2)$

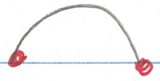
Rmk Everything above holds when 2 is replaced by $n \geq 2$, giving $\mathbb{Z}/2\mathbb{Z} \cong \pi_1(\mathbb{R}P^n)$ for $n \geq 2$.

Rmk $n = 1$ $\mathbb{Z}/2\mathbb{Z} \cong \underbrace{\pi_1(\mathbb{R}P^1)}_{\mathbb{Z}} / \underbrace{p_*(\pi_1(S^1))}_{2\mathbb{Z}} \cong \mathbb{Z}/2\mathbb{Z}$



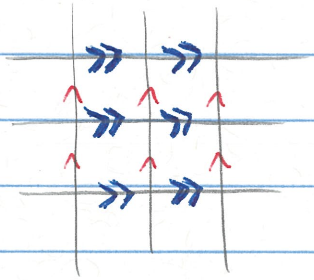
$Y = S^1$

$\downarrow$



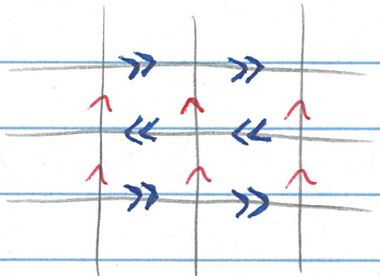
$Y/G = S^1/(\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{R}P^1$

Ex 1.42 $G \curvearrowright Y$ $Y = \mathbb{R}^2$ with decorated lines



$G \cong \mathbb{Z} \times \mathbb{Z}$

Y/G is the torus



$G \cong \langle a, b \mid abab^{-1} \rangle$ $ab = ba^{-1}$

a: translation to right

b: glide reflection up

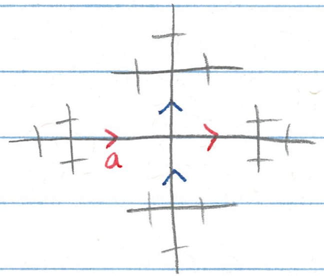
Y/G is the Klein bottle

There's a subgroup $H \subseteq G$ with $H \cong \mathbb{Z} \times \mathbb{Z}$ generated by a, b^2 such that Y/H is a torus.

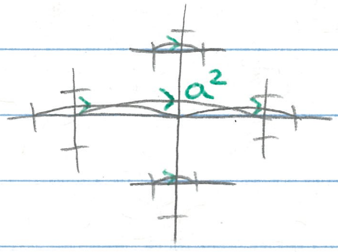
$$\begin{tikzcd}
Y \arrow[r] & Y/H \arrow[r] \arrow[d, phantom, "\shortparallel"] & Y/G \arrow[d, phantom, "\shortparallel"] \\
& \text{torus} & \text{Klein bottle}
\end{tikzcd}$$

Cayley graphs Let $G = \langle g_\alpha \mid r_\beta \rangle$ be a presentation of a group. Form a graph with vertex set G, and an edge $[g, gg_\alpha]$ $\forall g \in G$, $\forall \alpha$.

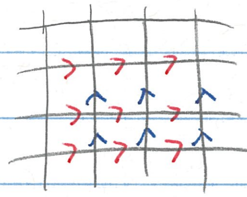
Ex $G = \langle a, b \rangle$



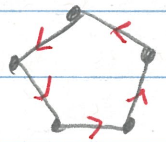
Ex $G = \langle a, b, a^2 \rangle$



Ex $G = \langle a, b \mid aba^{-1}b^{-1} \rangle$



Ex $G = \langle a \mid a^5 \rangle$



Rmk G acts on its Cayley graph.

Introduction to geometric group theory

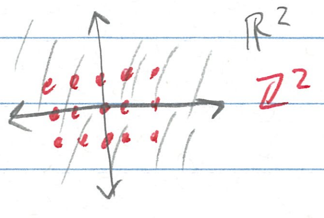
Fact Two different finite presentations of the same group have quasi-isometric Cayley graphs (with the shortest path metric)

Def Metric spaces X, Y are quasi-isometric if $\exists$ $f: X \to Y$ and constants K, C such that

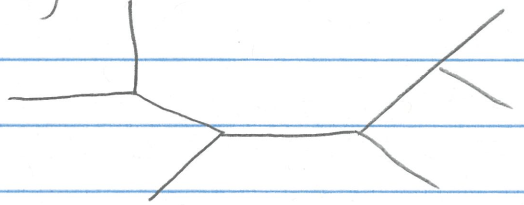
$$\frac{1}{K} d_X(x,x') - C \leq d_Y(f(x), f(x')) \leq K d_X(x,x') + C \quad \forall x, x' \in X$$

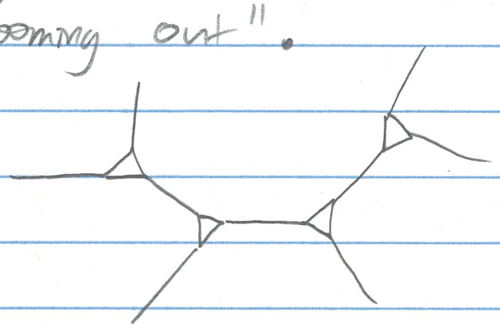
Ex $\langle a, b \rangle$ and $\langle a, b, a^2 \rangle$ have quasi-isometric Cayley graphs

Ex $\mathbb{R}^2$ and $\mathbb{Z}^2$ are quasi-isometric



Ex The two below graphs are quasi-isometric: they look the same after "zooming out".

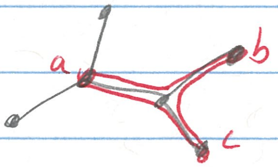




Def (Gromov) A geodesic metric space X is δ-hyperbolic if all triangles are δ-thin:

Note ac is contained in these two δ-nbhds. δ-thin means we have this for all three sides!

Ex Trees are 0-thin



Ex The hyperbolic plane and nice quotients thereof (orientable surfaces of genus $g \geq 2$)

Ex "Almost every group is hyperbolic". Few-relator model of random groups: fix $m \geq 2$ and $k \geq 1$. Then $\langle g_1, \dots, g_m \mid r_1, \dots, r_k \rangle$ is hyperbolic with probability 1 as $l \to \infty$, where each relation is selected uniformly at random from all words of length $\leq l$ in g_i, g_i^{-1}.

The word problem asks: given a finite presentation $\langle g_1, \ldots, g_m \mid r_1, \ldots, r_k \rangle$, is there an algorithm to decide if a word in this group is the identity?

In general no: this problem is undecidable for arbitrary groups.

But for hyperbolic groups, there is a linear time algorithm to the word problem!

2/14/18

Chapter 2: Homology

Section 2.1 Simplicial and Singular Homology

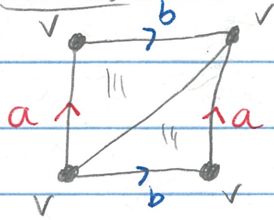
Δ-Complexes

We have inclusions

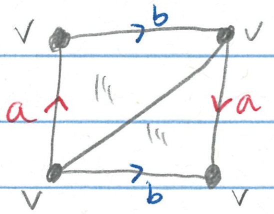
$$\text{Simplicial complexes} \subseteq \Delta\text{-complexes} \subseteq \text{CW complexes}$$

Ex The following are Δ-complexes but not simplicial complexes:

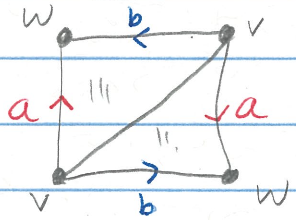
Torus



Klein bottle



$\mathbb{R}P^2$

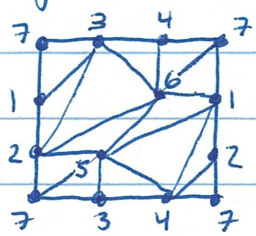


These have one 0-simplex, three 1-simplices, and two 2-simplices.

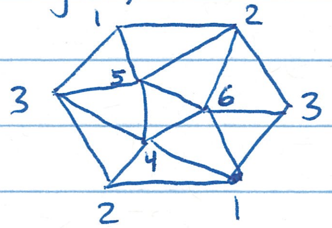
EDIT: $\mathbb{R}P^2$ above has two 0-simplices!

By contrast, in a simplicial complex each simplex is uniquely identified by its vertices.

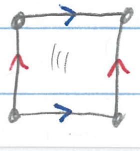
A simplicial complex for a torus needs at least 7 vertices, 21 edges, 14 triangles.

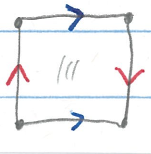


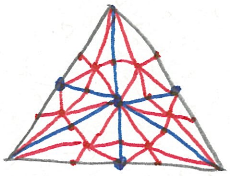
A simplicial complex for $\mathbb{R}P^2$ needs at least 6 vertices, 15 edges, 10 triangles.



By contrast, CW complex models have fewer (non-simplex) cells.







Aside Ex 23 on page 133 of Hatcher shows the second barycentric subdivision of a Δ-complex is a simplicial complex.

Aside Thm 2C.5 shows every CW complex is homotopy equivalent to a simplicial complex.

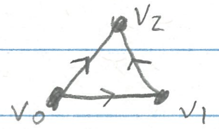
Rmk The attaching maps are simplest for simplicial complexes, more complicated for Δ-complexes, and the most general for CW complexes. This makes simplicial homology (for simplicial or Δ-complexes) easier than cellular homology (for CW complexes).

An "n-simplex" Δ^n means an "n-simplex with ordered vertices"

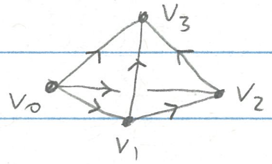
0-simplex

1-simplex

2-simplex



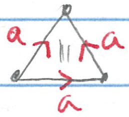
3-simplex



Def $\mathring{\Delta}^n = \Delta^n - \partial\Delta^n$ is the "open simplex"

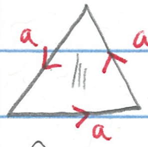
Def A Δ-complex structure on topological space X is a collection of maps $\sigma_\alpha : \Delta^n \to X$ such that

- $\sigma_\alpha|_{\mathring{\Delta}^n}$ is injective and their images partition X.
- Restricting $\sigma_\alpha : \Delta^n \to X$ to a face gives some $\sigma_\beta : \Delta^{n-1} \to X$, where $\Delta^{n-1} \subseteq \Delta^n$ is order-preserving.
- $A \subseteq X$ is open $\iff$ $\sigma_\alpha^{-1}(A)$ is open in Δ^n $\forall \alpha$.

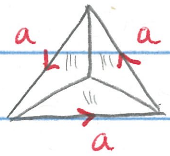
Ex The dunce hat 

= 

is a Δ-complex.

Ex 

is not a Δ-complex since the restricted face maps are not order-preserving.

Ex 

is a Δ-complex.

Aside The dunce cap is contractible (we've attached a disk to a circle via a map homotopy equivalent to a map wrapping around once).
However, it's not collapsible (no free face).
Zeeman Conjecture If a 2-complex X is contractible, then $X \times I$ is collapsible. (This is true for the dunce cap.)

Surprisingly, Zeeman's conjecture implies the...
Poincaré Conjecture Every simply-connected closed 3-manifold is homeomorphic to the three-sphere. ↑ compact w/o boundary.

What's known about Zeeman's conjecture:
- Replacing I by I^6 works.
- Perelman's proof of the Poincaré conjecture shows Zeeman's is true for "standard 2-polyhedra that are splines of 3-manifolds".
- X can be of arbitrarily high dimension if I is replaced by I^n for some n.

2/16/18

Simplicial Homology

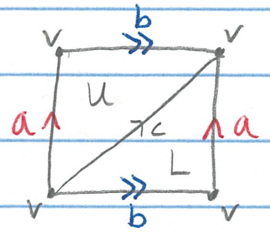
Let X be a Δ-complex (more general than a simplicial complex).
Let $\Delta_n(X)$ be the group of n-chains, the free abelian group with basis the n-simplices of X:

$$\Delta_n(X) = \{\textstyle\sum_\alpha n_\alpha \sigma_\alpha \mid \sigma_\alpha : \Delta^n \to X,\ n_\alpha \in \mathbb{Z},\ \text{finitely many coefficients nonzero}\}.$$

Rmk Hatcher reserves $C_n(X)$ for the singular n-chains.

Rmk Recall each $\sigma_\alpha : \Delta^n \to X$ is oriented.

Ex 2.3 X = torus

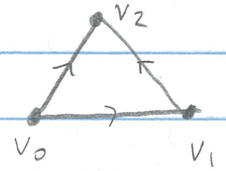


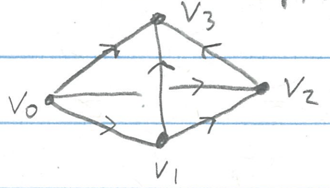
$\Delta_0(X) \cong \mathbb{Z}$
$\Delta_1(X) \cong \mathbb{Z}^3$
$\Delta_2(X) \cong \mathbb{Z}^2$

Let the boundary homomorphism $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$ be defined by $\partial_n(\sigma_\alpha) = \sum_{i=0}^{n} (-1)^i \sigma_\alpha|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$ (and extending linearly).

↑ leave out

Here $v_0, \dots, v_n$ are the vertices of the n-simplex





Ex 2.3 $\partial_0(v) = 0$
$\partial_1(a) = v - v = 0$
$\partial_1(b) = v - v = 0$
$\partial_1(c) = v - v = 0$

$\partial_2(U) = b - c + a = a + b - c$
$\partial_2(L) = a - c + b = a + b - c$

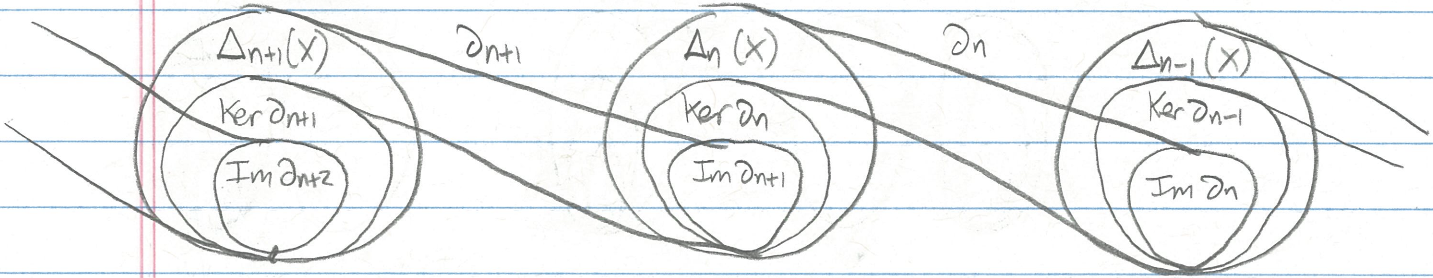
Lemma 2.1 The composition $\partial_n \circ \partial_{n+1} : \Delta_{n+1}(X) \to \Delta_{n-1}(X)$ is zero $\forall n$.

Hence we get a chain complex

$$\dots \to \Delta_{n+1}(X) \xrightarrow{\partial_{n+1}} \Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \to \dots \to \Delta_1(X) \xrightarrow{\partial_1} \Delta_0(X) \xrightarrow{\partial_0} 0$$

Note $\partial_n \partial_{n+1} = 0 \Rightarrow \overset{\text{"n-boundaries"}}{\operatorname{Im}\partial_{n+1}} \subseteq \overset{\text{"n-cycles"}}{\operatorname{Ker}\partial_n}$

Hence we can define the n^{th} simplicial homology group as $H_n^\Delta(X) = \operatorname{Ker}\partial_n / \operatorname{Im}\partial_{n+1}$.



Two n-cycles representing the same element of $H_n^\Delta(X)$ are homologous.

Ex 2.3

$$\begin{tikzcd} \Delta_3(X) \arrow[r, "\partial_3"] \arrow[d, phantom, "\cong"] & \Delta_2(X) \arrow[r, "\partial_2"] \arrow[d, phantom, "\cong"] & \Delta_1(X) \arrow[r, "\partial_1"] \arrow[d, phantom, "\cong"] & \Delta_0(X) \arrow[r, "\partial_0"] \arrow[d, phantom, "\cong"] & 0 \\ 0 & \mathbb{Z}^2 & \mathbb{Z}^3 & \mathbb{Z} & \end{tikzcd}$$

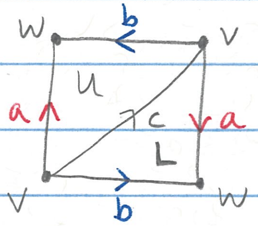
$\operatorname{Ker}\partial_0$ has basis v and $\operatorname{Im}\partial_1 = 0 \Rightarrow H_0^\Delta(X) \cong \mathbb{Z}$.

$\operatorname{Ker}\partial_1$ has basis $\{a,b,c\}$ or $\{a, b, a+b-c\}$ and $\operatorname{Im}\partial_2$ has basis $a+b-c$ $\Rightarrow H_1^\Delta(X) \cong \mathbb{Z}^2$.

$\operatorname{Ker}\partial_2$ has basis $U-L$ and $\operatorname{Im}\partial_3$ has basis 0
$\Rightarrow H_2^\Delta(X) \cong \mathbb{Z}$.

$\uparrow$ Proof $\partial_2(pU+qL) = p(a+b-c)+q(a+b-c) = 0$ if and only if $q=-p$.

Ex 2.4 $X = \mathbb{R}P^2$



$$\begin{tikzcd}
\Delta_3(X) \arrow[r, "\partial_3"] \arrow[d, phantom, sloped, "\cong"] & \Delta_2(X) \arrow[r, "\partial_2"] \arrow[d, phantom, sloped, "\cong"] & \Delta_1(X) \arrow[r, "\partial_1"] \arrow[d, phantom, sloped, "\cong"] & \Delta_0(X) \arrow[r, "\partial_0"] \arrow[d, phantom, sloped, "\cong"] & 0 \\
0 & \mathbb{Z}^2 & \mathbb{Z}^3 & \mathbb{Z}^2 &
\end{tikzcd}$$

$u \longmapsto b-a+c$
$\quad = -a+b+c$

$L \longmapsto a-b+c$

$a \longmapsto w-v$
$b \longmapsto w-v$
$c \longmapsto v-v=0$

$v \longmapsto 0$
$w \longmapsto 0$

$\operatorname{Ker}\partial_0$ has basis $\{v,w\}$ or $\{v, w-v\}$ and
$\operatorname{Im}\partial_1$ has basis $w-v$ $\Rightarrow$ $H_0^\Delta(X) \cong \mathbb{Z}$

$\operatorname{Ker}\partial_1$ has basis $\{a-b, c\}$ or $\{a-b+c, c\}$
$\operatorname{Im}\partial_2$ has basis $\{-a+b+c, a-b+c\}$ or $\{a-b+c, \underset{(-a+b+c)+(a-b+c)}{\underset{\shortparallel}{2c}}\}$
Hence $H_1^\Delta(X) \cong \mathbb{Z}/2\mathbb{Z}$

$\operatorname{Ker}\partial_2 = 0 \quad \Rightarrow \quad H_2^\Delta(X) \cong 0.$

Rmk for later With $\mathbb{Z}/2\mathbb{Z}$ instead of $\mathbb{Z}$ coefficients, we'd get
$\partial_2(u) = \partial_2(L)$ and hence
$H_0^\Delta(X;\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \quad H_1^\Delta(X;\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \quad H_2^\Delta(X;\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}.$

Rmk An algorithm for deducing $H_*^\Delta(X;\mathbb{Z}/p\mathbb{Z})$ from $H_*^\Delta(X;\mathbb{Z})$ is given in Corollary 3A.6 (b) of the universal coefficient theorem for homology (pg 266)

2/19/18

Singular Homology

A singular n-simplex in topological space X is a map $\sigma:\Delta^n\to X$.

Let $C_n(X)$ be the group of n-chains

$$C_n(X)=\{\text{finite sums } \textstyle\sum_\alpha n_\alpha\sigma_\alpha \mid \sigma_\alpha:\Delta^n\to X,\ n_\alpha\in\mathbb{Z}\}$$

As before, $\partial_n:C_n(X)\to C_{n-1}(X)$ via $\partial_n(\sigma)=\sum_{i=0}^{n}(-1)^i\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}$.

$$\cdots\to C_{n+1}(X)\xrightarrow{\partial_{n+1}}C_n(X)\xrightarrow{\partial_n}C_{n-1}(X)\to\cdots$$

As before $\partial^2=0$, meaning we have a chain complex.

The n^{th} singular homology group is $H_n(X)=\underset{\substack{\uparrow\\ \text{cycles}}}{\operatorname{Ker}\partial_n}/\underset{\substack{\uparrow\\ \text{boundaries}}}{\operatorname{Im}\partial_{n+1}}$.

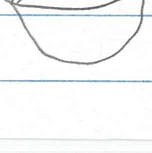
Rmk A disadvantage of singular homology is that $C_n(X)$ often has an uncountable basis, and it's not even clear that $H_n(X)$ is finitely generated for X a finite CW complex (though that turns out to be true.)

Rmk An advantage of singular homology is that homeomorphism invariance is clear. (In fact we have homotopy invariance: Corollary 2.11)

Rmk For X a Δ-complex, $H_n^\Delta(X)\cong H_n(X)$. (Thm 2.27)

Rmk Singular homology can be regarded as a special case of simplicial homology: for X an arbitrary space, let the singular complex S(X) be the Δ-complex with one n-simplex $\sigma:\Delta^n\to S(X)$ for each singular n-simplex $\sigma:\Delta^n\to X$.

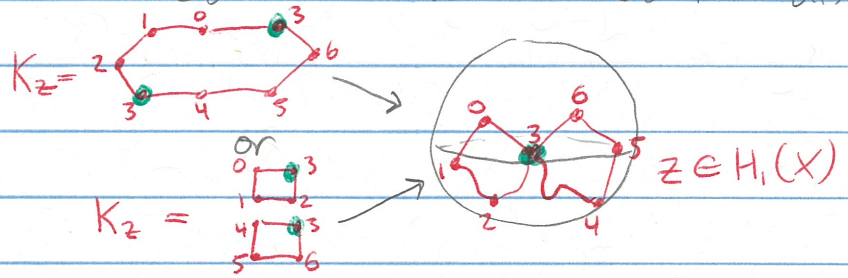
Ex $X=S^2$



S(X) is a Δ-complex with an uncountable # of n-simplices ∀n.

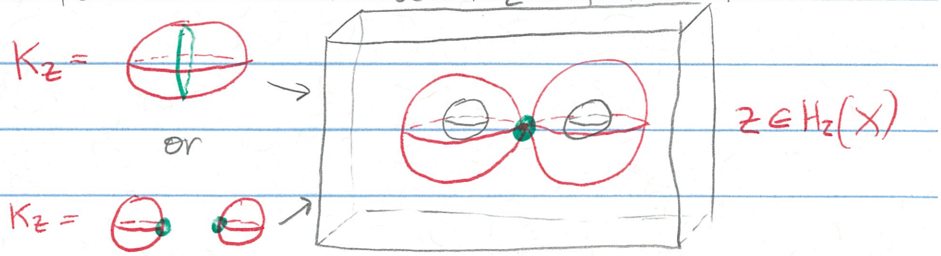
Rmk $H_n(X)$ is defined algebraically, and this is the "right" approach. Geometric reinterpretations work best for $n \leq 3$:

An n-cycle $z \in \mathrm{Ker}\, \partial_n$ can be written $z = \sum_i \varepsilon_i \sigma_i$ with $\varepsilon_i = \pm 1$ and the σ_i not distinct.

