Motivating question: When are two spaces equivalent up to stretching or bending, not allowing ripping, tearing, or gluing?

That is, when are two spaces homeomorphic (\(\cong\)), or homotopy equivalent (\(\cong\)), or ambient isotopic?

\[
\begin{align*}
\text{Ex} & \quad \circ \cong \quad \Rightarrow \quad \Omega \cong \quad \Rightarrow \quad \Omega \cong \\
\text{Ex} & \quad \circ \cong \quad \Omega \cong \\
\text{Ex} & \quad \circ \cong \\
\end{align*}
\]

If surfaces \(\circ\) and \(\Omega\) were made of very flexible rubber, how do you deform one to get the other? This is an ambient isotopy.

Point-set topology: This is a generalization of metric spaces that allows us to define spaces and continuity without a notion of distance.

Given spaces \(X\) and \(Y\), one can often show \(X \cong Y\) or \(X \cong Y\) by giving an explicit deformation between them. But how would you show \(X \not\cong Y\) or \(X \not\cong Y\)? It's hard to consider/rule out all possible deformations.
Algebraic topology

Given a space $X$, we'll associate to $X$ families of algebraic invariants: the homotopy groups $\pi_i(X)$ and homology groups $H_i(X)$.

Roughly speaking, $\pi_i(X)$ and $H_i(X)$ "count the # of $i$-dimensional holes in $X".\]

- "3 1-d holes"  "1 2-d hole"  "2 1-d holes"
  "1 2-d hole via $H_2$"  "No 2-d holes via $\pi_2$"

- If $X \simeq Y$, then $\pi_i(X) \cong \pi_i(Y)$ and $H_i(X) \cong H_i(Y)$ for all $i$.

  Hence if $\pi_i(X) \neq \pi_i(Y)$ or $H_i(X) \neq H_i(Y)$ for some $i$, then $X \not\simeq Y$.

- Homotopy groups are easier to define and harder to compute.
- Homology groups are harder to define and easier to compute.
- These are examples of categories and functors.
- An example non-negative result is Whitehead's Theorem: If a continuous map $f: X \to Y$ between CW complexes induces an isomorphism $f_* : \pi_i(X) \to \pi_i(Y)$ for all $i$, then $X \simeq Y$.\]
Chapter 2: Topological Spaces

Def A topology $T$ on a set $X$ is a collection of subsets of $X$ (called open sets) such that

(i) $X$ and $\emptyset$ are open
(ii) A finite intersection of open sets is open
(iii) An arbitrary union of open sets is open.

The resulting topological space is denoted $(X, T)$ or simply $X$.

Every metric space is a topological space (where the open sets are unions of open balls).

Ex

$\mathbb{R}^k = \{-n/n\} = \{0\}$ is not open in $\mathbb{R}$

Def The discrete topology on $X$ is $T = \mathcal{P}(X)$

The trivial or indiscrete topology on $X$ is $T = \{\emptyset, X\}$

Ex $X = \{1, 2, 3\}$

- Discrete topology
- Indiscrete topology
- Bizarre topology

Question If $X = \{1, 2, 3\}$ were a metric space, which topology must it have?

Rmk Any subset $X$ of $\mathbb{R}^n$ (with the Euclidean metric) is a topological space.
Def: A subset $C$ of topological space $X$ is **closed** if its complement $X \setminus C$ is open.

$X \setminus \emptyset$ is open in $X$.

(Not open in $\mathbb{R}^2$ but that's irrelevant)

Properties: If $X$ is a topological space, then

(i) $\emptyset$ and $X$ are closed.

(ii) A finite union of closed sets is closed.

(iii) An arbitrary intersection of closed sets is closed.

Proof (i) $X \setminus \emptyset = X$ is open; $X \setminus X = \emptyset$ is open.