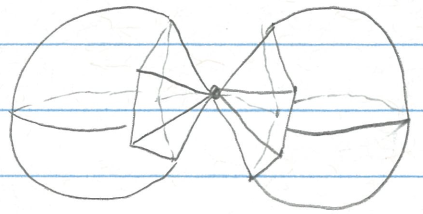


$\partial_n z = 0$ means we can build a Δ-complex K_z by (arbitrarily) choosing canceling pairs of $(n-1)$ faces to glue together.

Fact For $n = 0,1,2$ an n-cycle z can be interpreted as a map from an oriented n-manifold K_z into X.



This breaks down: For $n \geq 3$ K_z may not be an oriented n-manifold (though $K_z - K_z^{(n-3)}$ is).



K_z can have non-manifold points in its $(n-3)$-skeleton

(picture should be at least one dimension higher.)

Prop 2.8 For X a point, $H_0(X) \cong \mathbb{Z}$ and $H_n(X)=0$ for $n>0$

Pf There is a unique singular n-simplex σ_n $\forall n$.

$$\Delta^2 \xrightarrow{\sigma_2} \bullet \quad X$$

Note $\partial(\sigma_n) = \sum_{i=0}^{n} (-1)^i \sigma_{n-1} = \begin{cases} 0 & n \text{ odd} \\ \sigma_{n-1} & n \text{ even} \end{cases}$

$$\begin{tikzcd} \cdots \arrow[r] & C_4(X) \arrow[r, "\cong"] \arrow[d, phantom, sloped, "\cong"] & C_3(X) \arrow[r, "0"] \arrow[d, phantom, sloped, "\cong"] & C_2(X) \arrow[r, "\cong"] \arrow[d, phantom, sloped, "\cong"] & C_1(X) \arrow[r, "0"] \arrow[d, phantom, sloped, "\cong"] & C_0(X) \arrow[r, "0"] \arrow[d, phantom, sloped, "\cong"] & 0 \\ & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \end{tikzcd}$$

Hence $H_n(X) = \operatorname{Ker} \partial_n / \operatorname{Im} \partial_{n+1} \cong 0$ except for $H_0(X) \cong \mathbb{Z}$.

2/21/18

Reduced homology $\tilde{H}_n(X)$

TL;DR: $H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z}$ $\quad H_n(X) \cong \tilde{H}_n(X)$ for $n \geq 1$.

Def We let $\tilde{H}_n(X)$ be the homology groups of the augmented chain complex

$$\cdots \to C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \xrightarrow{0} 0$$

where $\varepsilon(\sum_i n_i \sigma_i) = \sum_i n_i$.

Note $H_n(X) = \operatorname{Ker} \partial_n / \operatorname{Im} \partial_{n+1} \cong \tilde{H}_n(X)$ for $n \geq 1$

Note $\varepsilon \partial_1 = 0$ so $\tilde{H}_0(X)$ is well-defined.
Furthermore, ε induces a (split) surjective homomorphism $\begin{tikzcd} H_0(X) \arrow[r, "\varepsilon"] & \mathbb{Z} \arrow[l, bend left] \end{tikzcd}$ with kernel $\tilde{H}_0(X)$, giving $H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z}$.

Chain maps and homotopies (pgs 111-113)

(Homological algebra) ↓

Def A chain complex is a sequence of homomorphisms of abelian groups

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots$$

such that $\partial_n \partial_{n+1} = 0 \quad \forall n.$

Def A chain map is a collection of homomorphisms $f_n : C_n \to C_n'$ such that $\partial_n' f_n = f_{n-1} \partial_n \quad \forall n.$

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & C_{n+1}' \arrow[r, "\partial_{n+1}'"] & C_n' \arrow[r, "\partial_n'"] & C_{n-1}' \arrow[r] & \cdots
\end{tikzcd}$$

Prop 2.9 A chain map induces homomorphisms between the homology groups of the chain complexes.

Pf $f_n(\operatorname{Ker} \partial_n) \subseteq \operatorname{Ker} \partial_n'$ since

$\partial_n z = 0 \Rightarrow \partial_n' f_n z = f_{n-1} \partial_n z = 0.$

$f_n(\operatorname{Im} \partial_{n+1}) \subseteq \operatorname{Im} \partial_{n+1}'$ since

$b = \partial_{n+1} c \Rightarrow f_n(b) = f_n(\partial_{n+1} c) = \partial_{n+1}'(f_{n+1} c).$

So $f_n : C_n \to C_n'$ induces

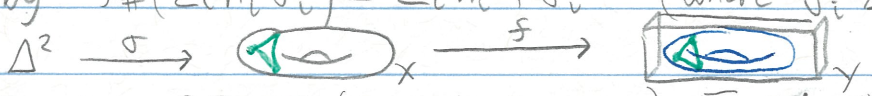
$f_n : \operatorname{Ker}\partial_n / \operatorname{Im}\partial_{n+1} \longrightarrow \operatorname{Ker}\partial_n' / \operatorname{Im}\partial_{n+1}'.$

Homological algebra ↑

Topological interlude

Topology ↓

Ex $f : X \to Y$ a continuous map of spaces induces $f_\# : C_n(X) \to C_n(Y) \ \forall n$ defined by $f_\#(\sum_i n_i \sigma_i) = \sum_i n_i f\sigma_i$ (where $\sigma_i : \Delta^n \to X$).



Note $f_\# \partial = \partial f_\#$ since $f_\# \partial \sigma = f_\#(\sum_i (-1)^i \sigma|[v_0, \dots, \hat{v}_i, \dots, v_n]) = \sum_i (-1)^i f\sigma|[v_0, \dots, \hat{v}_i, \dots, v_n] = \partial f_\# \sigma.$

Hence we get induced maps $f_*: H_n(X) \to H_n(Y)$ $\forall n$.

It's not hard to check homology is a functor:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "gf"'] & Y \arrow[r, "g"] & Z & \rightsquigarrow & H_n(X) \arrow[r, "f_*"] \arrow[rr, bend right, "(gf)_* = g_* f_*"'] & H_n(Y) \arrow[r, "g_*"] & H_n(Z) \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, "1_X"'] & X & \rightsquigarrow & H_n(X) \arrow[r, "(1_X)_* = 1_{H_n(X)}"'] & H_n(X) \end{tikzcd}$$

Topology ↑

[redacted]

Homological algebra ↓

Def Two chain maps $f_n, g_n : C_n \to C_n'$ are <u>chain homotopic</u> if there exist homomorphisms $P_n : C_n \to C_{n+1}'$ such that

$$\partial_{n+1}' P_n + P_{n-1} \partial_n = g_n - f_n \quad \forall n.$$

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, shift right, "f_{n+1}"'] \arrow[d, shift left, "g_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, shift right, "f_n"'] \arrow[d, shift left, "g_n"] \arrow[dl, "P_n"'] & C_{n-1} \arrow[r] \arrow[d, shift right, "f_{n-1}"'] \arrow[d, shift left, "g_{n-1}"] \arrow[dl, "P_{n-1}"'] & \cdots \\ \cdots \arrow[r] & C_{n+1}' \arrow[r, "\partial_{n+1}'"'] & C_n' \arrow[r, "\partial_n'"'] & C_{n-1}' \arrow[r] & \cdots \end{tikzcd}$$

Prop 2.12 Chain homotopic maps induce the same homomorphism on homology.

Pf If $z \in \operatorname{Ker} \partial_n$, then

$$\begin{aligned} g_n(z) - f_n(z) &= \partial_{n+1}' P_n(z) + P_{n-1} \partial_n(z) \\ &= \partial_{n+1}' P_n(z) + 0 \\ &\in \operatorname{Im} \partial_{n+1}'. \end{aligned}$$

Hence $g_n(z)$ and $f_n(z)$ represent the same equivalence class of $\ker \partial_n' / \operatorname{Im} \partial_{n+1}'$.

Homological algebra ↑

Advanced remark Chain maps are a "model category", meaning you can do homotopy theory on them! Other model categories include topological spaces, simplicial sets, and spectra.

Homotopy Invariance of Singular Homology

Thm 2.10 If two maps $f, g : X \to Y$ of spaces are homotopic, then they induce the same homomorphism $f_* = g_* : H_n(X) \longrightarrow H_n(Y)$.

Corollary 2.11 If $f : X \to Y$ is a homotopy equivalence, then $f_* : H_n(X) \to H_n(Y)$ is an isomorphism $\forall n$

Pf of 2.11

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$$

$$gf \simeq \mathbb{1}_X \Rightarrow g_* f_* = \mathbb{1}_{H_n(X)}$$
$$fg \simeq \mathbb{1}_Y \Rightarrow f_* g_* = \mathbb{1}_{H_n(Y)}.$$

Pf of 2.10 We'll find a chain homotopy $P : C_n(X) \to C_{n+1}(X)$ with $\partial P + P\partial = g_\# - f_\#$.

2/23/18

Homotopy Invariance of Singular Homology

Thm 2.10 If two maps $f, g : X \to Y$ of spaces are homotopic, then they induce the same homomorphism

$$f_* = g_* : H_n(X) \longrightarrow H_n(Y).$$

Corollary 2.11 If $f : X \to Y$ is a homotopy equivalence, then $f_* : H_n(X) \to H_n(Y)$ is an isomorphism $\forall n$

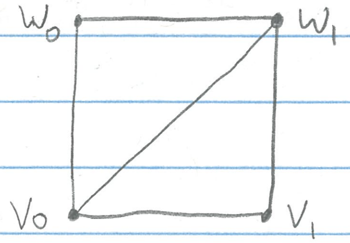
Pf of 2.11

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$$

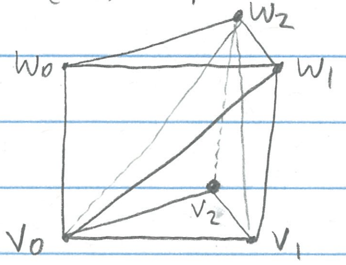
$$gf \simeq \mathbb{1}_X \Rightarrow g_* f_* = \mathbb{1}_{H_n(X)}$$
$$fg \simeq \mathbb{1}_Y \Rightarrow f_* g_* = \mathbb{1}_{H_n(Y)}.$$

Pf of 2.10 Essential step: divide $\Delta^n \times I$ into $(n+1)$-simplices.

$n=1$



$n=2$



$\Delta^n \times \{0\} = [v_0, \dots, v_n]$
$\Delta^n \times \{1\} = [w_0, \dots, w_n]$

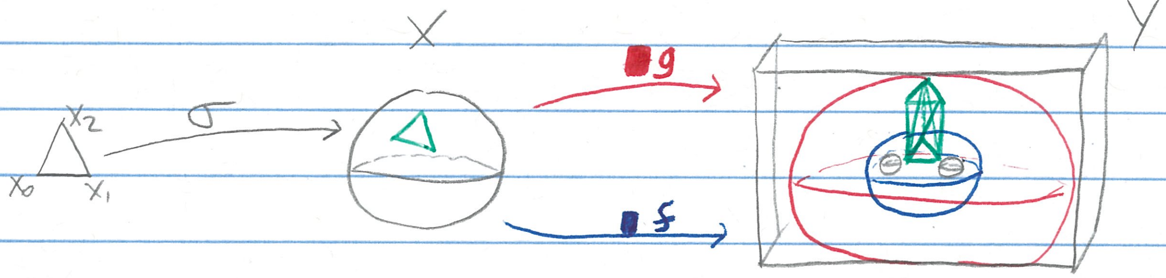
We've decomposed $\Delta^n \times I$ into the $n+1$ different $(n+1)$-simplices

$[v_0, \dots, v_n, w_n]$
$[v_0, \dots, v_{n-1}, w_{n-1}, w_n]$
$\vdots$
$[v_0, \dots, v_i, w_i, \dots, w_n]$
$\vdots$
$[v_0, w_0, \dots, w_n].$

Let $F: X \times I \to Y$ be a homotopy with $F(\cdot, 0) = f$ and $F(\cdot, 1) = g$.

Given a singular simplex $\sigma: \Delta^n \to X$, note

$$F \circ (\sigma \times \mathbb{1}): \Delta^n \times I \longrightarrow X \times I \longrightarrow Y$$



Define the prism operators $P_n: C_n(X) \to C_{n+1}(Y)$ by

$$P_n(\sigma) = \sum_{i=0}^{n} (-1)^i F \circ (\sigma \times \mathbb{1})\big|_{[v_0, \ldots, v_i, w_i, \ldots, w_n]}.$$

We'll show this gives a chan homotopy, and hence $f_* = g_*: H_n(X) \to H_n(Y)$ by Prop 2.12.

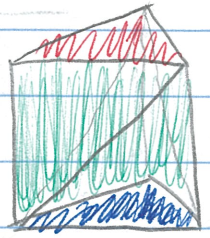
Recall A chan homotopy satisfies

$$\partial^Y_{n+1} P_n + P_{n-1} \partial^X_n = g_\# - f_\# \qquad \text{i.e.} \quad \partial P = g_\# - f_\# - P\partial$$

top bottom sides

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1}(X) \arrow[r, "\partial^X_{n+1}"] & C_n(X) \arrow[r, "\partial^X_n"] \arrow[dl, "P_n"'] \arrow[d, shift right, "f_\#"'] \arrow[d, shift left, "g_\#"] & C_{n-1}(X) \arrow[r] \arrow[dl, "P_{n-1}"'] & \cdots \\
\cdots \arrow[r] & C_{n+1}(Y) \arrow[r, "\partial^Y_{n+1}"'] & C_n(Y) \arrow[r, "\partial^Y_n"'] & C_{n-1}(Y) \arrow[r] & \cdots
\end{tikzcd}$$

$\partial P(\sigma)$



We calculate

$$\partial P(\sigma) = \sum_{j \leq i} (-1)^i (-1)^j F \circ (\sigma \times \mathbb{1})|[v_0, \dots, \hat{v}_j, \dots, v_i, w_i, \dots, w_n]$$
$$+ \sum_{j \geq i} (-1)^i (-1)^{j+1} F \circ (\sigma \times \mathbb{1})|[v_0, \dots, v_i, w_i, \dots, \hat{w}_j, \dots, w_n]$$

The terms with "$i=j$" cancel

Canceling interior n-simplices

$$\left\{ \left(\begin{array}{l} [v_0, \dots, v_{i-1}, \cancel{v_i}, w_i, \dots, w_n] \\ -[v_0, \dots, v_{i-1}, \cancel{w_{i-1}}, w_i, \dots, w_n] \end{array} \right) \begin{array}{l} \text{First sum, } j=i \\ \text{Second sum, } j=i-1 \end{array} \right.$$

except for

$$F \circ (\sigma \times \mathbb{1})|[\hat{v}_0, w_0, \dots, w_n] = g \circ \sigma = g_\sharp \sigma \quad \text{(top)}$$

and

$$-F \circ (\sigma \times \mathbb{1})|[v_0, \dots, v_n, \hat{w}_n] = -f \circ \sigma = -f_\sharp \sigma \quad \text{(bottom)}$$

The terms with "$i \neq j$" are exactly $-P\partial(\sigma)$ since

$$P\partial(\sigma) = P\left(\sum_{j=0}^{n} (-1)^j \sigma|[\mathbf{x}_0, \dots, \hat{\mathbf{x}}_j, \dots, \mathbf{x}_n] \right)$$
$$= \sum_{i<j} (-1)^i (-1)^j F \circ (\sigma \times \mathbb{1})|[v_0, \dots, v_i, w_i, \dots, \hat{w}_j, \dots, w_n]$$
$$+ \sum_{i>j} (-1)^{i-1} (-1)^j F \circ (\sigma \times \mathbb{1})|[v_0, \dots, \hat{v}_j, \dots, v_i, w_i, \dots, w_n].$$

(sides)

2/26/18

Exact sequences and excision

Def A chain complex $\cdots \to A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \to \cdots$ is exact if $\operatorname{Ker} \partial_n = \operatorname{Im} \partial_{n+1}$ $\forall n$.

Rmk IE, if $H_n = 0$ $\forall n$.

So homology measures how far a chain complex is from being exact.

Ex (i) $0 \to A \xrightarrow{\alpha} B$ is exact $\Longleftrightarrow$ $\operatorname{Ker} \alpha = 0$

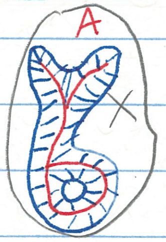
(ii) $A \xrightarrow{\alpha} B \to 0$ is exact $\Longleftrightarrow$ $\operatorname{Im} \alpha = B$

(iii) $0 \to A \xrightarrow{\alpha} B \to 0$ is exact $\Longleftrightarrow$ α is an isomorphism.

(iv) $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is exact (called a short exact sequence or SES) $\Longleftrightarrow$ α is injective, β is surjective, and $\operatorname{Ker} \beta = \operatorname{Im} \alpha$.

In this case $C \cong B/\operatorname{Ker}\beta = B/\operatorname{Im}\alpha \cong B/A$.

Thm 2.13 Let X be a space and A be a nonempty closed subspace that is a deformation retract of some neighborhood in X. (X,A) is a good pair.



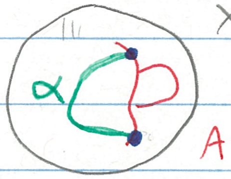
Then there is a long exact sequence (LES)

$$\begin{tikzcd}
\tilde{H}_n(A) \arrow[r, "i_*"] & \tilde{H}_n(X) \arrow[r, "j_*"] & \overset{\text{quotient space}}{\tilde{H}_n(X/A)} \arrow[dll, "\partial"] & \\
\tilde{H}_{n-1}(A) \arrow[r, "i_*"] & \tilde{H}_{n-1}(X) \arrow[r, "j_*"] & \tilde{H}_{n-1}(X/A) \arrow[dl, "\partial"] & \\
 & \vdots \arrow[dl] & & \\
\tilde{H}_0(A) \arrow[r, "i_*"] & \tilde{H}_0(X) \arrow[r, "j_*"] & \tilde{H}_0(X/A) \arrow[r] & 0
\end{tikzcd}$$

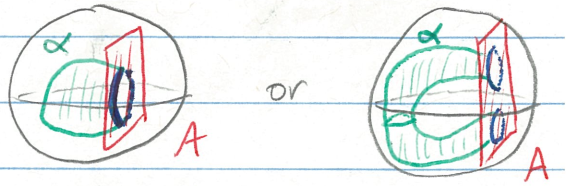
where $i : A \hookrightarrow X$ is the inclusion and $j : X \twoheadrightarrow X/A$ is the quotient.

Rmk The idea behind $\partial : \tilde{H}_n(X/A) \to \tilde{H}_{n-1}(A)$ is that an element $x \in \tilde{H}_n(X/A)$ can be represented by $\alpha \in C_n(X)$ with $\partial\alpha$ a cycle in A whose homology class is $[\partial\alpha] =: \partial x \in \tilde{H}_{n-1}(A)$.

$X = D^2$



$X = D^3$



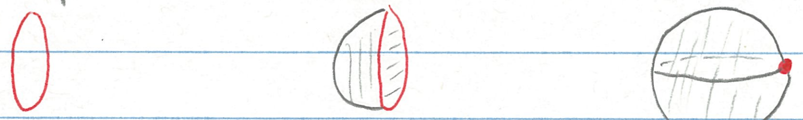
$\alpha \in C_1(X)$, $\partial\alpha \in H_0(A)$

$\alpha \in C_2(X)$, $\partial\alpha \in H_1(A)$

Corollary 2.14 $\tilde{H}_n(S^n) \cong \mathbb{Z}$ and $\tilde{H}_i(S^n) = 0$ for $i \neq n$.

Pf Base case $n=0$: $S^0 = \bullet \quad \bullet$

Inductive step: Assume true for S^{n-1}.



$$A = S^{n-1} \xhookrightarrow{i} X = D^n \xrightarrow{j} X/A = S^n$$

We get a LES

$$\cdots \xrightarrow{i_*} \cancelto{0}{\tilde{H}_i(D^n)} \xrightarrow{j_*} \tilde{H}_i(S^n) \xrightarrow{\partial} \tilde{H}_{i-1}(S^{n-1}) \xrightarrow{i_*} \cancelto{0}{\tilde{H}_{i-1}(D^n)} \xrightarrow{j_*} \cdots$$

By (iii) we have an isomorphism $\tilde{H}_i(S^n) \xrightarrow[\partial]{\cong} \tilde{H}_{i-1}(S^{n-1})$ $\forall i$ and hence $\tilde{H}_n(S^n) \cong \tilde{H}_{n-1}(S^{n-1}) \cong \mathbb{Z}$ and $\tilde{H}_i(S^n) \cong H_{i-1}(S^{n-1}) = 0$ for $i \neq n$.

Rmk The proof of Thm 2.13 is a long & important story.

- For $A \subseteq X$, we'll define <u>relative homology groups</u> $H_n(X,A)$.
- For $A \subseteq X$ arbitrary (no deformation retract property) we'll prove there's a LES ...

$$\begin{tikzcd} \cdots \arrow[r] & H_n(A) \arrow[r] & H_n(X) \arrow[r] & H_n(X,A) \arrow[r] & \cdots \end{tikzcd}$$

- For (X,A) a good pair, we have (Prop 2.22) $H_n(X,A) \cong \tilde{H}_n(X/A)$ $\forall n$.

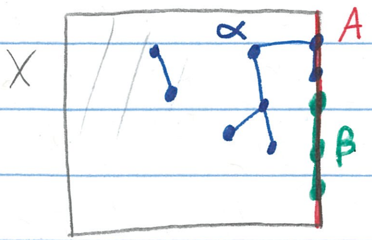
2/28/18

Relative Homology Groups

Let $A \subseteq X$ be spaces. By ignoring structure (chains in A) we can sometimes go further.

Let $C_n(X,A) = C_n(X)/C_n(A)$.

Ex



$\alpha \in C_1(X)$ $\beta \in C_1(A) \subseteq C_1(X)$.

$\beta = 0$ in $C_1(X,A)$.

$\alpha + \beta = \alpha$ in $C_1(X,A)$.

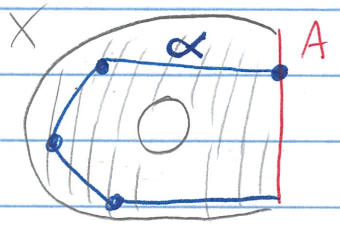
Note $\partial : C_n(X) \to C_{n-1}(X)$ takes $C_n(A)$ to $C_{n-1}(A)$, and so it induces $\partial : \underset{\substack{\| \\ C_n(X)/C_n(A)}}{C_n(X,A)} \longrightarrow \underset{\substack{\| \\ C_{n-1}(X)/C_{n-1}(A).}}{C_{n-1}(X,A)}$.

We get a chain complex ($\partial^2 = 0$)

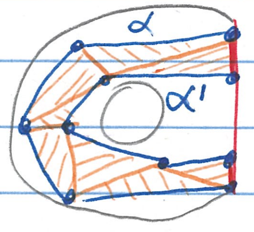
$$\cdots \to C_{n+1}(X,A) \xrightarrow{\partial} C_n(X,A) \xrightarrow{\partial} C_{n-1}(X,A) \to \cdots$$

Def The <u>relative homology group</u> $H_n(X,A)$ is the n-th homology of the above chain complex.

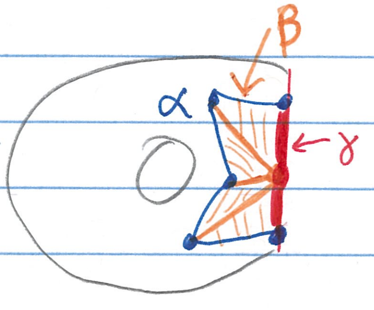
Rmk This is "homology of X modulo A"



$[\alpha] \in H_1(X,A)$



$[\alpha]=[\alpha'] \in H_1(X,A)$



$[\alpha]=0 \in H_1(X,A)$

- Elements of $H_n(X,A)$ are represented by <u>relative cycles</u>: n-chains $\alpha \in C_n(X)$ with $\partial\alpha \in C_{n-1}(A)$.
- A relative cycle α is trivial in $H_n(X,A)$ iff it is a <u>relative boundary</u>: $\alpha = \partial\beta + \gamma$ for some $\beta \in C_{n+1}(X)$ and $\gamma \in C_n(A)$.