Proof (ii) $C_1, \ldots, C_n$ closed $\Rightarrow X \setminus C_1, \ldots, X \setminus C_n$ open $\Rightarrow \bigcap_{i=1}^{n} (X \setminus C_i)$ is open

and $\bigcup_{i=1}^{n} C_i = X \setminus \left( \bigcap_{i=1}^{n} (X \setminus C_i) \right)$

Def: Let $X$ be a topological space and $A \subseteq X$.

The closure of $A$ in $X$ is the smallest closed set containing $A$:

$\bar{A} = \bigcap \{ C \subseteq X \text{ closed} : A \subseteq C \}$.
The interior of $A$ in $X$ is the largest open set contained in $A$:

$$\text{Int } A = \bigcup \{ U \subseteq X \mid U \subseteq X \text{ open and } U \subseteq A \}$$

The boundary of $A$ in $X$ is $\partial A = \overline{A} \cap (X \setminus \overline{A})$, or equivalently,

$$\partial A = X \setminus (\text{Int } A \cup (X \setminus \overline{A}))$$

**Ex**

Let $A = (1, 3]$ and $X = \mathbb{R}$. Then $\overline{A} = [1, 3]$, $\text{Int } A = (1, 3)$, $\partial A = \{1, 3\}$

**Ex**

Let $A = (1, 3]$ and $X = (1, \infty)$. Then $A = (1, 3]$, $\text{Int } A = (1, 3)$, $\partial A = \{1, 3\}$

---

**Convergence and Continuity**

Recall

If $X$ and $Y$ are metric spaces, then $f : X \to Y$ is continuous if:

1. For all $x \in X$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x, x') < \delta \Rightarrow d(f(x), f(x')) < \varepsilon$
2. For all $B(f(x), \varepsilon) \subseteq Y$ there exists $\delta > 0$ such that $B(x, \delta) \subseteq f^{-1}(B(f(x), \varepsilon))$
3. For all open balls $B$ in $Y$, $f^{-1}(B)$ is open in $X$
4. For all open sets $U$ in $Y$, $f^{-1}(U)$ is open in $X$

---

**Def**

If $X$ and $Y$ are topological spaces, then $f : X \to Y$ is continuous if for all open sets $U$ in $Y$, $f^{-1}(U)$ is open in $X$. 

- Continuous
- Not continuous
Def A **homeomorphism** between topological spaces $X$ and $Y$ is a bijection $f: X \to Y$ s.t. $f$ and $f^{-1}$ are continuous. If such a map exists, we say $X$ and $Y$ are **homeomorphic** $(X \cong Y)$.

**Ex**

$S^1 = \{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \}$ and $E = \{ (x,y) \in \mathbb{R}^2 \mid (x/3)^2 + y^2 = 1 \}$ are homeomorphic via $(x,y) \mapsto (3x, y)$.

**Rmk** The hypothesis that $f^{-1}$ is continuous is necessary.

Note $S^1 \to S^1$ via $f(t) = (\cos t, \sin t)$ is a continuous bijection that is not a homeomorphism since $f^{-1}$ is not continuous.

**Ex**

Let $B^n = \{ x \in \mathbb{R}^n \mid ||x|| < 1 \}$. Can you find a homeomorphism $f: B^n \to \mathbb{R}^n$?

**Ans**

$f(x) = \frac{x}{1-||x||}$, $f^{-1}(y) = \frac{y}{1+||y||}$.

Check $f^{-1} \circ f = \text{id}_X$ and $f \circ f^{-1} = \text{id}_Y$.

Hence "boundedness" is not a "topological property".

**Ex**

$\square \cong \square$ and $\bigcirc \cong \square$

"Corners" or "smoothness" is not a topological property.

**Rmk** For $f: X \to Y$ a homeomorphism, note $U$ is open in $X \iff f(U)$ is open in $Y$. 
Since arbitrary topological spaces can be wild (for instance a convergent sequence may have multiple limit points), we now study nicer spaces.