Theorem (No number) For $A \subseteq X$ any pair of spaces, we have a LES

$$\begin{tikzcd}
{} \arrow[d, bend right] & & & \\
H_n(A) \arrow[r] & H_n(X) \arrow[r] & H_n(X,A) \arrow[dll, out=0, in=180] & \\
H_{n-1}(A) \arrow[r] & H_{n-1}(X) \arrow[r] & H_{n-1}(X,A) \arrow[dll, out=0, in=180] & \\
{} & \vdots & & \\
{} & {} \arrow[r] & H_0(X,A) \arrow[r] & 0
\end{tikzcd}$$

Pf The diagram

$$\begin{tikzcd}
& {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d, "\partial"'] \arrow[dr, phantom, "\circlearrowright"] & C_n(X) \arrow[r] \arrow[d, "\partial"'] \arrow[dr, phantom, "\circlearrowright"] & C_n(X,A) \arrow[r] \arrow[d, "\partial"'] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] \arrow[d] & C_{n-1}(X) \arrow[r] \arrow[d] & C_{n-1}(X,A) \arrow[r] \arrow[d] & 0 \\
& {} & {} & {} &
\end{tikzcd}$$

is called a <u>short exact sequence of chain complexes</u> since each column is a chain complex and since each row is exact.

The fact now follows from a more general theorem:

Thm 2.16 Let

$$\begin{tikzcd} & b \arrow[r, mapsto] \arrow[d, mapsto] & c \arrow[d, mapsto] \\ a \arrow[r, mapsto, "i"] & \partial b & 0 \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & A_n \arrow[r, "i"] \arrow[d, "\partial"] & B_n \arrow[r, "j"] \arrow[d, "\partial"] & C_n \arrow[r] \arrow[d, "\partial"] & 0 \\ 0 \arrow[r] & A_{n-1} \arrow[r, "i"] \arrow[d, "\partial"] & B_{n-1} \arrow[r, "j"] \arrow[d, "\partial"] & C_{n-1} \arrow[r] \arrow[d, "\partial"] & 0 \\ 0 \arrow[r] & A_{n-2} \arrow[r, "i"] & B_{n-2} \arrow[r, "j"] & C_{n-2} \arrow[r] & 0 \end{tikzcd}$$

or $0 \to A_\bullet \xrightarrow{i} B_\bullet \xrightarrow{j} C_\bullet \to 0$

be any short exact sequence of chain complexes. Then we get the following LES of homology groups:

$$\begin{tikzcd} H_n(A) \arrow[r, "i_*"] & H_n(B) \arrow[r, "j_*"] & H_n(C) \arrow[dll, "\partial"] \\ H_{n-1}(A) \arrow[r, "i_*"] & H_{n-1}(B) \arrow[r, "j_*"] & H_{n-1}(C) \arrow[dll, "\partial"] \\ {} & & \end{tikzcd}$$

Pf i_* and j_* are well-defined since i and j are chain maps.

Defining $\partial : H_n(C) \longrightarrow H_{n-1}(A)$ takes work! See diagram above, where $\partial[c] := [a]$.

- $\partial a = 0$ since $i(\partial a) = \partial i a = \partial\partial b = 0$ and i is injective.
- Also $\partial : H_n(C) \longrightarrow H_{n-1}(A)$ is well-defined up to

– the choice of b $\quad j(b) = j(b') \Rightarrow b - b' \in \operatorname{Ker} j = \operatorname{Im} i \Rightarrow b - b' = i(a_n)$ for $a_n \in A_n$

– the choice of c $\quad$ Say $c' \in C_{n+1}$ with [redacted] $j(b') = c'$.

Then [redacted] $c + \partial c' = c + \partial j b' = c + j \partial b' = j(b + \partial b')$.

Note $\partial(b + \partial b') = \partial b + \partial^2 b' = \partial b$.

Exactness follows by checking:

(1) Im i_* $\subseteq$ Ker j_*. True since $ji = 0 \Rightarrow j_* i_* = 0$

(2) Im j_* $\subseteq$ Ker ∂. When defining $\partial[c]$ in this case, we have that b is a cycle, hence $\partial b = 0$.

(3) Im ∂ $\subseteq$ Ker i_* Here $i_*\partial = 0$ since $i_*\partial[c] = i_*[a] = [\partial b] = 0$.

(4) Ker j_* $\subseteq$ Im i_*

(5) Ker ∂ $\subseteq$ Im j_*

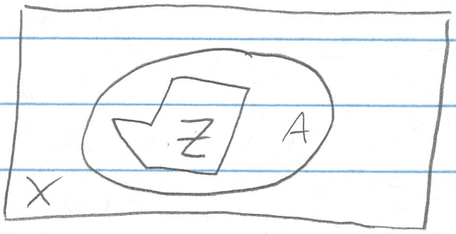
(6) Ker i_* $\subseteq$ Im ∂

(You are assigned to check (1)-(6) on HW 7) $\square$

3/2/18

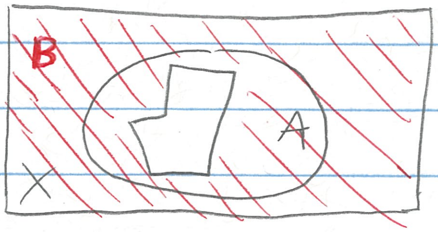
closure interior

Thm 2.20 (Excision Theorem) If $Z \subseteq A \subseteq X$ with $\text{cl}\, Z \subseteq \text{int}\, A$, then the inclusion $(X-Z, A-Z) \hookrightarrow (X,A)$ induces isomorphisms $H_n(X-Z, A-Z) \xrightarrow{\cong} H_n(X,A)$ $\forall n$.



Equivalently, for $A, B \subseteq X$ with $X = \text{int}\, A \cup \text{int}\, B$, the inclusion $(B, A\cap B) \hookrightarrow (X,A)$ induces isomorphisms $H_n(B, A\cap B) \xrightarrow{\cong} H_n(X,A)$ $\forall n$.

(Translation: $B = X - Z$.
So $X - \text{int}\, B = \text{cl}\, Z$.
So $\text{cl}\, Z \subseteq \text{int}\, A \iff X = \text{int}\, A \cup \text{int}\, B$.)



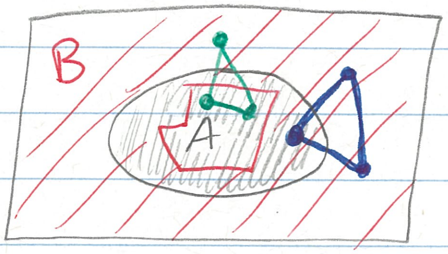
$B = X \setminus Z$

For the proof, we need the following machinery.

Let $C_n(A+B)$ be the subgroup of $C_n(X)$ consisting of chains $\sum_i n_i \sigma_i$ such that each $\sigma_i : \Delta^n \to X$ has image contained in A or B.

$\sigma \in C_2(A+B)$

$\sigma' \notin C_2(A+B)$



Prop 2.21 (Special case) Let $A, B \subseteq X$ with $X = \text{int} A \cup \text{int} B$. Then the inclusion $\iota : C_n(A+B) \longrightarrow C_n(X)$ is a <u>chain homotopy equivalence</u>

($\exists\ \rho : C_n(X) \to C_n(A+B)$ with $\iota\rho$ chain homotopic to $\mathbb{1}_{C_n(X)}$, and in this case $\rho\iota = \mathbb{1}_{C_n(A+B)}$.)

$$\begin{tikzcd} C_\bullet(A+B) \arrow[r, bend left, "\iota"] & C_\bullet(X) \arrow[l, bend left, "\rho"] \end{tikzcd}$$

$$\begin{tikzcd} C_{n+1}(X) \arrow[r, "\partial"] & C_n(X) \arrow[r, "\partial"] \arrow[dl, "D_n"'] \arrow[d, shift right, "\iota\rho"'] \arrow[d, shift left, "\mathbb{1}_{C_n(X)}"] & C_{n-1}(X) \arrow[dl, "D_{n-1}"] \\ C_{n+1}(X) \arrow[r, "\partial"'] & C_n(X) \arrow[r, "\partial"'] & C_{n-1}(X) \end{tikzcd}$$

$$\partial D_n + D_{n-1}\partial = \mathbb{1}_{C_n(X)} - \iota\rho$$

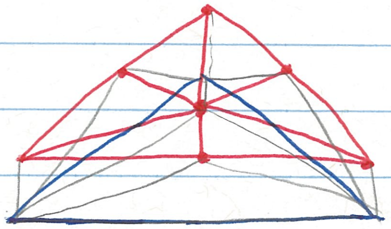
Not standard notation.

Hence ι induces isomorphisms $\boxed{H_n(A+B)} \cong H_n(X)$ $\forall n$.

Rmks The proof is 4½ pages in Hatcher.

- Note $\iota\rho$ must "split chains in X into smaller pieces in either A or B".
- D_n is constructed via stacked applications of

barycentric subdivision $sd(\Delta^n)$ "$\iota\rho$"



Prism is divided into copies of Δ^{n+1}

Δ^n

"$\mathbb{1}_{C_n(X)}$"

Pf of Thm 2.20 (Excision Theorem)

Let $X = \operatorname{int} A \cup \operatorname{int} B$. By Prop 2.21 we have
$\iota : C_n(A+B) \to C_n(X)$ and $\rho : C_n(X) \to C_n(A+B)$
with $\rho\iota = \mathbb{1}_{C_n(A+B)}$ and $\partial D + D\partial = \mathbb{1}_{C_n(X)} - \iota\rho$.

All maps above take chains in A to chains in A, giving

chain homotopy equivalence

isomorphism

$$\begin{tikzcd}
C_{n+1}(X)/C_{n+1}(A) \arrow[r, "\partial"] & C_n(X)/C_n(A) \arrow[r, "\partial"] & C_{n-1}(X)/C_{n-1}(A) \\
C_{n+1}(A+B)/C_{n+1}(A) \arrow[r, "\partial"] \arrow[u, "\iota"] & C_n(A+B)/C_n(A) \arrow[r, "\partial"] \arrow[u, "\iota"] & C_{n-1}(A+B)/C_{n-1}(A) \arrow[u, "\iota"] \\
C_{n+1}(B)/C_{n+1}(A\cap B) \arrow[r, "\partial"] \arrow[u, "\cong"] & C_n(B)/C_n(A\cap B) \arrow[r, "\partial"] \arrow[u, "\cong"] & C_{n-1}(B)/C_{n-1}(A) \arrow[u, "\cong"]
\end{tikzcd}$$

- ι is still an isomorphism on homology after modding out by A
  (since we still have the same chain homotopy D, modded out by A)

- The map $C_n(B)/C_n(A\cap B) \longrightarrow C_n(A+B)/C_n(A)$ induced by inclusion is an isomorphism of chain complexes since both quotient groups are free w/ basis the singular n-simplices in B but not contained in A.

This is true by the second isomorphism theorem of groups:

$$\frac{H}{H\cap N} \cong \frac{HN}{N}$$

Hence we get $H_n(B, A\cap B) \cong \overset{\text{Not standard notation}}{\boxed{H_n(A+B, A)}} \underset{\substack{\uparrow \\ \text{chain homotopy}}}{\cong} H_n(X, A)$.

3/5/18 Let $A \subseteq X$ be an arbitrary pair of spaces. On 2/28 in the "no number theorem" we proved there is a LES

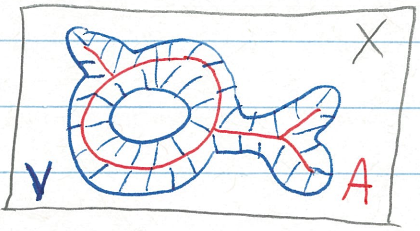
$$\cdots \longrightarrow H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X,A) \longrightarrow \cdots$$

We will now prove Thm 2.13, which says that for (X,A) a <u>good pair</u> (A is a deformation retract of some neighborhood in X), we also have a LES

$$\cdots \longrightarrow \tilde{H}_n(A) \longrightarrow \tilde{H}_n(X) \longrightarrow \tilde{H}_n(X/A) \longrightarrow \cdots$$

<u>Prop 2.22</u> For (X,A) a good pair, the quotient map $q: (X,A) \longrightarrow (X/A, A/A)$ induces isomorphisms $q_*: H_n(X,A) \longrightarrow H_n(X/A, A/A) \cong \tilde{H}_n(X/A)$ $\forall n$.

<u>Pf</u> Let V be a neighborhood that deformation retracts onto A. We have



$\boxed{\Rightarrow \cong}$

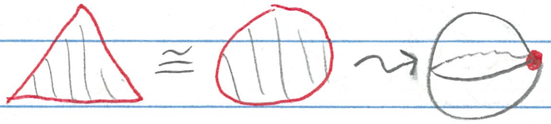
$$\begin{tikzcd}
H_n(X,A) \arrow[r, "\cong"] \arrow[d, "q_*"'] \arrow[dr, phantom, "\curvearrowright"] & H_n(X,V) \arrow[d, "q_*"'] \arrow[dr, phantom, "\curvearrowleft"] & H_n(X-A, V-A) \arrow[l, "\overset{\text{excision}}{\cong}"'] \arrow[d, "\cong"', "q_*"] \\
H_n(X/A, A/A) \arrow[r, "\cong"] & H_n(X/A, V/A) & H_n(X/A - A/A, V/A - A/A) \arrow[l, "\cong"', "\text{excision}"]
\end{tikzcd}$$

- The rightmost horizontal maps are ≅ by excision.
- The rightmost vertical map is ≅ since q is a homeomorphism on the complement of A (nothing to quotient by).
- It's plausible that the leftmost horizontal maps are ≅ since V deformation retracts to A

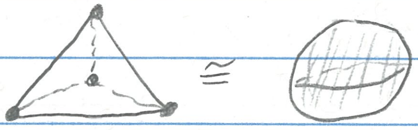
  (A rigorous proof uses the LES of a triple:
  $\cdots \longrightarrow H_n(V,A) \longrightarrow H_n(X,A) \longrightarrow H_n(X,V) \longrightarrow \cdots$).
- Hence the leftmost vertical map is ≅ by commutativity of the diagram.

Ex 2.23 $H_n(\Delta^k, \partial\Delta^k) \cong H_n(D^k, S^{k-1}) \cong \tilde{H}_n(S^k) \cong \begin{cases} \mathbb{Z} & n=k \\ 0 & \text{otherwise.} \end{cases}$

$k=2$



$k=3$



The equivalence of simplicial and singular homology

The Five Lemma Consider the following commutative diagram of abelian groups and exact rows.

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[dd, "\alpha"'] & B \arrow[r, "j"] \arrow[dd, "\beta"] & C \arrow[r, "k"] \arrow[dd, "\gamma"'] & D \arrow[r, "\ell"] \arrow[dd, "\delta"] & E \arrow[dd, "\varepsilon"'] \\
 & b \arrow[dd, mapsto] & c \arrow[r, mapsto] & d \arrow[r, mapsto] \arrow[dd, mapsto] & 0 \\
A' \arrow[r, "i'"] & B' \arrow[r, "j'"] & C' \arrow[r, "k'"] & D' \arrow[r, "\ell'"] & E' \\
 & b' & c' \arrow[r, mapsto] & k'c' &
\end{tikzcd}$$

If $\alpha, \beta, \delta, \varepsilon$ are isomorphisms, then so is γ.

Pf We'll show

(a) β, δ surjective and ε injective $\Rightarrow$ γ surjective

(b) β, δ injective and α surjective $\Rightarrow$ γ injective.

"Diagram chasing"

For (a), let $c' \in C'$. Construct $c \in C$ via "diagram chasing".
$k'(c'-\gamma c) = k'c' - k'\gamma c = k'c' - \delta k c = k'c' - \delta d = 0 \Rightarrow c' - \gamma c = j'b'$ for some $b' \in B'$.
β surjective $\Rightarrow$ $b' = \beta b$ for some $b \in B$.
Note $\gamma(c+jb) = \gamma c + \gamma j b = \gamma c + j'\beta b = \gamma c + j'b' = c'$,
so γ is surjective.

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[dd, "\alpha"'] & B \arrow[r, "j"] \arrow[dd, "\beta"'] & C \arrow[r, "k"] \arrow[dd, "\gamma"'] & D \arrow[r, "\ell"] \arrow[dd, "\delta"'] & E \arrow[dd, "\varepsilon"'] \\
a \arrow[dd, mapsto] & b \arrow[r, mapsto] \arrow[dd, mapsto] & c \arrow[r, mapsto] \arrow[dd, mapsto] & 0 & \\
A' \arrow[r, "i'"] & B' \arrow[r, "j'"] & C' \arrow[r, "k'"] & D' \arrow[r, "\ell'"] & E' \\
a' \arrow[r, mapsto] & \beta b \arrow[r, mapsto] & 0 & &
\end{tikzcd}$$

For (b), let $c \in C$ with $\gamma(c) = 0$.
Construct $a \in A$ via "diagram chasing".
$\beta(ia - b) = \beta ia - \beta b = i'\alpha a - \beta b = i'a' - \beta b = 0$
$\Rightarrow ia - b = 0$ since β injective.
Thus $ia = b$ and so $c = jb = jia = 0$ since $ji = 0$.
So γ is injective. □

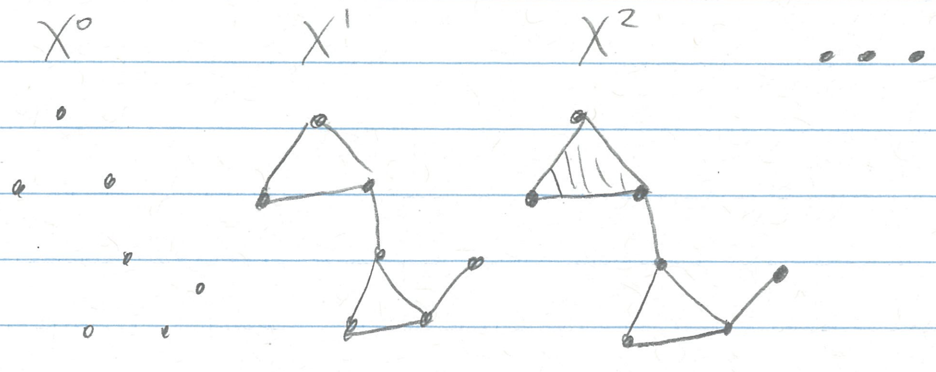
3/19/18

Thm 2.27 The homomorphisms $H_n^\Delta(X,A) \longrightarrow H_n(X,A)$ from simplicial homology to singular homology are isomorphisms $\forall n$ when (X,A) is a Δ-complex pair.

Rmk Taking $A = \emptyset$ gives $H_n^\Delta(X) \cong H_n(X)$

Rmk Just as $H_n(X,A)$ was defined using the chain complex $C_n(X)/C_n(A)$, similarly $H_n^\Delta(X,A)$ is defined using the chain complex $\Delta_n(X)/\Delta_n(A)$.

Pf We first do the case when X is finite-dimensional and $A = \emptyset$. Let X^k be the k-skeleton of X.



We have a commutative diagram with exact rows.

$$\begin{tikzcd}
{} \arrow[r] & H^{\Delta}_{n+1}(X^k, X^{k-1}) \arrow[r] \arrow[d, "\alpha"'] & H^{\Delta}_n(X^{k-1}) \arrow[r] \arrow[d, "\beta"'] & H^{\Delta}_n(X^k) \arrow[r] \arrow[d, "\gamma"'] & H^{\Delta}_n(X^k, X^{k-1}) \arrow[r] \arrow[d, "\delta"'] & H^{\Delta}_{n-1}(X^{k-1}) \arrow[d, "\varepsilon"'] \\
{} \arrow[r] & H_n(X^k, X^{k-1}) \arrow[r] & H_n(X^{k-1}) \arrow[r] & H_n(X^k) \arrow[r] & H^{\Delta}_n(X^k, X^{k-1}) \arrow[r] & H_n(X^{k-1})
\end{tikzcd}$$

We may assume β, ε are isomorphisms by induction on k.

Note $H^{\Delta}_n(X^k, X^{k-1}) \cong \Delta_n(X^k, X^{k-1}) \cong \begin{cases} \mathbb{Z}^{(\#\text{ of } k\text{-simplices})} & n=k \\ 0 & n \neq k \end{cases}$

For singular homology, recall (Ex 2.23) that
$H_n(\Delta^k, \partial\Delta^k) = \begin{cases} \mathbb{Z} & n=k \quad (\text{generated by } \Delta^k \to \Delta^k) \\ 0 & n \neq k \end{cases}$

The map $\Phi : \coprod_\alpha (\Delta^k_\alpha, \partial\Delta^k_\alpha) \longrightarrow (X^k, X^{k-1})$ formed by the attaching maps of k-simplices induces a homeomorphism $\vee_\alpha S^k = \vee_\alpha(\Delta^k_\alpha / \partial\Delta^k_\alpha) = \coprod_\alpha \Delta^k_\alpha / \coprod_\alpha \partial\Delta^k_\alpha \xrightarrow{\cong} X^k/X^{k-1}$ and hence an isomorphism on H_n.
Hence $H_n(X^k, X^{k-1}) \cong \begin{cases} \mathbb{Z}^{(\#\text{ of } k\text{-simplices})} & n=k \\ 0 & n \neq k \end{cases}$

and α, δ are also isomorphisms.

By the Five Lemma we get that γ is an isomorphism.

- The case when X is infinite dimensional requires more work.
- Now assume $A \neq \emptyset$. We have

$$\begin{tikzcd}
{} \arrow[r] & H_n^\Delta(A) \arrow[r] \arrow[d, "\cong"] & H_n^\Delta(X) \arrow[r] \arrow[d, "\cong"] & H_n^\Delta(X,A) \arrow[r] \arrow[d] & H_{n-1}^\Delta(A) \arrow[r] \arrow[d, "\cong"] & H_{n-1}^\Delta(X) \arrow[r] \arrow[d, "\cong"] & {} \\
{} \arrow[r] & H_n(A) \arrow[r] & H_n(X) \arrow[r] & H_n(X,A) \arrow[r] & H_{n-1}(A) \arrow[r] & H_{n-1}(X) \arrow[r] & {}
\end{tikzcd}$$

Applying the Five Lemma again finishes the proof of Thm 2.27 □.

3/21/18

Section 2.2 Computations and Applications

Degree

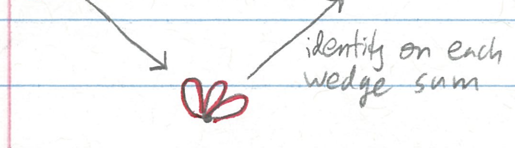
Def The *degree* of a map $f: S^n \to S^n$ $(n \geq 1)$ is the integer d s.t. $f_*: \underset{\mathbb{Z}}{\underset{\cong}{H_n(S^n)}} \longrightarrow \underset{\mathbb{Z}}{\underset{\cong}{H_n(S^n)}}$ is of the form $f_*(\alpha) = d\alpha$. This integer is denoted $\deg(f)$

Ex $f: S^1 \longrightarrow S^1$

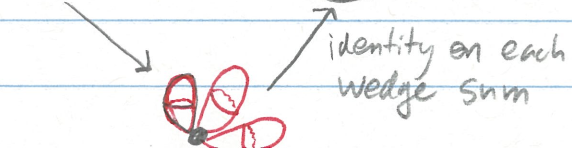
$\deg(f) = 2$

Rmk For more pictures, see our Math 570 notes on 12/4/17.

Ex $f: S^1 \to S^1$ with $\deg(f) = 3$

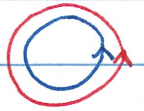


Ex $f: S^2 \to S^2$ with $\deg(f) = 3$



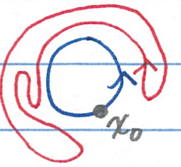
Basic Properties

(a) $\deg(\mathbb{1}) = 1$, since $\mathbb{1}_* = \mathbb{1}$.



$\deg(\mathbb{1}) = 1$

(b) $\deg(f) = 0$ if f is not surjective.



$\deg(f)=0$

Pf Pick $x_0 \notin f(S^n)$.

$$\begin{tikzcd} S^n \arrow[r, "f"] \arrow[rr, bend left, "f"] & S^n - \{x_0\} \arrow[r, hook] & S^n \end{tikzcd}$$

Apply H_n to get

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "f_*"] \arrow[rr, bend left, "f_*"] & 0 \arrow[r] & \mathbb{Z} \end{tikzcd}$$

So $f_* = 0$ since f_* factors through $H_n(S^n - \{x_0\}) = 0$.

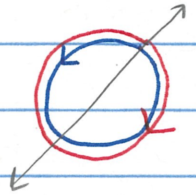
(c) If $f \simeq g$, then $\deg f = \deg g$ since $f_* = g_*$.
This is in fact an if-and-only-if (Corollary 4.25)!

(d) $\deg(fg) = \deg f \deg g$ since $(fg)_* = f_* g_*$.

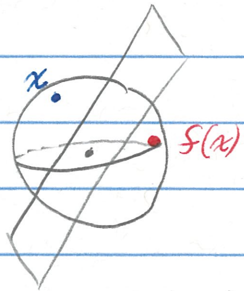
As a consequence, $\deg f = \pm 1$ if f is a homotopy equivalence, since $fg \simeq \mathbb{1} \Rightarrow \deg f \deg g = \deg fg = \deg \mathbb{1} = 1$.

(e) $\deg f = -1$ if f is a reflection of S^n

Picture $n=1$



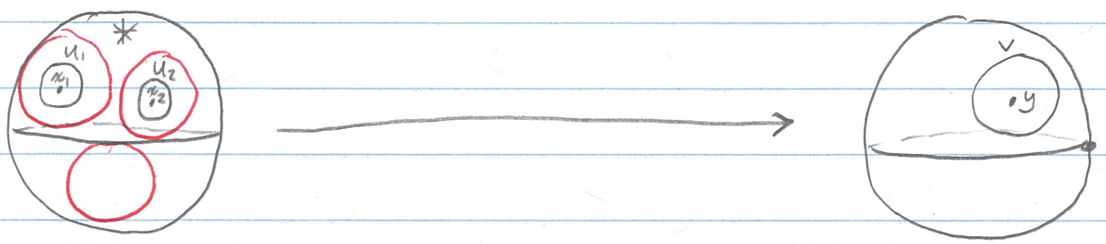
$n=2$



(f) The antipodal map $-\mathbb{1}: S^n \to S^n$ defined by $x \longmapsto -x$ has degree $\deg(-\mathbb{1}) = (-1)^{n+1}$, since it is a compositon of $n+1$ reflections (one for each coordinate axis in $\mathbb{R}^{n+1}$).

Local degree

Suppose $f: S^n \to S^n$ has the property that for some $y \in S^n$, $f^{-1}(y) = \{x_1, \dots, x_m\}$ is finite. Choose disjoint neighborhoods $U_i \ni x_i$ and $V \ni y$ with $f(U_i) \subseteq V$ $\forall i$.



For all i we have

$$\begin{tikzcd}
 & H_n(U_i, U_i - x_i) \arrow[r, "f_*"] \arrow[dl, "\cong \text{ by excision}"'] & H_n(V, V-y) \arrow[d, "\cong \text{ by excision: cut out } S^n - V"] \\
H_n(S^n, S^n - x_i) & & H_n(S^n, S^n - y) \\
 & H_n(S^n) \arrow[r, "f_*"] \arrow[ul, "\cong \text{ by LES of pair}"] & H_n(S^n) \arrow[u, "\cong \text{ by LES of pair since } S^n - y \simeq *"']
\end{tikzcd}$$

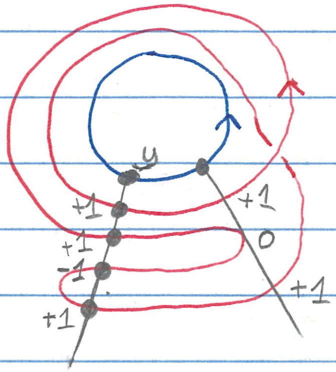
Hence all graps above are isomorphic to $\mathbb{Z}$. The top map is multiplication by an integer which we define to be the local degree of f at x_i, written $\deg f|x_i$.

Prop 2.30 For any $y \in S^n$ with $f^{-1}(y) = \{x_1, \dots, x_m\}$ finite, $\deg f = \sum_i \deg f|x_i$.

Pf Omitted

Rmk If f maps U_i homeomorphically onto V, then $\deg f|x_i = \pm 1$.

Ex



$\deg f = \sum_i \deg f|x_i$
$= 1+1-1+1$
$= 2$

$\deg f = \sum_i \deg f|x_i$
$= 1+0+1$
$= 2$

Rmk deg(f) can also be defined for any map $f: M \to N$ between closed orientable manifolds of the same dimension.

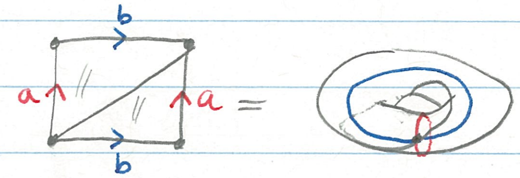
Prop 2.33 $\deg Sf = \deg f$, where $Sf: S^{n+1} \to S^{n+1}$ is the suspension of the map $f: S^n \to S^n$.

3/23/18

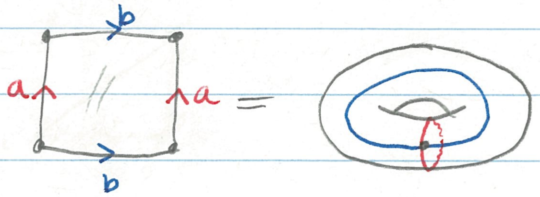
Cellular homology

Recall Simplicial complexes $\subsetneq$ Δ-complexes $\subsetneq$ CW complexes

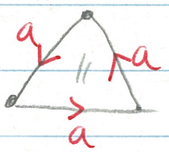
Δ-complex but not a simplicial complex:



CW complex but not a Δ-complex:



Another example is



(the face maps aren't order-preserving).

We've already defined simplicial homology on Δ-complexes, and observed $H_n^{\Delta}(X) \cong H_n(X)$ for X a Δ-complex.

$$\underset{\text{simplicial}}{\overset{\uparrow}{H_n^{\Delta}}}(X) \cong \underset{\text{singular}}{\overset{\uparrow}{H_n}}(X)$$

Using degree, we'll now define cellular homology for CW complexes, and observe $H_n^{CW}(X) \cong H_n(X)$ for X a CW-complex.

$$\underset{\text{cellular}}{\overset{\uparrow}{H_n^{CW}}}(X) \cong \underset{\text{singular}}{\overset{\uparrow}{H_n}}(X)$$

Rmk Hence people drop the Δ and CW decorations on $H_n(X)$!

For X a CW complex, cellular homology is defined as the homology of a chain complex $H_n^{CW}(X) := \operatorname{Ker} d_n / \operatorname{Im} d_{n+1}$.

$$\cdots \to H_{n+1}(X^{n+1}, X^n) \xrightarrow{d_{n+1}} \overbrace{H_n(\underset{\substack{\uparrow \\ \text{n-skeleton of } X}}{X^n}, X^{n-1})}^{\substack{\mathbb{Z}^{(\#\text{ n-cells})} \\ \cong}} \xrightarrow{d_n} H_{n-1}(X^{n-1}, X^{n-2}) \xrightarrow{d_{n-1}} \cdots \to H_0(X^0, \phi) \xrightarrow{d_0} 0$$

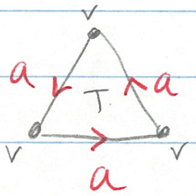
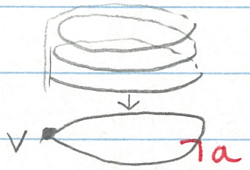
(Sometimes called "homology squared")

We postpone a definition of d_n, and a verification that $d_{n-1} d_n = 0$.

Cellular Boundary Formula For e^n_α an n-cell in X, we have $d_n(e^n_\alpha) = \Sigma_\beta\, d_{\alpha\beta}\, e^{n-1}_\beta$, where $d_{\alpha\beta}$ is

- (roughly speaking) the number of times the attaching map for e^n_α "wraps around" e^{n-1}_β
- (more precisely) the degree of the map

$$\underset{\text{attaching map of } e^n_\alpha}{S^{n-1}_\alpha \longrightarrow X^{n-1}} \underset{\substack{\text{quotient map collapsing} \\ X^{n-1} - e^{n-1}_\beta \text{ to a point}}}{\longrightarrow} S^{n-1}_\beta$$

Ex $X =$  $=$ 

disk attached, wrapping around 3 times

Chain complex

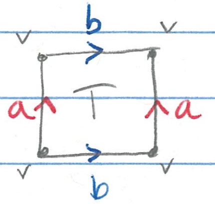
$$\to \underset{\text{3-cells}}{0} \xrightarrow{d_3} \underset{\text{2-cells}}{\mathbb{Z}} \xrightarrow{d_2} \underset{\text{1-cells}}{\mathbb{Z}} \xrightarrow{d_1} \underset{\text{0-cells}}{\mathbb{Z}} \xrightarrow{d_0} 0$$

$T \longmapsto 3a \qquad a \longmapsto 0 \qquad v \longmapsto 0$

$H_0^{CW}(X) = \operatorname{Ker} d_0 / \operatorname{Im} d_1 \cong \mathbb{Z}$

$H_1^{CW}(X) = \operatorname{Ker} d_1 / \operatorname{Im} d_2 \cong \mathbb{Z}/3\mathbb{Z}$

$H_2^{CW}(X) = \operatorname{Ker} d_2 / \operatorname{Im} d_3 \cong 0$

Ex $X =$ 

$X =$ torus

2-cells, 1-cells, 0-cells

Chain complex $\to 0 \longrightarrow \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^2 \xrightarrow{d_1} \mathbb{Z} \xrightarrow{d_0} 0$

$T \mapsto 0$

$a \mapsto 0$

$b \mapsto 0$

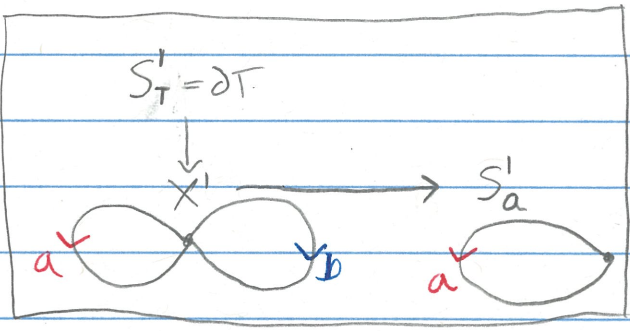
$v \mapsto 0$

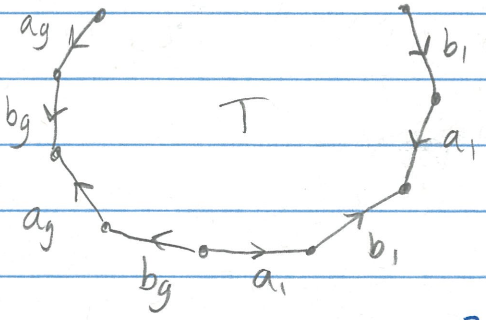
To see $d_2(T) = 0$, note $d_2(T) = d_{Ta}\, a + d_{Tb}\, b$ by the cellular boundary formula, where T is attached along $aba^{-1}b^{-1}$, and so $d_{Ta} = 1 - 1 = 0$ and $d_{Tb} = 1 - 1 = 0$.

$H_0^{CW}(X) = \operatorname{Ker} d_0 / \operatorname{Im} d_1 \cong \mathbb{Z}$

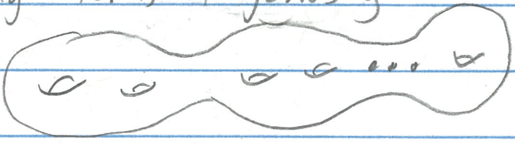
$H_1^{CW}(X) = \operatorname{Ker} d_1 / \operatorname{Im} d_2 \cong \mathbb{Z}^2$

$H_2^{CW}(X) = \operatorname{Ker} d_2 / \operatorname{Im} d_3 \cong \mathbb{Z}$



Ex $X =$ 

$X = M_g =$ torus of genus g



2-cells, 1-cells, 0-cells

Chain complex $\to 0 \longrightarrow \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^{2g} \xrightarrow{d_1} \mathbb{Z} \xrightarrow{d_0} 0$

$T \mapsto 0$

$a_1 \mapsto 0$

$b_1 \mapsto 0$

$\vdots$

$a_g \mapsto 0$

$b_g \mapsto 0$

$v \mapsto 0$

To see $d_2(T) = 0$, note $d_2(T) = \sum_{i=1}^{g} d_{Ta_i} a_i + \sum_{i=1}^{g} d_{Tb_i} b_i$ where $d_{Ta_i} = 1 - 1 = 0 = d_{Tb_i}$ for all i.

Hence $H_0^{CW}(X) \cong \mathbb{Z}$ $\quad H_1^{CW}(X) \cong \mathbb{Z}^{2g}$ $\quad H_2^{CW}(X) \cong \mathbb{Z}$

3/26/18 For X a CW complex, cellular homology is defined as the homology of a chain complex $H_n^{CW}(X) := \operatorname{Ker} d_n / \operatorname{Im} d_{n+1}$

$$\begin{tikzcd}
 & & & & & 0 & & \\
 & & 0 \arrow[dr] & & H_n(X^{n+1}) \cong \underset{\text{singular homology}}{H_n(X)}\text{[redacted]} \arrow[ur] & & & \\
 & & & H_n(X^n) \arrow[ur, "i_{n+1}"] \arrow[dr, "j_n"] & & & & {} \\
\cdots \arrow[rr] & & H_{n+1}(X^{n+1},X^n) \arrow[rr, "d_{n+1}"] \arrow[ur, "\partial_{n+1}"] & & H_n(X^n,X^{n-1}) \arrow[rr, "d_n"] \arrow[dr, "\partial_n"] \arrow[d, phantom, sloped, "\cong"] & & H_{n-1}(X^{n-1},X^{n-2}) \arrow[r] \arrow[ur, "\partial_{n-1}"] & \cdots \\
 & H_{n+1}(X^{n+1}) \arrow[ur, "j_{n+1}"] & & & \mathbb{Z}^{(\# n\text{-cells})} & H_{n-1}(X^{n-1}) \arrow[ur, "j_{n-1}"] \arrow[dr] & & \\
0 \arrow[ur] & & & & 0 \arrow[ur] & & H_{n-1}(X^n) \arrow[dr] & \\
 & & & & & & & 0
\end{tikzcd}$$

Here $d_n := j_{n-1} \partial_n$ is the cellular boundary map. The diagonal lines are the LES's for the pairs (X^{n+1}, X^n) and (X^n, X^{n-1}) etc.

To see this is a chain complex, note

$$d_n d_{n+1} = \partial_{n+1} \underbrace{j_n \partial_n}_{\uparrow 0} j_{n-1} = 0$$

0 since this is two adjacent maps in an exact sequence.

Thm 2.35 For X a CW complex, $H_n^{CW}(X) \cong H_n(X)$.

Pf Note

$$H_n(X) \underset{\substack{\uparrow \\ \text{n+2 cells and higher} \\ \text{don't affect } H_n}}{\cong} H_n(X^{n+1}) \underset{\substack{\uparrow \\ \text{1st isomorphism} \\ \text{theorem}}}{\cong} H_n(X^n)/\operatorname{Ker} i_{n+1} \underset{\substack{\uparrow \\ \text{exactness}}}{=} H_n(X^n)/\operatorname{Im} \partial_{n+1}$$

We'll now show j_n induces an isomorphism

$$j_n : H_n(X^n)/\operatorname{Im}\partial_{n+1} \xrightarrow{\cong} \operatorname{Ker} d_n / \operatorname{Im} d_{n+1} =: H_n^{CW}(X)$$

Indeed, since j_n is injective it maps $\operatorname{Im}\partial_{n+1}$ isomorphically onto $\operatorname{Im}(j_n\partial_{n+1}) = \operatorname{Im} d_{n+1}$, and $H_n(X^n)$ isomorphically onto $\operatorname{Im} j_n = \operatorname{Ker}\partial_n \underset{\substack{\uparrow \\ \text{since } j_{n-1} \text{ injective}}}{=} \operatorname{Ker} d_n$

$\square$

Immediate applications

(i) $H_n(X) = 0$ if X is a CW complex with no n-cells (in which case $H_n(X^n, X^{n-1}) \cong \mathbb{Z}^{(\#\ n\text{-cells})} = 0$).

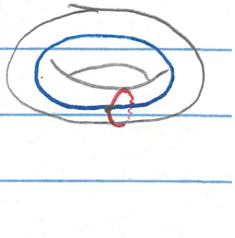
(ii) More generally, if X has k n-cells, then $H_n(X)$ is generated by at most k elements.

(iii) If X is a CW complex having no two of its cells in adjacent dimensions, then $H_n(X) \cong \mathbb{Z}^{(\#\ n\text{-cells})}$ for all n.
This is because all cellular boundary maps d_n are zero in this case.

Ex of (iii) $\mathbb{CP}^n$ has a CW structure with one cell of each even dimension $2k \leq 2n$. Hence

$$H_i(\mathbb{CP}^n) \cong \begin{cases} \mathbb{Z} & \text{for } i = 0, 2, 4, 6, \dots, 2n \\ 0 & \text{otherwise.} \end{cases}$$

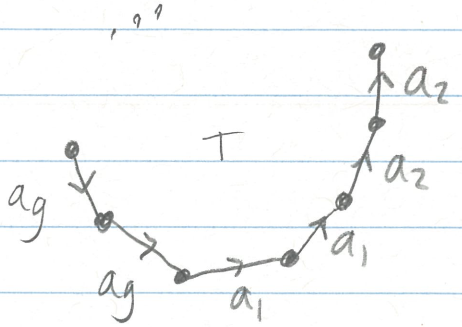
Ex of (iii) $S^n \times S^n$ has a CW structure of one 0-cell, two n-cells, one $2n$-cell.



Hence for $n > 1$ we have

$$H_i(S^n \times S^n) \cong \begin{cases} \mathbb{Z} & i = 0 \text{ or } 2n \\ \mathbb{Z}^2 & i = n \\ 0 & \text{otherwise.} \end{cases}$$

Ex 2.37 The nonorientable surface N_g of genus g has a CW structure with one 0-cell, g 1-cells, and one 2-cell attached by the word $a_1^2 a_2^2 \cdot \ldots \cdot a_g^2$.



Chain complex
$$\begin{tikzcd}
 & \text{3-cells} & \text{2-cells} & \text{1-cells} & \text{0-cells} & \\
 {} \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r, "d_2"] & \mathbb{Z}^g \arrow[r, "d_1"] & \mathbb{Z} \arrow[r, "d_0"] & 0
\end{tikzcd}$$
$$a_1 \longmapsto 0 \qquad v \longmapsto 0$$
$$\vdots$$
$$a_g \longmapsto 0$$

We compute $d_2(T) = 2a_1 + 2a_2 + \ldots + 2a_g$.

$H_0(N_g) = \text{Ker} d_0 / \text{Im} d_1 \cong \mathbb{Z}$

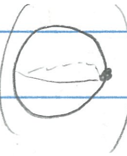
$H_1(N_g) = \text{Ker} d_1 / \text{Im} d_2 \cong \mathbb{Z}^{g-1} \oplus \mathbb{Z}/2\mathbb{Z}$ since
Ker d_1 has basis $\{a_1, \ldots, a_g\}$ or $\{a_1, \ldots, a_{g-1}, a_1 + \ldots + a_g\}$
Im d_2 has basis $\{2a_1 + \ldots + 2a_g\}$.

$H_2(N_g) = \text{Ker} d_2 / \text{Im} d_3 = 0$ since $\text{Ker} d_2 = 0$.

3/28/18

Euler characteristic

Def For X a finite CW complex, the Euler characteristic $\chi(X)$ is $\chi(X) = \sum_n (-1)^n c_n$, where c_n is the # of n-cells in X.

Ex $\chi\Big($  $\Big) = 1+0+1=2$

S^2 with one 0-cell, one 2-cell

Ex $\chi\Big($  $\Big) = 2-2+2=2$

S^2 with two 0-cells, two 1-cells, two 2-cells

Ex $\chi(S^n) = \underbrace{1}_{\text{one 0-cell}} + \underbrace{(-1)^n}_{\text{one n-cell}}$

Ex $\underbrace{\chi(M_g)}_{\text{torus of genus } g} = \underbrace{1}_{\text{one 0-cell}} - \underbrace{2g}_{\text{2g 1-cells}} + \underbrace{1}_{\text{one 2-cell}} = 2-2g$

Hence Euler characteristic is independent of CW structure, and also a homotopy invariant!

Thm 2.44 $\chi(X) = \sum_n (-1)^n \operatorname{rank} H_n(X)$

Rmk The rank of a finitely generated abelian group is the number of $\mathbb{Z}$ summands when the group is expressed as a direct sum of cyclic groups.

Ex $\operatorname{rank}\left(\mathbb{Z}/3\mathbb{Z} \oplus (\mathbb{Z}/6\mathbb{Z})^2 \oplus \mathbb{Z}/7\mathbb{Z} \oplus \mathbb{Z}^5\right) = 5.$

Rmk If $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ is a short exact sequence (SES) of finitely generated abelian groups, then $\operatorname{rank} B = \operatorname{rank} A + \operatorname{rank} C$ since

$$C = \operatorname{im}\beta \cong B/\operatorname{Ker}\beta = B/\operatorname{Im}\alpha \quad \text{with } \alpha \text{ injective}$$

$$\Longrightarrow \operatorname{rank} C = \operatorname{rank} B - \operatorname{rank} A$$

Pf of Thm 2.44

(Algebra Step)

Let $0 \to C_k \xrightarrow{d_k} C_{k-1} \xrightarrow{d_{k-1}} C_{k-2} \to \dots \to C_1 \xrightarrow{\partial_1} C_0 \to 0$ be any chain complex of finitely generated abelian groups.