**Hausdorff Spaces**

**Def** A neighborhood of point $x$ in topological space $X$ is an open set $U$ with $x \in U$.

**Def** A topological space $X$ is **Hausdorff** if given any distinct $x_1, x_2 \in X$, there exist disjoint open neighborhoods $U_1 \ni x_1$ and $U_2 \ni x_2$.

**Ex** A metric space $X$ is **Hausdorff**: given $x_1, x_2 \in X$ with $d(x_1, x_2) = r > 0$, consider the open balls $B(x_1, r/2)$ and $B(x_2, r/2)$.

**Ex** A discrete space $X$ is **Hausdorff**: given $x_1 \neq x_2 \in X$, consider the disjoint open neighborhoods $\{x_1\}$ and $\{x_2\}$.

**Prop 2.37** For $X$ a Hausdorff space,

(a) Every finite subset is closed

(b) A convergent sequence has a unique limit

**PF (a)** We first show a single point $\{x_0\}$ is closed. Let $x \in X \setminus \{x_0\}$; since $X$ Hausdorff we can find an open neighborhood $U \ni x$ contained in $X \setminus \{x_0\}$. Hence $X \setminus \{x_0\}$ is open and $\{x_0\}$ is closed.
Finally, note any finite set is a finite union of closed points and hence closed.

**Bases and Countability**

Let $X$ be a topological space. Instead of writing down all the open subsets of $X$, a more convenient way to describe the topology is to give a basis.

**Def** A collection $\mathcal{B}$ of open subsets of $X$ is a **basis** for $X$ if every open set in $X$ is a union of sets in $\mathcal{B}$.

**Ex** $M$, a metric space. The collection of open balls is a basis for $M$ since every open set is a union of open balls.

**Ex** In a discrete topological space, the collection of singletons is a basis.

\[ \emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\} \]

**Def** A topological space is **second countable** if it has a countable basis.

**Ex** How do you see that $\mathbb{R}$ is second countable?

Consider the collection of all balls with rational centers and radii.

**Ex** $\mathbb{R}^n$ is second countable, by the same argument!
We'll define a manifold to be a second countable Hausdorff space that is "locally Euclidean".

Hausdorff will rule out the line with two origins:

\[ \bullet \quad o' \rightarrow \bullet \quad o \]

Open sets consist of all open sets \( U \) in \( \mathbb{R} \), along with \((U \setminus o') \cup o'\) if \( 0 \notin U \).

Not Hausdorff since any open sets about \( 0, o' \) intersect.

Second countable will rule out the long line:
Whereas \( \mathbb{R} \) is a countable union of half-open intervals \((0,1])\) laid end-to-end, the long line is an uncountable such union.

Homework A second countable space has a countable dense subset (i.e. is "separable").

It is easy to believe that the long line is not separable.

Picture \( \mathbb{R} = \mathbb{Z} \times (0,1] \) with order topology

Long ray = \( \mathbb{R} \times (0,1] \) with order topology

open interval
Manifolds

Def. A space $M$ is locally Euclidean of dimension $n$ if every point $p \in M$ has a neighborhood homeomorphic to an open subset of $\mathbb{R}^n$.

Ex.

Lemma 2.52. Could equivalently replace (i) “open subset of $\mathbb{R}^n$” with (ii) “open ball of $\mathbb{R}^n$” or (iii) “$\mathbb{R}^n$”.

- $(ii) \Rightarrow (i)$ and $(iii) \Rightarrow (i)$ clear.
- $(ii) \Leftrightarrow (iii)$ since $B(x,r) \cong \mathbb{R}^n$.

To see $(i) \Rightarrow (ii)$, consider some $B(s(p),r) \subseteq V$ and its preimage $S^{-1}(B(s(p),r)) \subseteq U$.

Def. An $n$-dimensional manifold is a second countable Hausdorff space that is locally Euclidean of dimension $n$.

Ex. $S^n = \{ x \in \mathbb{R}^{n+1} | \|x\| = 1 \}$ is an $n$-sphere.
Ex: Projective space \( \mathbb{RP}^n = S^n / \sim \), where \( \sim \) is the equivalence relation \( \alpha \sim -\alpha \).