Let $Z_n = \text{Ker } d_n$ (cycles)

$B_n = \text{Im } d_{n+1}$ (boundaries)

$H_n = Z_n / B_n$ (homology)

We have SES's

$$0 \to Z_n \hookrightarrow C_n \to B_{n-1} \to 0$$

and

$$0 \to B_n \hookrightarrow Z_n \to H_n \to 0.$$

Hence rank C_n = rank Z_n + rank B_{n-1}

and rank Z_n = rank B_n + rank H_n

Substitution gives rank C_n = rank B_n + rank H_n + rank B_{n-1}

Cancellation in the alternating sum gives

$\Sigma_n (-1)^n$ rank C_n = $\Sigma_n (-1)^n$ rank H_n

(Topology Step) Let $C_n = H_n(X^n, X^{n-1}) \cong \mathbb{Z}^{(\# \text{ n-cells})} \cong \mathbb{Z}^{c_n}$.

$$\chi(X) = \Sigma_n (-1)^n c_n = \Sigma_n (-1)^n \text{rank } C_n = \Sigma_n (-1)^n \text{rank } H_n$$
$$= \Sigma_n (-1)^n \text{rank } H_n(X).$$

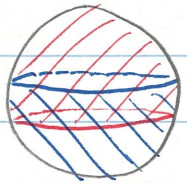
3/30/18 <u>Mayer-Vietoris sequence</u>

<u>Thm (page 149)</u> Let X be a space and $A, B \subseteq X$ such that $X = \text{int}(A) \cup \text{int}(B)$. Then there is a long exact sequence (LES)

$$\begin{tikzcd}
H_n(A\cap B) \arrow[r] & H_n(A)\oplus H_n(B) \arrow[r] & H_n(X) \arrow[dll, rounded corners] \\
H_{n-1}(A\cap B) \arrow[r] & H_{n-1}(A)\oplus H_{n-1}(B) \arrow[r] & H_{n-1}(X) \arrow[r] & {} \\
& \vdots & \\
{} \arrow[r] & H_0(A\cap B) \arrow[r] & H_0(A)\oplus H_0(B) \arrow[r] & H_0(X) \arrow[r] & 0
\end{tikzcd}$$

(or also with $\tilde{H}_0$ instead of H_0)

<u>Ex 2.46</u>



$X = S^2$

$A = D^2$ $\quad A \cap B \simeq S^1$

$B = D^2$

$$\begin{tikzcd}
H_2(A\cap B) \arrow[r] & \overset{0}{\overset{\shortparallel}{H_2(A)}}\oplus \overset{0}{\overset{\shortparallel}{H_2(B)}} \arrow[r] & H_2(S^2) \arrow[dll, rounded corners, "\partial \text{ is } \cong"] \\
\underset{\mathbb{Z}}{\underset{\wr\|}{H_1(A\cap B)}} \arrow[r] & \underset{0}{\underset{\shortparallel}{H_1(A)}}\oplus \underset{0}{\underset{\shortparallel}{H_1(B)}} &
\end{tikzcd}$$

The zeros imply this map is injective and surjective.

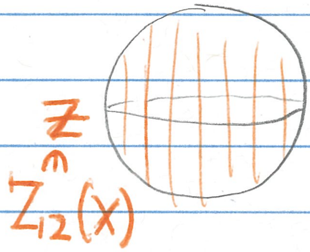
So $H_1(A\cap B) \cong H_1(S^1) \cong \mathbb{Z}$ implies $H_2(S^2) \cong \mathbb{Z}$.

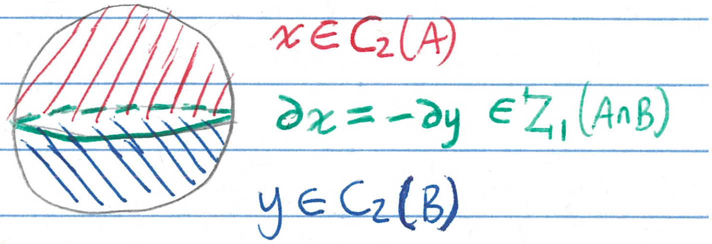
More generally, we could choose $X = S^n$, $A = D^n$, $B = D^n$, and $A \cap B \simeq S^{n-1}$ to get that $H_{n-1}(S^{n-1}) \cong \mathbb{Z}$ implies $H_n(S^n) \cong \mathbb{Z}$.

Rmk The connecting homomorphism $\partial : H_n(X) \longrightarrow H_{n-1}(A \cap B)$ can be made explicit.

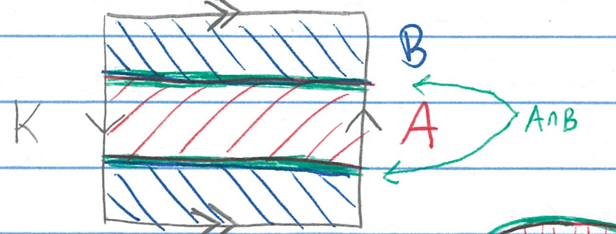
Different maps

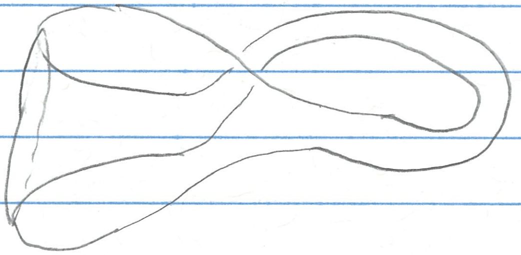
Let $[z] \in H_n(X)$, where z is a cycle in X $(\partial z = 0)$.
It turns out we can write $z = x + y$ for x a chain in A and y a chain in B.
Note $\partial z = 0 \Rightarrow \partial(x+y) = 0 \Rightarrow \partial x = -\partial y$.
We have $\partial[z] = [\partial x] = [-\partial y] \in H_{n-1}(A \cap B)$.

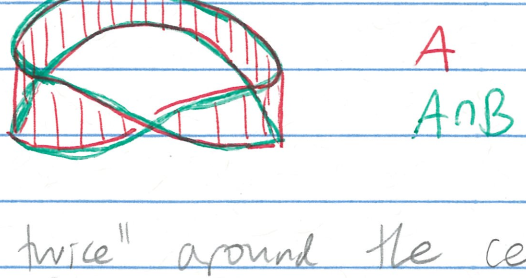




Ex 2.47 The Klein bottle K is the union of two Möbius bands glued together along their boundary circle.







Note $A \cap B$ "wraps twice" around the central circle of A (or of B)

Mayer-Vietoris gives a LES

$$\begin{tikzcd}
 & \overset{\cong 0}{H_2(A)\oplus H_2(B)} \arrow[r] & H_2(K) \arrow[dll, "\partial"] \\
\overset{\cong \mathbb{Z}}{H_1(A\cap B)} \arrow[r, "\Phi"] & \overset{\cong \mathbb{Z}}{H_1(A)} \oplus \overset{\cong \mathbb{Z}}{H_1(B)} \arrow[r, "\Psi"] & H_1(K) \arrow[ddll] \\
1 \arrow[r, mapsto] & (2\ ,\ -2) & \\
\overset{\cong 0}{\tilde{H}_0(A\cap B)} & &
\end{tikzcd}$$

The negative sign in $1 \longmapsto (2,-2)$ will be explained later.

Φ injective $\Longrightarrow$ $\partial = 0$ (since $\operatorname{Im}\partial = \operatorname{Ker}\Phi = 0$) $\Longrightarrow H_2(K) = 0$

$$H_1(K) \cong \operatorname{Im}\Psi \cong \frac{H_1(A)\oplus H_1(B)}{\operatorname{Ker}\Psi} \cong \frac{\overset{\text{basis } \{(1,-1),(1,0)\}}{\mathbb{Z}\oplus\mathbb{Z}}}{\underset{\text{basis } \{(2,-2)\}}{\operatorname{Im}\Phi}} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}.$$

4/2/18

Verification that we have a Mayer-Vietoris LES

We have a SES of chain complexes

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
0 \arrow[r] & C_n(A\cap B) \arrow[r, "\varphi"] \arrow[d] & C_n(A)\oplus C_n(B) \arrow[r, "\psi"] \arrow[d] & C_n(A+B) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A\cap B) \arrow[r, "\varphi"] \arrow[d] & C_{n-1}(A)\oplus C_{n-1}(B) \arrow[r, "\psi"] \arrow[d] & C_{n-1}(A+B) \arrow[r] \arrow[d] & 0 \\
 & {} & {} & {} &
\end{tikzcd}$$

Here $\varphi(x) := (x,-x)$ and $\psi(x,y) := x+y$

Here $C_n(A+B)$ is the subgroup of $C_n(X)$ consisting of all sums of chains in A and chains in B.

By def^n, ψ is surjective.

Also φ is injective.

$\psi\varphi(x) = \psi(x,-x) = x-x = 0$ so $\operatorname{Ker}\psi \subseteq \operatorname{Im}\varphi$.

Also, $\operatorname{Im}\varphi \le \operatorname{Ker}\psi$ since $\psi(x,y)=0 \Rightarrow x=-y$

$\Rightarrow$ x and y are chains in $A\cap B$ $\quad (x=-y\in C_n(A\cap B))$

$\Rightarrow$ $(x,y) = (x,-x) = \varphi(x)$.

So we've verified exactness.

This SES of chain complexes now gives a LES of homology groups

$$\begin{tikzcd}
 & & & \cdots \arrow[dlll, rounded corners] \\
{} \arrow[r] & H_n(A\cap B) \arrow[r, "\Phi"] & H_n(A)\oplus H_n(B) \arrow[r, "\Psi"] & \underline{H_n(A+B)} \arrow[dlll, rounded corners] \\
{} \arrow[r] & \cdots & &
\end{tikzcd}$$

Not standard notation

Recall from Prop 2.21 (special case) that since $X = \text{int}(A) \cup \text{int}(B)$, the inclusion $C_n(A+B) \longrightarrow C_n(X)$ is a chain homotopy equivalence, inducing $\underset{\text{Not standard notation}}{\underline{H_n(A+B)}} \cong H_n(X)$ $\forall n$.

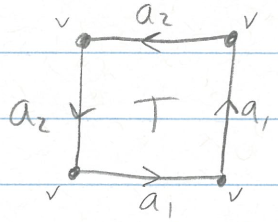
This gives the LES

$$\begin{tikzcd}
 & & \cdots \arrow[dll, rounded corners] \\
H_n(A\cap B) \arrow[r] & H_n(A)\oplus H_n(B) \arrow[r] & H_n(X) \arrow[dll, rounded corners] \\
\cdots & &
\end{tikzcd}$$

Homology with coefficients

So far, we have been doing homology with $\mathbb{Z}$ coefficients: $H_n(X) = H_n(X;\mathbb{Z})$. This can be generalized to homology $H_n(X;G)$ with G coefficients, where G is any abelian group.

Ex Cellular homology of the Klein bottle K with $G=\mathbb{Z}$ and $G=\mathbb{Z}/2\mathbb{Z}$



$G=\mathbb{Z}$

$$\begin{array}{ccccccccc} & & \text{2-cells} & & \text{1-cells} & & \text{0-cells} & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\ & & T \longmapsto 2a_1+2a_2 & & \begin{array}{l} a_1 \longmapsto 0 \\ a_2 \longmapsto 0 \end{array} & & v \longmapsto 0 & & \end{array}$$

$$H_i(K;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z} & i=1 \\ 0 & \text{otherwise} \end{cases}$$

$G=\mathbb{Z}/2\mathbb{Z}$

$$\begin{array}{ccccccccc} & & \text{2-cells} & & \text{1-cells} & & \text{0-cells} & & \\ 0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & (\mathbb{Z}/2\mathbb{Z})^2 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 \\ & & T \longmapsto \underset{\displaystyle 0}{\underset{\|}{2a_1+2a_2}} & & \begin{array}{l} a_1 \longmapsto 0 \\ a_2 \longmapsto 0 \end{array} & & v \longmapsto 0 & & \end{array}$$

$$H_i(K;\mathbb{Z}/2\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & i=0 \text{ or } 2 \\ (\mathbb{Z}/2\mathbb{Z})^2 & i=1 \\ 0 & \text{otherwise} \end{cases}$$

Note $H_i(K;\mathbb{Z}/2\mathbb{Z}) \cong H_i(\text{torus};\mathbb{Z}/2\mathbb{Z})$

4/4/18 In singular homology with coefficients, the chain groups $C_n(X) = C_n(X;\mathbb{Z}) \cong \mathbb{Z}^{(\#\text{ singular } n\text{-simplices})}$ are replaced by $C_n(X;G) \cong G^{(\#\text{ singular } n\text{-simplices})}$

$$\cdots \to C_n(X;G) \xrightarrow{\partial_n} C_{n-1}(X;G) \xrightarrow{\partial_{n-1}} C_{n-2}(X;G) \to \cdots$$

The boundary operator formula $\partial : C_n(X;G) \to C_{n-1}(X;G)$ remans unchanged.

$\partial(\sum_i n_i \sigma_i) = \sum_i n_i \partial\sigma_i$ with $\partial\sigma_i = \sum_{j=0}^{n} (-1)^j \sigma_i|[v_0, \ldots, \hat{v}_j, \ldots, v_n]$ except with $n_i \in G$ instead of $n_i \in \mathbb{Z}$.

The most common coefficients are $G = \mathbb{Z}, \mathbb{Z}/m\mathbb{Z}, \mathbb{Q}$, or $\mathbb{R}$.

$H_n(X;\mathbb{Z}/m\mathbb{Z})$ is sometimes easier to compute than $H_n(X;\mathbb{Z})$, for example as we saw with the Klein bottle.

From $H_n(X;\mathbb{Z})$ one can almost always determine $H_n(X;\mathbb{Z}/m\mathbb{Z})$ $\forall m$ and $H_n(X;\mathbb{Q})$, as explained on the following page.

Universal coefficient theorem for homology

(Beyond the scope of this class)

Thm 3A.3 If C is a chain complex of free abelian groups, then there are natural SES's

$$0 \to H_n(C)\otimes G \to H_n(C;G) \to \mathrm{Tor}(H_{n-1}(C),G) \to 0$$

for all n and G.

Corollary 3A.6 (a) $H_n(X;\mathbb{Q}) \cong H_n(X;\mathbb{Z})\otimes\mathbb{Q}$

So for $H_n(X;\mathbb{Z})$ finitely generated, $\dim_{\mathbb{Q}} H_n(X;\mathbb{Q}) = \mathrm{rank}\, H_n(X;\mathbb{Z})$.

(b) If $H_n(X;\mathbb{Z})$ and $H_{n-1}(X;\mathbb{Z})$ are finitely generated and p is prime, then $H_n(X;\mathbb{Z}/p\mathbb{Z})$ consists of a $\mathbb{Z}/p\mathbb{Z}$ summand

(i) for each $\mathbb{Z}$ summand of $H_n(X;\mathbb{Z})$

(ii) for each $\mathbb{Z}/p^k\mathbb{Z}$ summand of $H_n(X;\mathbb{Z})$

(iii) for each $\mathbb{Z}/p^k\mathbb{Z}$ summand of $H_{n-1}(X;\mathbb{Z})$.

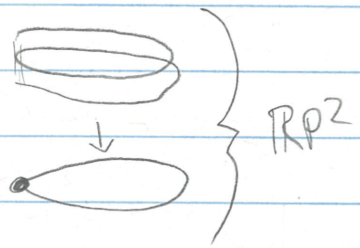
Ex We saw $H_n(\text{Klein bottle};\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z} & n=1 \\ 0 & n\geq 2. \end{cases}$

Corollary 3A.6 then implies

$$H_n(\text{Klein bottle};\mathbb{Z}/2\mathbb{Z}) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & n=0 \\ \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z} & n=1 \\ \mathbb{Z}/2\mathbb{Z} & n=2 \\ 0 & n\geq 3 \end{cases}$$

<u>Rmk</u> Homology with coefficients can sometimes tell us more than ordinary homology.

For $X = \mathbb{R}P^2$, we have a quotient map $\mathbb{R}P^2 \longrightarrow \mathbb{R}P^2/(\mathbb{R}P^2)^{(1)} = \mathbb{R}P^2/S^1 \cong S^2$ obtained by collapsing the 1-skeleton to a point.



<u>Question</u> Is q homotopy equivalent to a constant map (which induces the zero map on reduced homology)?

<u>$\mathbb{Z}$ coefficients</u> $\tilde{H}_n(\mathbb{R}P^2) \overset{\text{HW9}}{\cong} \begin{cases} \mathbb{Z}/2\mathbb{Z} & n=1 \\ 0 & \text{otherwise.} \end{cases}$

$$\tilde{H}_n(S^2) = \begin{cases} \mathbb{Z} & n=1 \\ 0 & \text{otherwise.} \end{cases}$$

Since $q_* : \tilde{H}_n(\mathbb{R}P^2) \to \tilde{H}_n(S^2)$ is necessarily zero $\forall n$, we can't conclude if q is homotopy equivalent to the constant map or not.

<u>$\mathbb{Z}/2\mathbb{Z}$ coefficients</u> $H_2(\mathbb{R}P^2) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & n=0,1,2 \\ 0 & n \geq 3. \end{cases}$

We have a LES of the (good) pair

$$\to \underset{\substack{\| \\ 0}}{H_2(S^1;\mathbb{Z}/2\mathbb{Z})} \to \underset{\substack{\wr\| \\ \mathbb{Z}/2\mathbb{Z}}}{H_2(\mathbb{R}P^2;\mathbb{Z}/2\mathbb{Z})} \xhookrightarrow{q_*} H_2(\mathbb{R}P^2/S^1;\mathbb{Z}/2\mathbb{Z}) \to$$

Now q_* is not zero for H_2 with $\mathbb{Z}/2\mathbb{Z}$ coefficients, and so q is <u>not</u> homotopy equivalent to a constant map.

Rmk Sometimes you can "go backwards" and get $H_n(X;\mathbb{Z})$ from $H_n(X;\mathbb{Q})$ and $H_n(X;\mathbb{Z}/p\mathbb{Z})$ $\forall$ primes p:

Corollary 3A.7

(a) $\tilde{H}_n(X;\mathbb{Z})=0$ $\forall n$ $\iff$ $\tilde{H}_n(X;\mathbb{Q})=0$ and $\tilde{H}_n(X;\mathbb{Z}/p\mathbb{Z})=0$ $\forall n$ and $\forall$ primes p.

(b) A map $f:X\to Y$ induces isomorphisms on homology with $\mathbb{Z}$ coefficients $\iff$ it induces isomorphisms on homology with $\mathbb{Q}$ and $\mathbb{Z}/p\mathbb{Z}$ coefficients $\forall$ primes p.

4/6/18

Chapter 3: Cohomology

The biggest differences between homology and cohomology are

- Homology is a covariant functor whereas cohomology is contraviant: a continuous map $f: X \to Y$ of spaces induces a homomorphism $f_*: H_i(X) \to H_i(Y)$ on homology, but a homomorphism $f^*: H^i(Y) \to H^i(X)$ on cohomology.

(subscript)
(superscript)

- Cohomology has a natural product structure $H^i(X) \times H^j(X) \xrightarrow{\cup} H^{i+j}(X)$ defined as a composition (called the cup product)

$$H^i(X) \times H^j(X) \longrightarrow H^{i+j}(X \times X) \longrightarrow H^{i+j}(X)$$

In both cohomology and homology, given spaces Z and W, one can construct natural maps $H^i(Z) \times H^j(W) \to H^{i+j}(Z \times W)$ and $H_i(Z) \times H_j(W) \to H_{i+j}(Z \times W)$

The diagonal map $X \to X \times X$ defined by $x \mapsto (x,x)$ induces this map on cohomology. By contravariance! No nice analogue for homology.

Organization of Chapter 3:

The Idea of Cohomology (pg 186)

Section 3.1 Cohomology Groups

The Universal Coefficient Theorem

Cohomology of Spaces (pg 197)

We'll start here

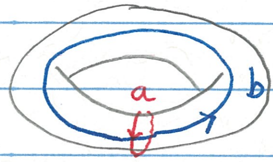
Section 3.2 Cup Product

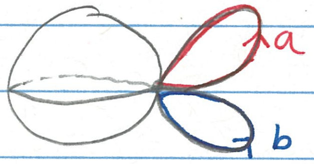
Section 3.3 Poncaré Duality

For M an n-dimensional orientable closed manifold, we have $H^k(M) \cong H_{n-k}(M)$ ∀k.

Example of the cup product

Homology can't tell the difference between the torus $S^1 \times S^1$ and $S^2 \vee S^1 \vee S^1$.





$$H_i(S^1\times S^1) \cong \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}\oplus\mathbb{Z} & i=1 \\ \mathbb{Z} & i=2 \\ 0 & i\geq 3 \end{cases} \qquad H_i(S^2\vee S^1\vee S^1) \cong \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}\oplus\mathbb{Z} & i=1 \\ \mathbb{Z} & i=2 \\ 0 & i\geq 3 \end{cases}$$

Follows from $\tilde{H}_i(X\vee Y) \cong \tilde{H}_i(X) \oplus \tilde{H}_i(Y)$

The cup product structure on cohomology can tell the difference!

$$H^i(S^1\times S^1) \cong \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}\times\mathbb{Z} & i=1 \\ \mathbb{Z} & i=2 \\ 0 & i\geq 3 \end{cases} \qquad H^i(S^2\vee S^1\vee S^1) \cong \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}\times\mathbb{Z} & i=1 \\ \mathbb{Z} & i=2 \\ 0 & i\geq 3 \end{cases}$$

Let $[a]$ and $[b]$ be generators for H^1, with $[T]$ a generator for H^2. We have

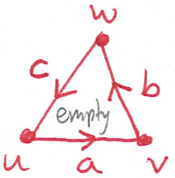
$$[a] \cup [b] = [T]$$

Let $[a]$ and $[b]$ be generators for H^1. We have

$$[a] \cup [b] = 0$$

$$H^1(X) \times H^1(X) \xrightarrow{\cup} H^2(X)$$

Running example $X =$



For X a Δ-complex, recall simplicial homology:

$$\begin{tikzcd}
{} \arrow[r] & \Delta_{n+1}(X) \arrow[r, "\partial_{n+1}"] & \Delta_n(X) \arrow[r, "\partial_n"] & \cdots \arrow[r] & \Delta_2(X) \arrow[r, "\partial_2"] \arrow[d, equal] & \Delta_1(X) \arrow[r, "\partial_1"] \arrow[d, phantom, "\rotatebox{-90}{$\cong$}"] & \Delta_0(X) \arrow[r, "\partial_0"] \arrow[d, phantom, "\rotatebox{-90}{$\cong$}"] & 0 \\
& & & & 0 & \mathbb{Z}^{\oplus 3} & \mathbb{Z}^{\oplus 3} &
\end{tikzcd}$$

Let G be an abelian group (think $G = \mathbb{Z}$).

Def The <u>n-cochains with coefficients in G</u> is the group $\Delta^n(X;G) := \text{Hom}(\Delta_n(X), G)$

Rmk For A, B abelian groups, $\text{Hom}(A,B)$ is the group of homomorphisms $f : A \to B$.
The group structure is defined, for $f, g \in \text{Hom}(A,B)$, by $(f+g) : A \to B$ via $(f+g)(a) = f(a) + g(a)$

Ex For X above, $\Delta^1(X;\mathbb{Z}) \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, with $f \in \Delta^1(X;\mathbb{Z})$ determined by $f(a), f(b), f(c) \in \mathbb{Z}$.

4/9/18 We have a <u>cochain complex</u>

$$\begin{tikzcd}
{} & \Delta^{n+1}(X;G) \arrow[l] & \Delta^n(X;G) \arrow[l, "\delta^n"'] & \cdots \arrow[l, "\delta^{n-1}"'] & \Delta^2(X;G) \arrow[l, "\delta^2"'] \arrow[d, equal] & \Delta^1(X;G) \arrow[l, "\delta^1"'] \arrow[d, phantom, "\rotatebox{-90}{$\cong$}"] & \Delta^0(X;G) \arrow[l, "\delta^0"'] \arrow[d, phantom, "\rotatebox{-90}{$\cong$}"] & 0 \arrow[l] \\
& & & & 0 & \mathbb{Z}^{\times 3} & \mathbb{Z}^{\times 3} & (G=\mathbb{Z})
\end{tikzcd}$$

where the <u>coboundary map</u> $\delta^n : \Delta^n(X;G) \to \Delta^{n+1}(X;G)$ is defined, for $f \in \Delta^n(X;G)$, by $\delta f = f\partial$.
$(\delta^n f = f\partial_{n+1})$

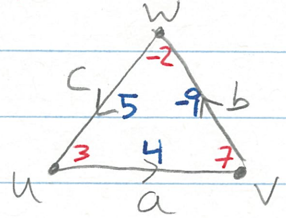
$$\begin{tikzcd}
\Delta_{n+1}(X) \arrow[r, "\partial"] \arrow[rr, bend right, "f\partial"', "(\text{in } \Delta^{n+1}(X;G))"] & \Delta^n(X) \arrow[r, "f", "(\text{in } \Delta^n(X;G))"'] & G
\end{tikzcd}$$

Ex For X above, let $f \in \Delta^0(X;\mathbb{Z})$ be determined by $f(u)=3$, $f(v)=7$, $f(w)=-2$.
Then $\delta f \in \Delta^1(X;\mathbb{Z})$ is determined by

$\delta f(a) = f\partial(a) = f(v-u) = f(v)-f(u) = 7-3=4$

$\delta f(b) = f\partial(b) = f(w-v) = f(w)-f(v) = -2-7=-9$

$\delta f(c) = f\partial(c) = f(u-w) = f(u)-f(w) = 3-(-2)=5$.



More generally, for $f \in \Delta^n(X;G)$ and $\sigma: \Delta^{n+1} \to X$ an $(n+1)$-simplex in X, we have

$$\delta f(\sigma) = f\partial(\sigma) = f\left(\sum_{i=0}^{n+1} (-1)^i \sigma|_{[v_0,\ldots,\hat{v_i},\ldots,v_{n+1}]}\right)$$

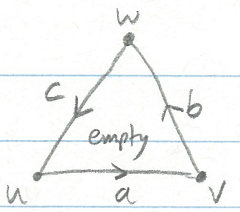
$$= \sum_{i=0}^{n+1} (-1)^i f\left(\sigma|_{[v_0,\ldots,\hat{v_i},\ldots,v_{n+1}]}\right)$$

Note $\delta \circ \delta = 0$ since $\partial \circ \partial = 0$

$$\left(\begin{array}{l}\text{More explicitly, for } f \in \Delta^n(X;G), \text{ we have} \\ \delta^{n+1}\delta^n f = \delta^{n+1} f\partial_{n+1} = f\partial_{n+1}\partial_{n+2} = 0\end{array}\right)$$

Def The cohomology group $H^n(X;G)$ of X with coefficients in G is $\mathrm{Ker}(\delta^n)/\mathrm{Im}(\delta^{n-1})$

Ex $X =$



$$\begin{tikzcd}
\cdots & \Delta^2(X;\mathbb{Z}) \arrow[l, "\delta^2"'] \arrow[d, equal] & \Delta^1(X;\mathbb{Z}) \arrow[l, "\delta^1"'] \arrow[d, equal] & \Delta^0(X;\mathbb{Z}) \arrow[l, "\delta^0"'] \arrow[d, equal] & 0 \arrow[l] \\
& \mathrm{Hom}(\Delta_2(X),\mathbb{Z}) \arrow[d, phantom, "\cong" rotate=90] & \mathrm{Hom}(\Delta_1(X),\mathbb{Z}) \arrow[d, phantom, "\cong" rotate=90] & \mathrm{Hom}(\Delta_0(X),\mathbb{Z}) \arrow[d, phantom, "\cong" rotate=90] & \\
& \mathrm{Hom}(0,\mathbb{Z}) \arrow[d, equal] & \mathrm{Hom}(\mathbb{Z}^{\oplus 3},\mathbb{Z}) \arrow[d, phantom, "\cong" rotate=90] & \mathrm{Hom}(\mathbb{Z}^{\oplus 3},\mathbb{Z}) \arrow[d, phantom, "\cong" rotate=90] & \\
& 0 & \mathbb{Z}^{\times 3} & \mathbb{Z}^{\times 3} &
\end{tikzcd}$$

$$H^0(X;\mathbb{Z}) = \frac{\mathrm{Ker}(\delta^0)}{\mathrm{Im}(\delta^{-1})} = \mathrm{Ker}(\delta^0) \cong \mathbb{Z}$$

since if $f \in \Delta^0(X;\mathbb{Z})$ with $\delta f = 0$, then

$0 = \delta f(a) \Rightarrow 0 = f(v) - f(u) \Rightarrow f(v) = f(u)$

$0 = \delta f(b) \Rightarrow 0 = f(w) - f(v) \Rightarrow f(w) = f(v)$

$0 = \delta f(c) \Rightarrow 0 = f(u) - f(w) \Rightarrow f(u) = f(w)$

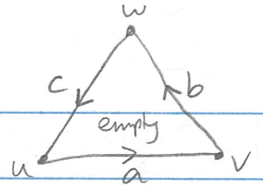
So f is a constant function determined by a single $c \in \mathbb{Z}$ $(c = f(u) = f(v) = f(w))$.

__More generally__ For X a Δ-complex,

$H^0(X;\mathbb{Z}) \cong \mathbb{Z}^{\times (\#\text{ connected components of } X)}$ and

$H^0(X;G) \cong G^{\times (\#\text{ connected components of } X)}$

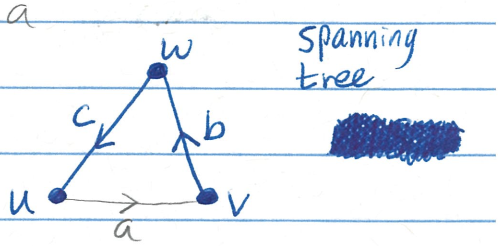
Ex Again with $X =$



we have

$$H^1(X;\mathbb{Z}) = \frac{\operatorname{Ker}\delta^1}{\operatorname{Im}\delta^0} = \frac{\Delta^1(X;\mathbb{Z})}{\operatorname{Im}\delta^0} \quad \text{since } \delta^1 = 0 \text{ here.}$$

To understand this quotient, chose a spanning tree for X.



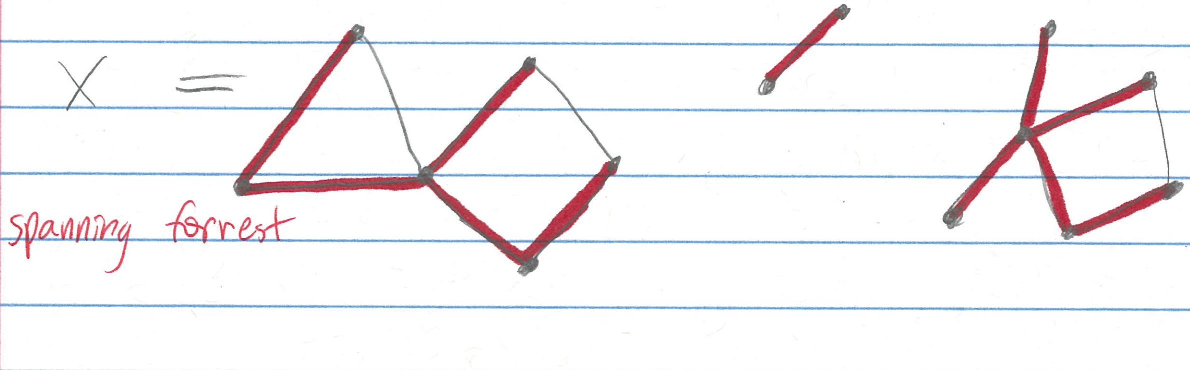
Let $g \in \Delta^1(X;\mathbb{Z})$.
We can always find $f \in \Delta^0(X;\mathbb{Z})$ with $(\delta^0 f)(b) = g(b)$ and $(\delta^0 f)(c) = g(c)$ by choosing $f(u), f(v), f(w)$ so that
$f(w) - f(v) = g(b)$ and $f(u) - f(w) = g(c)$.
But these choices determine $(\delta^0 f)(a) = f(v) - f(u)$
$(\neq g(a)$ in general$)$

This shows $H^1(X;\mathbb{Z}) = \frac{\Delta^1(X;\mathbb{Z})}{\operatorname{Im}\delta^0} \cong \mathbb{Z}$ here.

4/11/2018

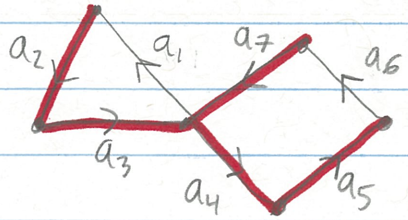
More generally Let X be a graph with k edges not in a spanning forrest for X. Then
$H^1(X;\mathbb{Z}) \cong \mathbb{Z}^{\times k}$
$H^1(X;G) \cong G^{\times k}$

$X =$



$H^1(X;\mathbb{Z}) \cong \mathbb{Z}^{\times 3} = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$

Generators for cohomology For X a graph, $H^1(X;\mathbb{Z})$ is generated by the cocycles that assign 1 to a single (oriented) edge not in a spanning forrest, and 0 to every other edge

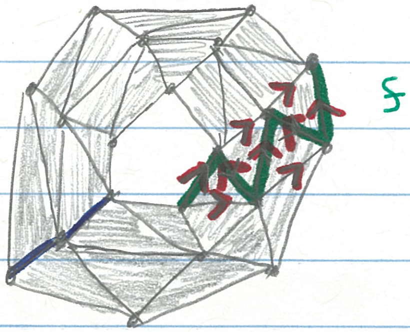
$X =$ 

$H^1(X;\mathbb{Z})$ is generated by $f_{a_1}, f_{a_6} \in \operatorname{Ker}\delta^1 \subseteq \Delta^1(X;\mathbb{Z})$, where $f_{a_1}: \Delta_1(X) \to \mathbb{Z}$ via $f_{a_1}(c_1a_1 + \dots + c_7a_7) = c_1$ and $f_{a_6}: \Delta_1(X) \to \mathbb{Z}$ via $f_{a_6}(c_1a_1 + \dots + c_7a_7) = c_6$.

Here $f_{a_6}(a_6) = 1$ (so $f_{a_6}(-a_6) = -1$) and $f_{a_6}(a_i) = 0$ $\forall i \neq 6$.
We say f_{a_6} is the dual cochain to edge a_6.

Ex $X =$ 2-dimensional complex (triangles filled in)

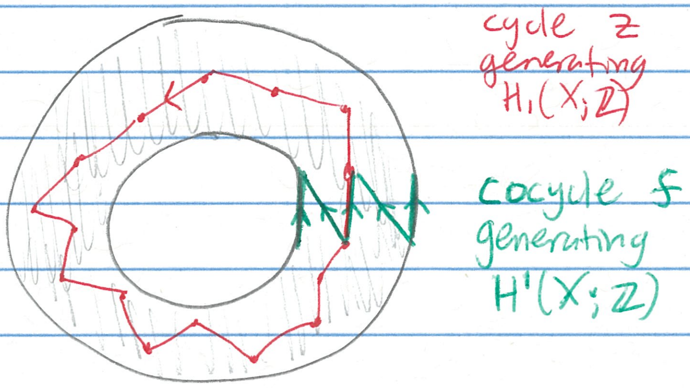
Aside: The cochain g in blue is not a cocycle since $\delta^1 g \neq 0$.



$H^1(X;\mathbb{Z})$ is generated by the cocycle f assigning 1 to each (oriented) edge in green, and 0 to every other edge.

To see that f is a cocycle ($f \in \operatorname{Ker}\delta^1$), note $\delta^1 f(T) = f\partial_2(T) = \begin{cases} 1-1+0=0 & \text{if } T \text{ is a 2-simplex bordering two green edges} \\ 0+0+0=0 & \text{if } T \text{ is any other 2-simplex.} \end{cases}$

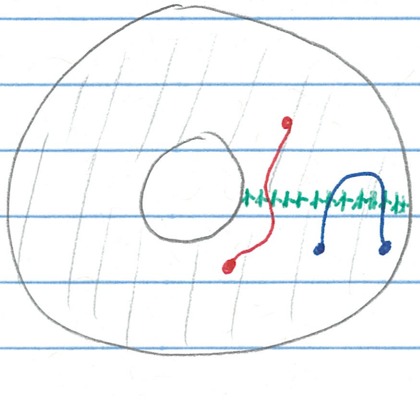
This generator for simplicial $H^1(X;\mathbb{Z})$ is "dual" to a generator for $H_1(X;\mathbb{Z})$



Note $f(z) = 1$, even if we replace z with a homologous cycle, or if we replace f with a cohomologous cocycle.

Life tip Always prepare a question you "could" ask at the end of any math talk, so that if nobody else asks a question, you can help the speaker by asking yours. One question that makes sense at the end of any talk is "what happens if you dualize?"

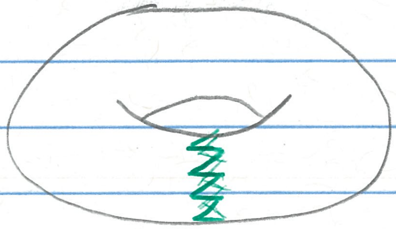
Rmk For singular $H^1(X;\mathbb{Z})$ where X is again the annulus, a generating cocycle f assigns to each singular edge an "oriented count" of the number of times it crosses the green line.



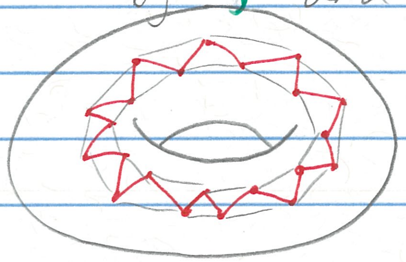
$f(\text{red edge}) = 1$

$f(\text{blue edge}) = 1 - 1 = 0$

Ex $X=$ torus. Simplicial $H^1(X;\mathbb{Z}) \cong \mathbb{Z}\times\mathbb{Z}$ generated by f and g

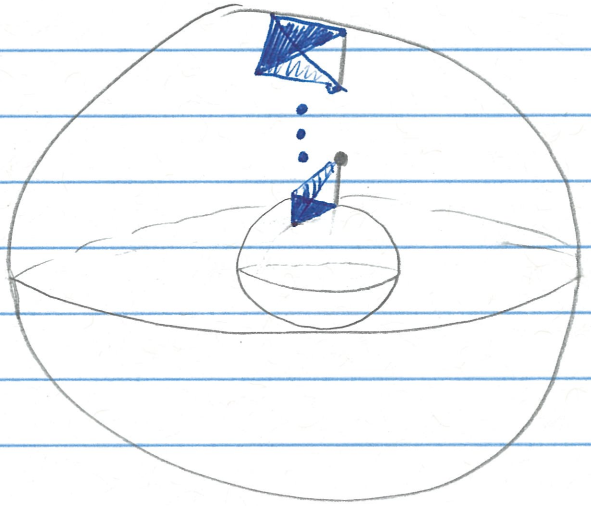


cocycle f



cocycle g

Ex $X = S^2 \times I$ Simplicial $H^2(X;\mathbb{Z}) \cong \mathbb{Z}$



generating cocycle f (with $\delta^2 f = 0$)

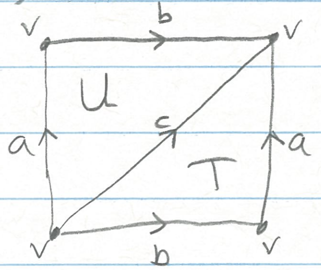
Here each tetrahedron T satisfies

$$\delta^2 f(T) = f\partial_3(T) = \begin{cases} 1-1+0+0=0 & T \text{ borders two blue triangles} \\ 0+0+0+0=0 & \text{otherwise} \end{cases}$$

4/13/18

Example simplicial cohomology calculations (coefficients in $\mathbb{Z}$)

Ex X is a torus with the Δ-complex structure



Here $\partial_2 U = b - c + a$
and $\partial_2 T = a - c + b$

Cochain complex

$$\begin{tikzcd}
\cdots & \Delta^3(X;\mathbb{Z}) \arrow[l, "\delta^3"'] \arrow[d, equal, "\text{3-cochains}"] & \Delta^2(X;\mathbb{Z}) \arrow[l, "\delta^2"'] \arrow[d, equal, "\text{2-cochains}"] & \Delta^1(X;\mathbb{Z}) \arrow[l, "\delta^1"'] \arrow[d, equal, "\text{1-cochains}"] & \Delta^0(X;\mathbb{Z}) \arrow[l, "\delta^0"'] \arrow[d, equal, "\text{0-cochains}"] & 0 \arrow[l] \\
 & 0 & \mathrm{Hom}(\Delta_2(X),\mathbb{Z}) \arrow[d, phantom, "\cong"] & \mathrm{Hom}(\Delta_1(X),\mathbb{Z}) \arrow[d, phantom, "\cong"] & \mathrm{Hom}(\Delta_0(X),\mathbb{Z}) \arrow[d, phantom, "\cong"] & \\
 & & \mathrm{Hom}(\mathbb{Z}^{\oplus 2},\mathbb{Z}) \arrow[d, phantom, "\cong"] & \mathrm{Hom}(\mathbb{Z}^{\oplus 3},\mathbb{Z}) \arrow[d, phantom, "\cong"] & \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \arrow[d, phantom, "\cong"] & \\
 & & \mathbb{Z}^{\times 2} & \mathbb{Z}^{\times 3} & \mathbb{Z} &
\end{tikzcd}$$

Generated by U^* and T^*, where U^* is the 2-cochain given by $U^*(U) = 1$, $U^*(T) = 0$. That is, U^* is dual to U. Similarly, T^* is dual to T.

Generated by a^*, b^*, and c^*. For example, $a^*(a) = 1$, $a^*(b) = 0$, $a^*(c) = 0$.

Generated by v^*

Note $\delta^0 v^* = 0$ since

$\delta^0 v^*(a) = v^* \partial_1(a) = v^*(v - v) = v^*(0) = 0$ and

$\delta^0 v^*(b) = v^* \partial_1(b) = \quad \cdots \quad = 0$ and

$\delta^0 v^*(c) = v^* \partial_1(c) = \quad \cdots \quad = 0$.

So $\delta^0 = 0$.

Note $\delta^1 a^* = U^* + T^*$ since

$\delta^1 a^*(U) = a^* \partial_2(U) = a^*(b-c+a) = 1$ and

$\delta^1 a^*(T) = a^* \partial_2(T) = a^*(a-c+b) = 1$.

Similarly, we compute $\delta^1 b^* = U^* + T^*$

and $\delta^1 c^* = -U^* - T^*$.

Note $\delta^2 = 0$ since $\Delta^3(X;\mathbb{Z}) = 0$.

We compute

$$H^0(X;\mathbb{Z}) = \text{Ker}\,\delta^0 \cong \mathbb{Z}$$

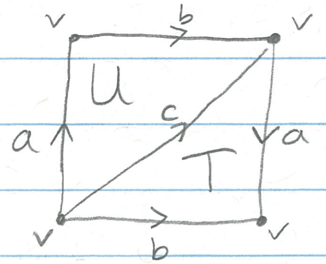
$$H^1(X;\mathbb{Z}) = \text{Ker}\,\delta^1 / \text{Im}\,\delta^0 \cong \text{Ker}\,\delta^1 \cong \mathbb{Z}^{\times 2}$$

(generated for example by $a^* + c^*$ and $b^* + c^*$)

$$H^2(X;\mathbb{Z}) = \text{Ker}\,\delta^2 / \text{Im}\,\delta^1 \cong \underset{\uparrow}{\Delta^2(X;\mathbb{Z})} / \text{Im}\,\underset{\uparrow}{\delta^1} \cong \mathbb{Z}$$

Generated by $\{U^*, T^*\}$ or $\{U^*, U^* + T^*\}$ Generated by $U^* + T^*$

Ex X is a Klein bottle with the Δ-complex structure



Here $\partial_2 U = b-c+a$
and $\partial_2 T = a-b+c$

Cocham complex

$$\begin{tikzcd}
\cdots & \Delta^3(X;\mathbb{Z}) \arrow[l, "\delta^3"'] \arrow[d, equal] & \Delta^2(X;\mathbb{Z}) \arrow[l, "\delta^2"'] \arrow[d, phantom, "\cong" rotate=90] & \Delta^1(X;\mathbb{Z}) \arrow[l, "\delta^1"'] \arrow[d, phantom, "\cong" rotate=90] & \Delta^0(X;\mathbb{Z}) \arrow[l, "\delta^0"'] \arrow[d, phantom, "\cong" rotate=90] & 0 \arrow[l] \\
 & 0 & \mathbb{Z}^{\times 2} & \mathbb{Z}^{\times 3} & \mathbb{Z} &
\end{tikzcd}$$

Generated by U^*, T^*

Generated by a^*, b^*, c^*

Generated by v^*

We have $\delta^0 = 0$, as before.

Again $\delta^1 a^* = U^* + T^*$, but now $\delta^1 b^* = U^* - T^*$ since
$\delta^1 b^*(U) = b^* \partial_2(U) = b^*(b-c+a) = 1$ and
$\delta^1 b^*(T) = b^* \partial_2(T) = b^*(a-b+c) = -1$.
Similarly $\delta^1 c^* = -U^* + T^*$.

Agam $\delta^2 = 0$.

Generated by $b^* + c^*$.

Generated by $\{U^*, T^*\}$ or $\{U^*, U^* + T^*\}$.

We compute $H^0(X;\mathbb{Z}) = \operatorname{Ker}\delta^0 \cong \mathbb{Z}$

Generated by $\{U^* + T^*, U^* - T^*\}$ or $\{2U^*, U^* + T^*\}$.

$H^1(X;\mathbb{Z}) = \operatorname{Ker}\delta^1 / \operatorname{Im}\delta^0 \cong \operatorname{Ker}\delta^1 \cong \mathbb{Z}$

$H^2(X;\mathbb{Z}) = \operatorname{Ker}\delta^2 / \operatorname{Im}\delta^1 \cong \Delta^2(X;\mathbb{Z}) / \operatorname{Im}\delta^1 \cong \mathbb{Z}/2\mathbb{Z}$

This is the first example we've seen where cohomology is not isomorphic to homology, since for X = Klein bottle we had

$$H_i(X;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & i=1 \\ 0 & i \geq 2. \end{cases}$$

(Indeed, see solutions to HW6 #3)

The fact that the torsion term has "jumped" from $H_1(X;\mathbb{Z})$ to $H^2(X;\mathbb{Z})$ is related to Corollary 3.3 in our book.

1/16/18

Section 3.2 Cup Product

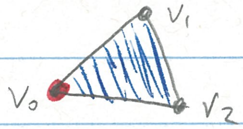
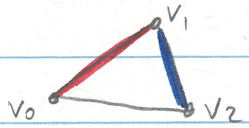
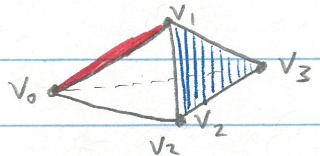
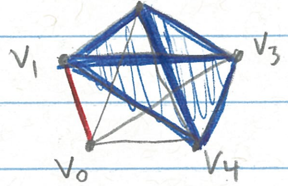
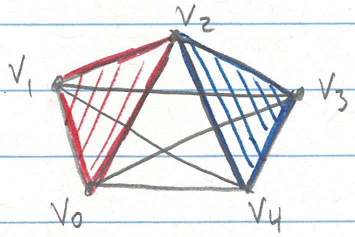
Let X be a space

Let R be a ring (typically $\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$, or $\mathbb{Q}$).

Def Given singular cochains $f \in C^k(X;R)$ and $g \in C^\ell(X;R)$ (or simplicial cochains $f \in \Delta^k(X;R)$ and $g \in \Delta^\ell(X;R)$), the <u>cup product</u> $f \smile g \in C^{k+\ell}(X;R)$ is given by

$$(f \smile g)(\sigma) = f(\sigma|_{[v_0, \ldots, v_k]}) \underset{\substack{\uparrow \\ \text{product in } R}}{} g(\sigma|_{[v_k, \ldots, v_{k+\ell}]})$$

Ex $f \in C^k(X;R)$ $\quad g \in C^\ell(X;R)$

k	ℓ	Picture of $f \smile g$
0	0	
0	1	
0	2	
1	1	
1	2	
1	3	
2	2	

This product feels fairly unnatural!

Certainly $f \smile g \neq g \smile f$

Rmk We'll see $\smile : C^k(X;R) \times C^\ell(X;R) \longrightarrow C^{k+\ell}(X;R)$

induces $\smile : H^k(X;R) \times H^\ell(X;R) \longrightarrow H^{k+\ell}(X;R)$

For R commutative, and $\alpha \in H^k(X;R)$, $\beta \in H^\ell(X;R)$, we'll have $\alpha \smile \beta = (-1)^{k\ell} \beta \smile \alpha$.

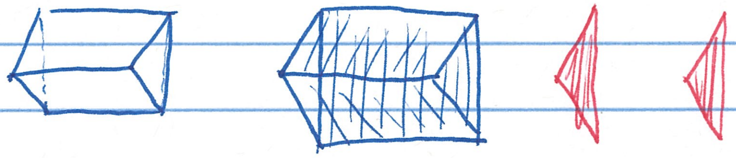
The following lemma will give the induced map on homology

Lemma 3.6 $\delta(f \smile g) = (\delta f \smile g) + (-1)^k (f \smile \delta g)$ for $f \in C^k(X;R)$ and $g \in C^\ell(X;R)$.

Aside This is loosly analogous to having $\partial(X \times Y) = (\partial X \times Y) \cup (X \times \partial Y)$ when X and Y are spaces.

Ex $\partial(I \times I) = (\partial I \times I) \cup (I \times \partial I)$

Ex $\partial(\Delta^2 \times I) = (\partial \Delta^2 \times I) \cup (\Delta^2 \times \partial I)$



Proof of Lemma 3.6 Let $\sigma : \Delta^{k+\ell+1} \longrightarrow X$.

$$\begin{aligned}
\delta(f \smile g)(\sigma) &= (f \smile g)(\partial \sigma) \\
&= \sum_{i=0}^{k+\ell+1} (-1)^i (f \smile g)(\sigma|_{[v_0, \ldots, \hat{v}_i, \ldots, v_{k+\ell+1}]}) \\
&= \sum_{i=0}^{k} (-1)^i f(\sigma|_{[v_0, \ldots, \hat{v}_i, \ldots v_{k+1}]}) g(\sigma|_{[v_{k+1}, \ldots, v_{k+\ell+1}]}) \\
&\quad + \sum_{i=k+1}^{k+\ell+1} (-1)^i f(\sigma|_{[v_0, \ldots, v_k]}) g(\sigma|_{[v_k, \ldots, \hat{v}_i, \ldots, v_{k+\ell+1}]})
\end{aligned}$$

Note $(\delta f \cup g)(\sigma) = (\delta f)(\sigma|_{[v_0, \dots, v_{k+1}]})\, g(\sigma|_{[v_{k+1}, \dots, v_{k+l+1}]})$

$$= \sum_{i=0}^{k+1} (-1)^i f(\sigma|_{[v_0, \dots, \hat{v}_i, \dots v_{k+1}]})\, g(\sigma|_{[v_{k+1}, \dots, v_{k+l+1}]})$$

and $(-1)^k (f \cup \delta g)(\sigma) = (-1)^k f(\sigma|_{[v_0, \dots, v_k]})\, (\delta g)(\sigma|_{[v_k, \dots, v_{k+l+1}]})$

$$= (-1)^k f(\sigma|_{[v_0, \dots, v_k]}) \sum_{i=k}^{k+l+1} (-1)^{i-k} g(\sigma|_{[v_k, \dots, \hat{v}_i, \dots, v_{k+l+1}]})$$

$$= \sum_{i=k}^{k+l+1} (-1)^i f(\sigma|_{[v_0, \dots, v_k]})\, g(\sigma|_{[v_k, \dots, \hat{v}_i, \dots, v_{k+l+1}]})$$

Hence we're done after checking that the $i=k+1$ term in $(\delta f \cup g)(\sigma)$ cancels with the $i=k$ term in $(-1)^k (f \cup \delta g)(\sigma)$.

Consequences

- If f and g are cocycles ($\delta f = 0$, $\delta g = 0$), then so is $f \cup g$ since
$$\delta(f \cup g) = (\underset{\substack{\shortparallel \\ 0}}{\delta f} \cup g) + (-1)^k (f \cup \underset{\substack{\shortparallel \\ 0}}{\delta g}) = 0$$

- The cup product of a cocycle and a coboundary is a coboundary

  Case 1 $\underset{\text{cocycle}}{f} \cup \underset{\text{coboundary}}{\delta g} = \pm \delta(f \cup g)$ if $\delta f = 0$

  Case 2 $\underset{\text{coboundary}}{\delta f} \cup \underset{\text{cocycle}}{g} = \delta(f \cup g)$ if $\delta g = 0$

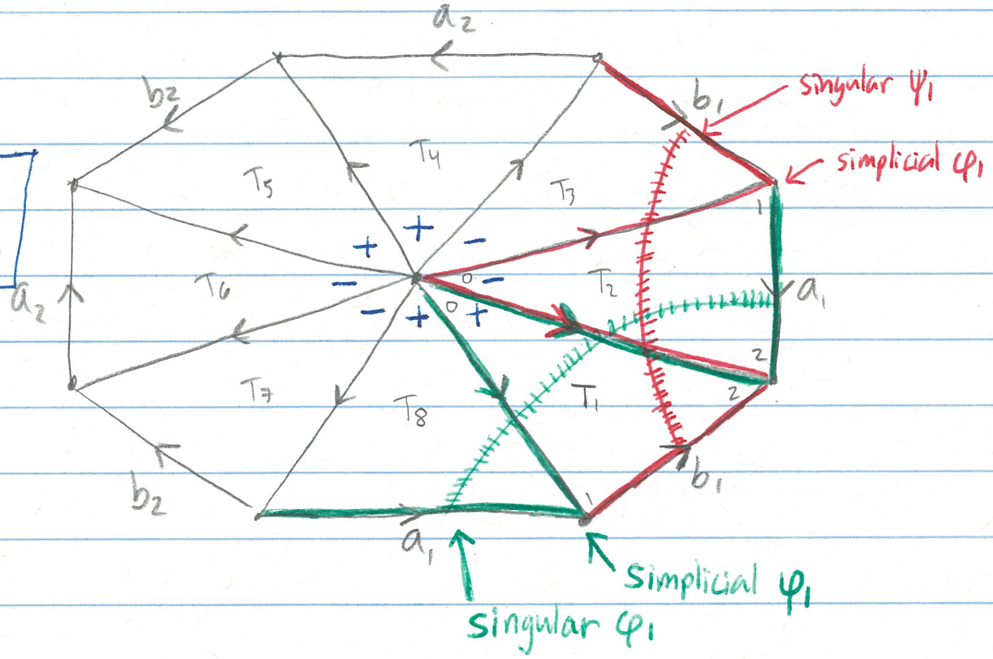
- Hence we get an induced map on cohomology
$$\cup : H^k(X;R) \times H^l(X;R) \longrightarrow H^{k+l}(X;R).$$

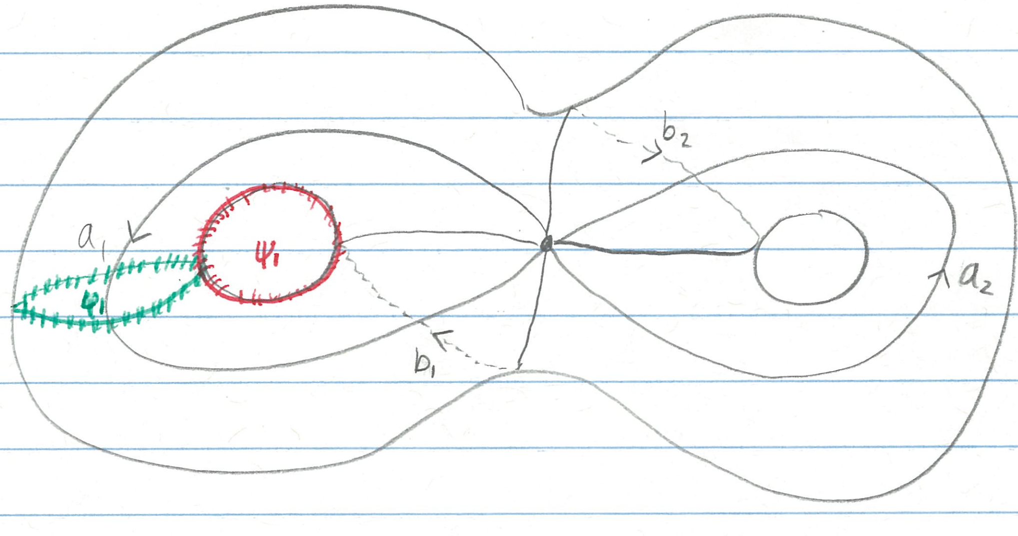
4/18/18 Cup product example

Ex 3.7 Let $M = M_g$ be the orientable surface of genus $g \geq 1$, with Δ-complex as sketched below.

Pic $g=2$

2-chain c is a 2-cycle since $\partial c = 0$





Let's use $\mathbb{Z}$ coefficients everywhere.

$$H_i(M) \cong \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}^{\oplus 2g} & i=1 \\ \mathbb{Z} & i=2 \\ 0 & i\geq 3 \end{cases} \qquad H^i(M) \cong \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}^{\times 2g} & i=1 \\ \mathbb{Z} & i=2 \\ 0 & i\geq 3 \end{cases}$$

$H_1(M) \cong \mathbb{Z}^{\oplus 2g}$ is generated by cycles $a_1, b_1, \ldots, a_g, b_g$ (really $[a_1], \ldots, [b_g]$).
$H^1(M) \cong \mathbb{Z}^{\times 2g}$ is generated by cocycles $\varphi_1, \psi_1, \ldots, \varphi_g, \psi_g$ (really $[\varphi_1], \ldots, [\psi_g]$).

Here $\varphi_i(a_i)=1$ and $\varphi_i(a_j)=0=\varphi_i(b_j)$ otherwise
Here $\psi_i(b_i)=1$ and $\psi_i(b_j)=0=\psi_i(a_j)$ otherwise

Note $\delta\varphi_i=0$ since each triangle meets either 0 or 2 (appropriately oriented) edges on which φ_i has the value 1.

Similarly, $\delta\psi_i=0$.

What is the cup product structure $H^1(M)\times H^1(M) \xrightarrow{\cup} H^2(M)$?

Note $(\varphi_1 \cup \psi_1)(T_1)=1$ and $(\varphi_1\cup\psi_1)(T_i)=0$ $\forall$ 2-simplices with $i\neq 1$.
Note $(\psi_1\cup\varphi_1)(T_2)=1$ and $(\psi_1\cup\varphi_1)(T_i)=0$ $\forall i\neq 2$.

$H_2(M)\cong\mathbb{Z}$ is generated by the 2-cycle c (really by $[c]$).
$H^2(M)\cong\mathbb{Z}$ is generated by the 2-cocycle γ (really by $[\gamma]$).
Here $\gamma(T_i)=\pm 1$ for a single i (and $\gamma(T_j)=0$ otherwise).
All such choices of representing cocycle γ differ by a coboundary.
Signs are such that $\gamma(c)=1$.

We compute $(\varphi_1 \cup \psi_1)(c) = 1$, meaning $[\varphi_1] \cup [\psi_1] = [\gamma]$

We compute $(\psi_1 \cup \varphi_1)(c) = -1$, meaning $[\psi_1] \cup [\varphi_1] = -[\gamma]$.

Similarly

$$[\varphi_i] \cup [\psi_j] = \left\{ \begin{array}{cc} [\gamma] & i=j \\ 0 & i \neq j \end{array} \right\} = -([\psi_j] \cup [\varphi_i])$$

$$[\varphi_i] \cup [\varphi_j] = 0 \quad \forall i,j$$
$$[\psi_i] \cup [\psi_j] = 0 \quad \forall i,j$$

By distributivity, this determines the cup product $H^1(M) \times H^1(M) \xrightarrow{\cup} H^2(M)$ completely.

Rmk Nonzero cup products occur here precisely when the corresponding singular loops intersect. This works even for $[\varphi_1] \cup [\varphi_1] = 0$ after deforming one copy of the loop to be disjoint from the other.

Intersecting 1-cocycles allows the cup product to evaluate to a non-zero number on some triangles.

4/20/18

Introduction to de Rham cohomology

Not on final.

Terry Tao: "The integration on forms concept is of fundamental importance in differential topology, geometry, and physics, and also yields one of the most important examples of cohomology, namely de Rham cohomology, which (roughly speaking) measures precisely the extent to which the fundamental theorem of calculus fails in higher dimensions and on general manifolds."

Big Picture M smooth manifold

The de Rham cochain complex of differential forms is

$$0 \to \Omega^0(M) \xrightarrow{d^0} \Omega^1(M) \xrightarrow{d^1} \Omega^2(M) \xrightarrow{d^2} \dots$$

Here $\Omega^0(M)$ is the space of C^∞ functions on M
$\Omega^1(M)$ is the space of 1-forms
$\Omega^2(M)$ is the space of 2-forms

Here d is the exterior derivative, which satisfies $d \circ d = 0$.

The n-th de Rham cohomology group is

$H^n_{dR}(M) = \operatorname{Ker}(d^n) / \operatorname{Im}(d^{n-1})$

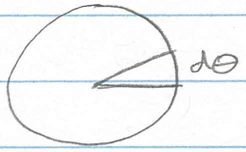
Forms in $\operatorname{Im}(d^{n-1})$ are exact
Forms in $\operatorname{Ker}(d^n)$ are closed
$d \circ d = 0$ says exact forms are closed.

The converse (closed n-forms are exact) is true if and only if $H^n_{dR}(M) = 0$.

Aside The vector space of $\underset{\substack{\uparrow \\ w(v_1,\dots,v_k)=0 \\ \text{if } v_i=v_j \text{ for } i\neq j}}{\text{alternating}}$, k-linear maps $\underbrace{V\times\dots\times V}_{k \text{ times}} \to \mathbb{R}$ is $\mathrm{Alt}^k(V)$

For $U \subseteq \mathbb{R}^n$, a differential k-form on U is a smooth map $w: U \to \mathrm{Alt}^k(\mathbb{R}^n)$. The vector space of all such maps is denoted $\Omega^k(U)$.

Ex Let $d\theta$ be the (closed) 1-form of angular measure on the unit circle $M = S^1$.



There's no function $\theta: S^1 \to \mathbb{R}$ whose derivative is $d\theta$ (We don't know whether to define $\theta(1,0) = 0$ or $\pm 2\pi$ or $\pm 4\pi$ or ...)

So $d\theta$ is a closed 1-form that's not exact.

So $H^1_{dR}(S^1) \neq 0$.

Book: "From Calculus to Cohomology" by Madsen & Tornehave

Restrict attention to $M = U$ with $U \subseteq \mathbb{R}^2$ open.

$$0 \longrightarrow \Omega^0(U) \xrightarrow{d^0} \Omega^1(U) \xrightarrow{d^1} \Omega^2(U) \longrightarrow 0$$

$$0 \longrightarrow C^\infty(U,\mathbb{R}) \xrightarrow{\text{grad}} C^\infty(U,\mathbb{R}^2) \xrightarrow{\text{rot}} C^\infty(U,\mathbb{R}) \longrightarrow 0$$

$$\text{grad}(\phi) = \left(\frac{\partial\phi}{\partial x_1}, \frac{\partial\phi}{\partial x_2}\right) \qquad \text{rot}(\phi_1,\phi_2) = \frac{\partial\phi_1}{\partial x_2} - \frac{\partial\phi_2}{\partial x_1}$$

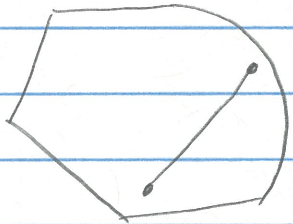
Note $\text{rot}\circ\text{grad} = 0$ since
$$\text{rot}(\text{grad}(\phi)) = \text{rot}\left(\frac{\partial\phi}{\partial x_1}, \frac{\partial\phi}{\partial x_2}\right) = \frac{\partial\phi}{\partial x_1\partial x_2} - \frac{\partial\phi}{\partial x_2\partial x_1} = 0$$

$$H^1_{dR}(U) = \mathrm{Ker}(\text{rot})/\mathrm{Im}(\text{grad}) \qquad H^0_{dR}(U) = \mathrm{Ker}(\text{grad})$$

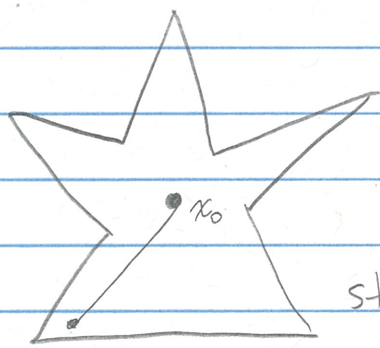
Question 1 Let $U \subseteq \mathbb{R}^2$ be open.
Let $f: U \to \mathbb{R}^2$ with $f = (f_1, f_2)$ be smooth.
Is there a smooth $F: U \to \mathbb{R}$ with $\frac{\partial F}{\partial x_1} = f_1$ and $\frac{\partial F}{\partial x_2} = f_2$?
(IE, is f exact)?

Necessary condition $$\frac{\partial^2 F}{\partial x_2 \partial x_1} = \frac{\partial^2 F}{\partial x_1 \partial x_2} \Rightarrow \frac{\partial f_1}{\partial x_2} = \frac{\partial f_2}{\partial x_1} \quad (\star)$$
(For f to be exact, it must be closed)

Theorem If $U \subseteq \mathbb{R}^2$ is convex, or more generally star-shaped, then Question 1 has a solution whenever f satisfies $(\star)$.



Convex



star-shaped

(IE, $H^1_{dR}(U) = 0$ if U is convex or star-shaped)

Ex $U = \mathbb{R}^2 \setminus \{(0,0)\}$. Let $f: U \to \mathbb{R}^2$ via
$$f(x_1, x_2) = \left(\frac{-x_2}{x_1^2 + x_2^2}, \frac{x_1}{x_1^2 + x_2^2} \right).$$
It's easy to check $(\star)$ is satisfied.

However, there's no $F: U \to \mathbb{R}$ with $\frac{\partial F}{\partial x_1} = f_1$ and $\frac{\partial F}{\partial x_2} = f_2$.

Indeed, such an F would satisfy

$$\int_0^{2\pi} \frac{d}{d\theta} F(\cos\theta, \sin\theta)\, d\theta = F(1,0) - F(1,0) = 0$$

Contradicting the fact that

$$\begin{aligned}\frac{d}{d\theta} F(\cos\theta, \sin\theta) &= \frac{\partial F}{\partial x_1}(-\sin\theta) + \frac{\partial F}{\partial x_2}(\cos\theta)\\ &= -f_1(\cos\theta, \sin\theta)\sin\theta + f_2(\cos\theta, \sin\theta)\cos\theta\\ &= \frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta + \sin^2\theta}\\ &= 1.\end{aligned}$$

(IE, $H^1_{dR}(\mathbb{R}^2 \setminus \{(0,0)\}) \neq 0$)

Restrict attention to $M = U$ with $U \subseteq \mathbb{R}^3$ open

$$0 \rightarrow \Omega^0(U) \xrightarrow{d^0} \Omega^1(U) \xrightarrow{d^1} \Omega^2(U) \xrightarrow{d^2} \Omega^3(U) \rightarrow 0$$

$$0 \rightarrow C^\infty(U, \mathbb{R}) \xrightarrow{\text{grad}} C^\infty(U, \mathbb{R}^3) \xrightarrow{\text{rot}} C^\infty(U, \mathbb{R}^3) \xrightarrow{\text{div}} C^\infty(U, \mathbb{R}) \rightarrow 0$$

$$\text{grad}(f) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right)$$
$$\text{rot}(f_1, f_2, f_3) = \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}, \frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}, \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right)$$
$$\text{div}(f_1, f_2, f_3) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}$$

Note $\text{rot} \circ \text{grad} = 0$ and $\text{div} \circ \text{rot} = 0$

$H^1_{dR} = \text{Ker}(\text{rot}) / \text{Im}(\text{grad})$ $\qquad$ $H^2_{dR} = \text{Ker}(\text{div}) / \text{Im}(\text{rot})$

Stokes' theorem $\int_M d\omega = \int_{\partial M} \omega$ expresses a duality between de Rham cohomology and the homology of chains.

<u>de Rham's Theorem (1931)</u> For M a smooth manifold, the map $I: H^n_{dR}(M) \longrightarrow H^n(M;\mathbb{R})$ is an isomorphism.

For any $\omega \in H^n_{dR}(M)$, let $I(\omega)$ be the element of $H^n(M;\mathbb{R}) \cong \mathrm{Hom}(H_n(M;\mathbb{R}),\mathbb{R})$ that acts via $H_n(M;\mathbb{R}) \ni c \longmapsto \int_c \omega$.

4/23/18

Section 3.3 Poincaré Duality

We'll start with an easier version; the formal version is Theorem 3.30.

Recall a closed manifold is compact with no boundary.

Poincaré Duality (easier version) Let M be a n-dimensional manifold. Let M be differentiable (or more generally, let M have a pair of "dual cell structures").

Then $H^k(M;\mathbb{Z}/2\mathbb{Z}) \cong H_{n-k}(M;\mathbb{Z}/2\mathbb{Z})$.

Furthermore, if M is orientable, then $H^k(M;\mathbb{Z}) \cong H_{n-k}(M;\mathbb{Z})$

Rmk In Thm 3.30, no assumption of differentiability or dual cell structures in needed.

In Thm 3.30, the isomorphism is given via taking "cap products" with the "fundamental class".

Orientable examples

$H^n(S^n;\mathbb{Z}) \cong \mathbb{Z} \cong H_0(S^n;\mathbb{Z})$ $\quad$ $H^0(S^n;\mathbb{Z}) \cong \mathbb{Z} \cong H_n(S^n;\mathbb{Z})$ $\quad n>1$

$H^k(S^n;\mathbb{Z}) \cong 0 \cong H_{n-k}(S^n;\mathbb{Z})$ for $0<k<n$

$H^k((S^1)^n;\mathbb{Z}) \cong \mathbb{Z}^{\binom{n}{k}} \cong \mathbb{Z}^{\binom{n}{n-k}} \cong H_{n-k}((S^1)^n;\mathbb{Z}) \quad \forall k$

$\uparrow$ n-dimensional torus

$$H^k(M_g;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & k=0,2 \\ \mathbb{Z}^{\times 2g} & k=1 \end{cases} \cong H_{n-k}(M_g;\mathbb{Z})$$

$\uparrow$ orientable surface genus g

$\mathbb{R}P^n$ is orientable for n odd

$$H_i(\mathbb{R}P^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i=0 \text{ or } n \text{ odd} \\ \mathbb{Z}/2\mathbb{Z} & 0<i<n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

$$H^i(\mathbb{R}P^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i=0 \text{ or } n \text{ odd} \\ \mathbb{Z}/2\mathbb{Z} & 0<i<n \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

So for n odd, $H^k(\mathbb{R}P^n;\mathbb{Z}) \cong H_{n-k}(\mathbb{R}P^n;\mathbb{Z})$

Non-orientable examples $\mathbb{R}P^n$ is not orientable for n even

$$H^k(\mathbb{R}P^n;\mathbb{Z}/2\mathbb{Z}) = \left\{ \begin{matrix} \mathbb{Z}/2\mathbb{Z} & 0\leq k\leq n \\ 0 & \text{otherwise} \end{matrix} \right\} \cong H_{n-k}(\mathbb{R}P^n;\mathbb{Z}/2\mathbb{Z})$$

N_g is a non-orientable surface of genus g

$$H^k(N_g;\mathbb{Z}/2\mathbb{Z}) = \left\{ \begin{matrix} \mathbb{Z}/2\mathbb{Z} & k=0,2 \\ (\mathbb{Z}/2\mathbb{Z})^g & k=1 \\ 0 & \text{otherwise} \end{matrix} \right\} \cong H_{n-k}(N_g;\mathbb{Z}/2\mathbb{Z})$$

For M a n-manifold, we have
$H^k(M;\mathbb{Z}/2\mathbb{Z}) \cong 0 \cong H_{n-k}(M;\mathbb{Z}/2\mathbb{Z})$
for $k>n$ or $k<0$, since one of
k or $n-k$ will be negative!

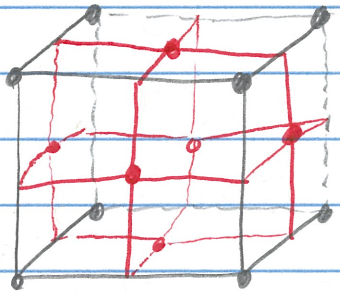
Rmk Somehow the local property defining manifolds (locally homeomorphic to $\mathbb{R}^n$) imposes strong control on the global properties of homology and cohomology (Poincaré duality).

A dual cell-decomposition of an n-manifold M is a pair of cell structures C, C^* such that

- each $(n-k)$-cell of C has a corresponding k-cell of C^*
- the boundary of an $(n-k)$-cell σ in C contains a $(n-k-1)$-cell of C^*

$\Longleftrightarrow$ the boundary of the dual $(k+1)$-cell in C^* contains the dual k-cell of C^*

Ex $M = S^2$

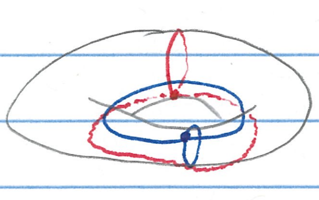


C - 8 vertices
12 edges
6 2-cells

C^* - 6 vertices
12 edges
8 2-cells

Besides the duality between the cube and octahedron, another example is the duality between the dodecahedron and icosahedron.

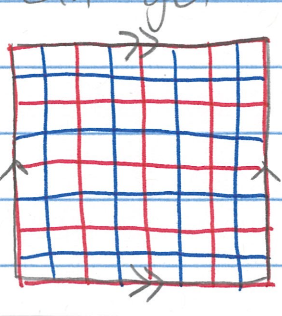
Ex $M = S^1 \times S^1$



C - 1 vertex
2 edges
1 2-cell

C^* - 1 vertex
2 edges
1 2-cell

We can get finer dual cell-decompositions via



C C^*

A similar example works for $S^1 \times S^1 \times S^1$ with all squares replaced by cubes.

Proof of this easier version of Poincaré duality

Use CW structure C to build a chain complex, and CW structure C^* to build a cochain complex

$$0 \to C_n(M;\tfrac{\mathbb{Z}}{2\mathbb{Z}}) \xrightarrow{\partial_n} C_{n-1}(M;\tfrac{\mathbb{Z}}{2\mathbb{Z}}) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_1(M;\tfrac{\mathbb{Z}}{2\mathbb{Z}}) \xrightarrow{\partial_1} C_0(M;\tfrac{\mathbb{Z}}{2\mathbb{Z}}) \xrightarrow{\partial_0} 0$$

$$0 \to C^{*0}(M;\tfrac{\mathbb{Z}}{2\mathbb{Z}}) \xrightarrow{\delta^0} C^{*1}(M;\tfrac{\mathbb{Z}}{2\mathbb{Z}}) \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^{n-2}} C^{*(n-1)}(M;\tfrac{\mathbb{Z}}{2\mathbb{Z}}) \xrightarrow{\delta^{n-1}} C^{*n}(M;\tfrac{\mathbb{Z}}{2\mathbb{Z}}) \xrightarrow{\delta^n} 0$$

Note for all k we have

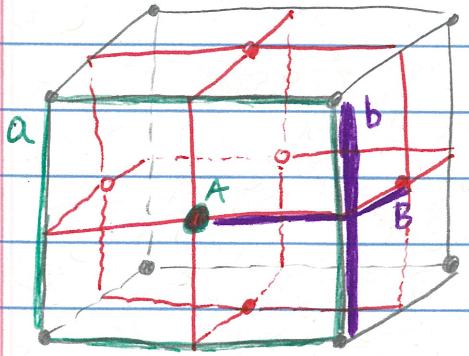
$$\begin{aligned} C_{n-k}(M;\tfrac{\mathbb{Z}}{2\mathbb{Z}}) &\cong \mathbb{Z}^{\oplus(\#(n-k)\text{-cells in } C)} \\ &\cong \mathbb{Z}^{\times(\#\, k\text{-cells in } C^*)} \\ &\cong C^{*k}(M;\tfrac{\mathbb{Z}}{2\mathbb{Z}}) \end{aligned}$$

Furthermore, the definition of the dual cell-decomposition gives $\partial_{n-k} = \delta^k$.

Hence $H^k(M;\mathbb{Z}/2\mathbb{Z}) \cong H_{n-k}(M;\mathbb{Z}/2\mathbb{Z})$.

A similar proof works with $\mathbb{Z}$ coefficients when M is orientable.

4/25/18 In the proof of the easier version of Poincaré duality, let's check that $\partial_{n-k} = \delta^k$ (after identifying $C_{n-k}(M; \frac{\mathbb{Z}}{2\mathbb{Z}})$ with $C^{*k}(M; \frac{\mathbb{Z}}{2\mathbb{Z}})$ $\forall k$)



C, C^* dual cell-decompositions

Let a be an $(n-k)$-cell in C and b be a $(n-k-1)$-cell in C

Let A be the corresponding k-cell in C^* and B be the corresponding $(k+1)$-cell in C^*

By the second bullet point defining dual cell decompositions,

b has coefficient 1 in ∂a

$\iff$ A has coefficient 1 in ∂B

$\iff$ B^* has coefficient 1 in δA^* (since $\delta A^*(B) = A^*(\partial B)$)

$\uparrow$ cochain assigning 1 to the $(k+1)$-cell B

$\uparrow$ cochain assigning 1 to the k-cell A

This is true for all a, b, A, B, giving $\partial_{n-k} = \delta^k$.

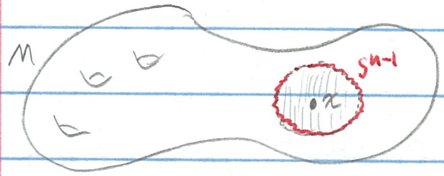
Let R be a commutative ring with identity (think $R = \mathbb{Z}, \mathbb{Z}/2\mathbb{Z}$)

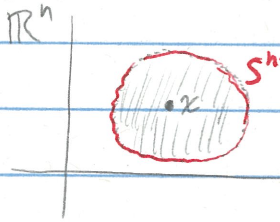
Thm 3.30 (Poincaré Duality) If M is a closed R-orientable n-manifold with fundamental class $[M] \in H_n(M;R)$, then the map $D : H^k(M;R) \longrightarrow H_{n-k}(M;R)$ defined by $D(\alpha) = [M] \frown \alpha$ is an isomorphism $\forall k$.

Rmk We need to define R-orientable, fundamental class, and the cap product $\frown$.

Def (local) An *R-orientation of M at $x \in M$* is a choice of generator/unit (element $\mu \in R$ with $R\mu = R$) of

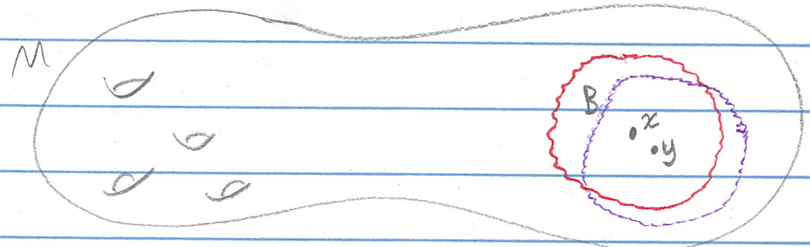
$$
\begin{aligned}
H_n(M, M-\{x\};R) &\cong H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\};R) && \text{by excision and the def}^{\text{n}} \text{ of } n\text{-manifold} \\
&\cong H_{n-1}(\mathbb{R}^n - \{x\};R) && \text{by the LES of the pair } (\mathbb{R}^n, \mathbb{R}^n-\{x\}) \text{ with } \mathbb{R}^n \simeq * \\
&\cong H_{n-1}(S^{n-1};R) && \text{since } \mathbb{R}^n - \{x\} \simeq S^{n-1} \\
&\cong R.
\end{aligned}
$$





Def (global) An *R-orientation of M* is a function $x \longmapsto \mu_x$ assigning to each $x \in M$ a local orientation $\mu_x \in H_n(M, M-\{x\};R)$, satisfying the consistency criterion that each $x \in M$ has an open neighborhood $x \in B \subseteq M$ with $\mu_B \in H_n(M, M-B;R)$, where $\forall y \in B$ we have

$$\begin{tikzcd}
H_n(M, M-B;R) \arrow[r] & H_n(M, M-\{y\};R) \\
\mu_B \arrow[r, mapsto] & \mu_y .
\end{tikzcd}$$



If an R-orientation exists then M is <u>R-orientable</u>.

<u>Ex</u> Any manifold is $\mathbb{Z}/2\mathbb{Z}$ orientable as there is only one choice of generator.

<u>Ex</u> Spheres and tori are $\mathbb{Z}$-orientable whereas the Klen bottle and $\mathbb{RP}^2$ are not.

<u>Def</u> A <u>fundamental class</u> for M is an element $[M] \in H_n(M;R)$ satisfying

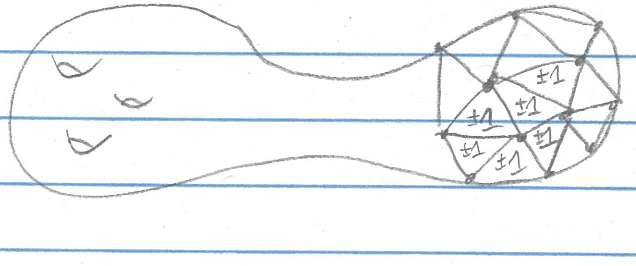
$$\begin{tikzcd} H_n(M;R) \arrow[r] & H_n(M, M-\{x\};R) \\ {[M]} \arrow[r, mapsto] & \mu_x \end{tikzcd}$$

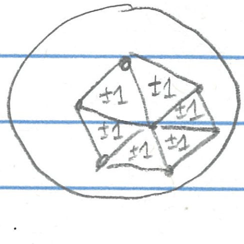
(local orientation)

$\forall x \in M$.

<u>Fact</u> M is R orientable $\Longleftrightarrow$ a fundamental class exists.

<u>Ex</u> When M is a Δ-complex and $R=\mathbb{Z}$, a fundamental class $[M] \in H_n(M;\mathbb{Z})$ is an n-cycle with coefficient ±1 on each n-simplex of M.





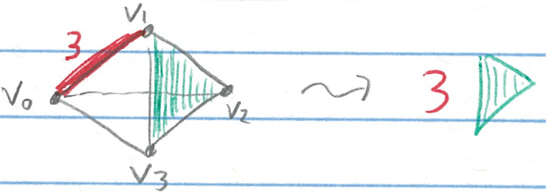
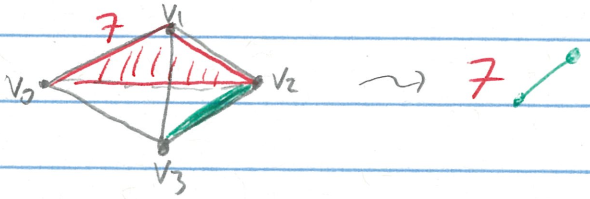
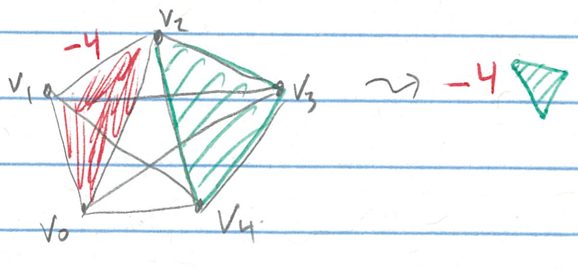
Cap product

Let X be an arbitrary space and R an arbitrary ring. We'll have a cap product $H_k(X;R)\times H^\ell(X;R) \xrightarrow{\frown} H_{k-\ell}(X;R)$.

This has close connections to the cup product $H^k(X;R)\times H^\ell(X;R) \xrightarrow{\smile} H^{k+\ell}(X;R)$.

For $k \geq \ell$, define the R-bilinear map
$C_k(X;R) \times C^\ell(X;R) \longrightarrow C_{k-\ell}(X;R)$ via

$$\overset{\rotatebox{90}{$\in$}}{\sigma} \frown \overset{\rotatebox{90}{$\in$}}{f} = f(\sigma|_{[v_0,\dots,v_\ell]})\,\sigma|_{[v_\ell,\dots,v_k]} \qquad \sigma:\Delta^k \to X$$

k	ℓ	Picture of $\sigma \frown f$
3	1	
3	2	
4	2	

Here f "eats" the first ℓ vertices to turn a k-simplex into a $(k-\ell)$-simplex.

1/27/18 One can check $\partial(\sigma \frown f) = (-1)^{\ell}(\partial\sigma \frown f - \sigma \frown \delta f)$.

Consequences

- The cap product of a cycle and a cocycle is a cycle.
- The cap product of a cycle and coboundary is a boundary since $\pm(\sigma \frown \delta f) = \partial(\sigma \frown f)$ if $\partial\sigma = 0$.
- The cap product of a boundary and cocycle is a boundary since $\pm(\partial\sigma \frown f) = \partial(\sigma \frown f)$ if $\delta f = 0$.

Hence we get an induced cap product

$$H_k(X;R) \times H^{\ell}(X;R) \xrightarrow{\frown} H_{k-\ell}(X;R).$$

Verification $\partial\sigma \frown f = \sum_{i=0}^{\ell}(-1)^i f(\sigma|[v_0,\dots,\hat{v}_i,\dots,v_{\ell+1}])\,\sigma|[v_{\ell+1},\dots,v_k]$
$+\sum_{i=\ell+1}^{k}(-1)^i f(\sigma|[v_0,\dots,v_\ell])\,\sigma|[v_\ell,\dots,\hat{v}_i,\dots,v_k]$

$$\sigma \frown \delta f = \sum_{i=0}^{\ell+1}(-1)^i f(\sigma|[v_0,\dots,\hat{v}_i,\dots,v_{\ell+1}])\,\sigma|[v_{\ell+1},\dots,v_k]$$

$$\partial(\sigma \frown f) = \sum_{i=\ell}^{k}(-1)^{i-\ell} f(\sigma|[v_0,\dots,v_\ell])\,\sigma|[v_\ell,\dots,\hat{v}_i,\dots,v_k]$$

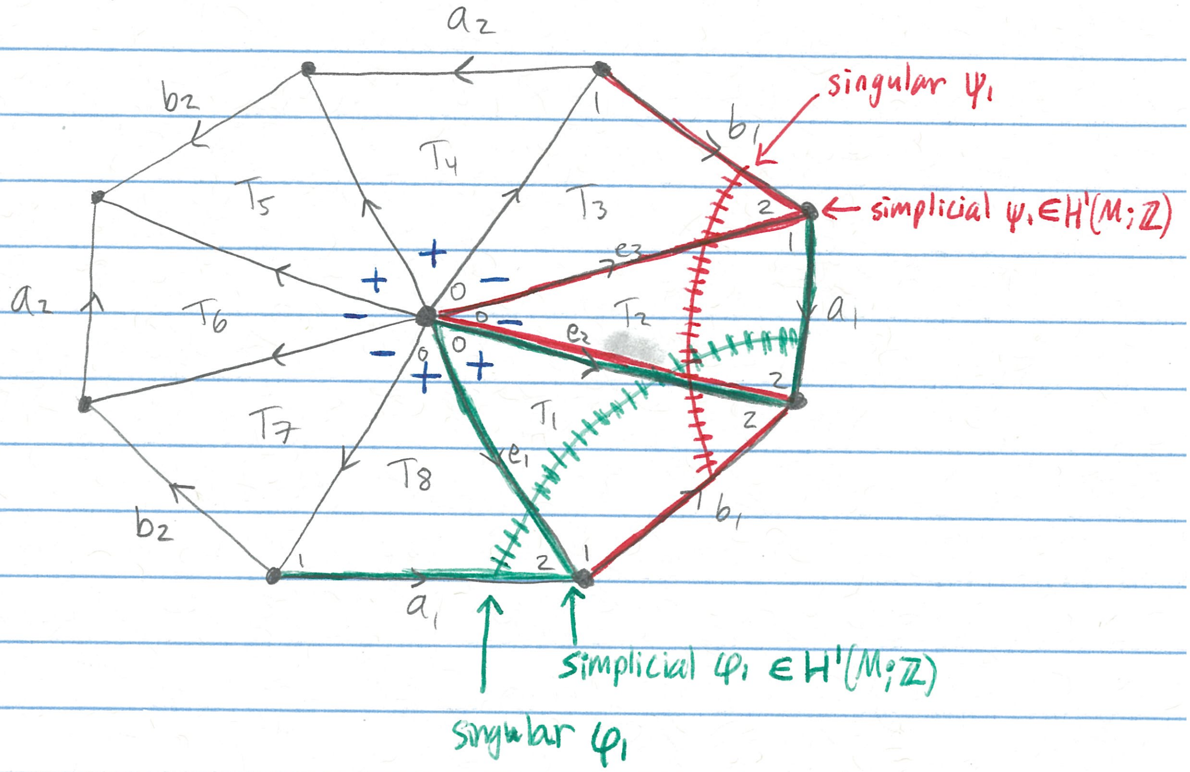
One can check that the $i=\ell+1$ term of $\sigma \frown \delta f$ and the $i=\ell$ term of $\partial(\sigma \frown f)$ cancel after considering all of the signs.

Poincaré Duality example

__Ex 3.31__ Let $M=M_g$ be the orientable surface of genus $g\geq 1$, with Δ-complex as sketched below.

Pic $g=2$

The fundamental class $[M]\in H_2(M_g;\mathbb{Z})$ is a linear combination of all 2-simplices, with signs chosen to get a 2-cycle.



We compute

$$[M]\frown\varphi_1=\Sigma_{j=1}^{8}\left(\pm T_j\frown\varphi_1\right)$$

$$=\Sigma_{j=1}^{8}(\pm\varphi_1)\left(T_j|_{[v_0,v_1]}\right)T_j|_{[v_1,v_2]}$$

$$=\varphi_1\left(T_1|_{[v_0,v_1]}\right)T_1|_{[v_1,v_2]}\quad\text{(all other terms are zero)}$$

$$=\varphi_1(e_1)\cdot b_1$$

$$=b_1$$

So the cohomology class $\varphi_1\in H^1(M;\mathbb{Z})$ is Poincaré dual to the homology class $b_1\in H_1(M;\mathbb{Z})$.

Similarly, the cohomology class $\psi_1 \in H^1(M;\mathbb{Z})$ is Poincaré dual to the homology class $-a_1 \in H_1(M;\mathbb{Z})$ since

$$[M] \frown \psi_1 = \sum_{j=1}^{8} (\pm T_j \frown \psi_1)$$

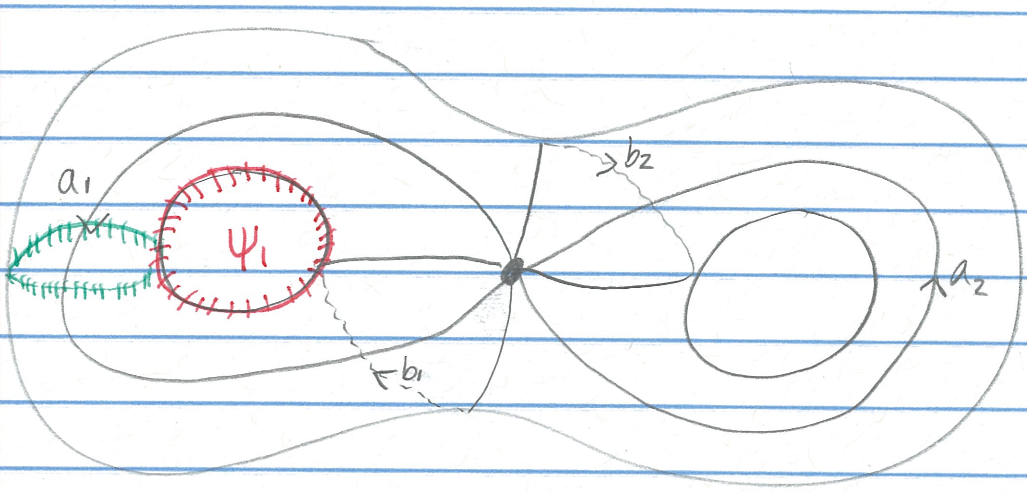
$$= \sum_{j=1}^{8} (\pm \psi_1)(T_j|_{[v_0,v_1]})\, T_j|_{[v_1,v_2]}$$

$$= -\psi_1 (T_2|_{[v_0,v_1]})\, T_2|_{[v_1,v_2]} \quad \text{(all other terms are zero)}$$

$$= -\psi_1(e_3) \cdot a_1$$

$$= -a_1 .$$

Geometrically, Poincaré duality is represented by the fact that $-a_1$ is homotopic to the (singular) ψ_1 "loop", and b_1 is homotopic to the (singular) φ_1 "loop".



Rmk In "simple cases" only, we have $H^n(X;\mathbb{Z}) \cong \text{Hom}(H_n(X), \mathbb{Z})$.

Indeed, the universal coefficient theorem for cohomology (Theorem 3.2) gives a SES

$$0 \longrightarrow \text{Ext}(H_{n-1}(X), \mathbb{Z}) \longrightarrow H^n(X;\mathbb{Z}) \longrightarrow \text{Hom}(H_n(X), \mathbb{Z}) \longrightarrow 0,$$

so $H^n(X;\mathbb{Z}) \cong \text{Hom}(H_n(X);\mathbb{Z}) \Longleftrightarrow H_{n-1}(X)$ has no torsion.

In "even simpler cases" where $H_n(X)$ has no torsion, we have $H^n(X;\mathbb{Z}) \cong \text{Hom}(H_n(X), \mathbb{Z}) \cong H_n(X;\mathbb{Z})$.

[The above is essentially the proof of Corollary 3.3:
$H^n(X;\mathbb{Z}) \cong (H_n(X)/T_n) \oplus T_{n-1}$ in the finitely generated case.]

In Ex 3.31, we are in this particularly simple case

$$\begin{aligned} H^1(X;\mathbb{Z}) &\cong \text{Hom}(H_1(X), \mathbb{Z}) \cong H_1(X;\mathbb{Z}) \\ \varphi_1 &\longleftrightarrow [a_1 \longmapsto 1] \quad \text{since } \varphi_1(a_1) = 1 \\ \psi_1 &\longleftrightarrow [b_1 \longmapsto 1] \quad \text{since } \psi_1(b_1) = 1 \end{aligned}$$

(Geometrically, this was since φ_1 and a_1 intersected, and since ψ_1 and b_1 intersected.)

The above isomorphism is very different from Poincaré duality, where we had

$$\begin{aligned} H^1(X;\mathbb{Z}) &\xrightarrow{[M]\frown -} H_1(X;\mathbb{Z}) \\ \varphi_1 &\longleftrightarrow b_1 \\ \psi_1 &\longleftrightarrow -a_1. \end{aligned}$$

The end!