\( \mathbb{RP}^1 \cong S^1 \)

Ex: Klein bottle

9/1/2017  Manifolds with boundary

\( \mathbb{B}^3 = \{ x \in \mathbb{R}^3 | \| x \| \leq 1 \} \)

Def: Upper half-space \( H^n = \{ (x_1, \ldots, x_n) \in \mathbb{R}^n | x_n \geq 0 \} \).

Def: An n-dimensional manifold with boundary is a second countable Hausdorff space in which every point has a neighborhood homeomorphic to an open subset of either \( \mathbb{R}^n \) or \( H^n \). (Could use only \( H^n \))
Def: The boundary of $H^n$ is $\partial H^n = \{x_1, \ldots, x_n \in H^n \mid x_n = 0\}$.

Def: For $M$ an $n$-dimensional manifold with boundary, the boundary of $M$ (denoted $\partial M$) is the set of all $p \in M$ such that some (hence all) homeomorphism $f$ from a neighborhood of $p$ to $H^n$ has $f(p) \in \partial H^n$.

Warning: The boundary of a manifold and the boundary of some subset $A \subseteq X$ (with $X$ a topological space) mean different things.

Chapter 3: New spaces from old
- subspaces, product spaces, disjoint unions, quotients

Subspaces

Def: Given a topological space $X$ and a subset $S \subseteq X$, the subspace topology on $S$ has as its open sets all $U \subseteq S$ such that $U = S \cap V$ for some open set $V$ in $X$.

Example open sets in $X$
Example open sets in $S$

Ex: $[0, 1)$ is not open in $\mathbb{R}$, but $[0, 1)$ is open in $[0, 2)$ since $[0, 1) = [0, 2) \cap (-1, 1)$.

Rank: For $S \subseteq \mathbb{R}^n$, the Euclidean and subspace topologies agree.
Corollary 3.10(a) If $X, Y$ are spaces, $f: X \to Y$ is continuous, and $S \subseteq X$ has the subspace topology, then $f|_S: S \to Y$ is continuous.

**Proof**

$V$ open in $Y \Rightarrow f^{-1}(V)$ open in $X$ (since $f$ is continuous).

Hence $S \cap f^{-1}(V) = S \cap f^{-1}(V)$ is open in $S$.

---

**Product Spaces**

**Def** Given topological spaces $X_1, \ldots, X_n$, a basis for the product topology on $X_1 \times \ldots \times X_n = \{ (x_1, \ldots, x_n) \mid x_i \in X_i \forall i \}$ is

$B = \{ \prod U_i \mid U_i \text{ is open in } X_i \forall i \}$

"for all"

---

**Pic**

$s^1 \times s^1$

open basis element

---

**Remk** The Euclidean and product topologies on $\mathbb{R}^n$ agree.

More generally,

**Def** Given an indexed family of topological spaces $\{X_\alpha\}_{\alpha \in \Lambda}$, a basis for the product topology on $\prod_{\alpha \in \Lambda} X_\alpha$ is

$B = \left\{ \prod U_\alpha \mid U_\alpha \text{ is open in } X_\alpha \forall \alpha \right\}$

and $U_\alpha = X_\alpha$ for all but finitely many $\alpha$.

---

**Remk** Removing the "$U_\alpha = X_\alpha$ for all but finitely many $\alpha$" gives the box topology, which is different for infinite products.

Note $(0,1)^n$ is open in $(\mathbb{R}^n, \text{box})$ but not $(\mathbb{R}^n, \text{product topology})$. 
The product topology is the "right one" since it gives the product in the category of topological spaces.

9/6/2017

Disjoint Unions

Def. Given an indexed family of topological spaces \( \{ X_\alpha \}_{\alpha \in A} \), the disjoint union topology on \( \bigcup_{\alpha \in A} X_\alpha \) declares a set to be open if its intersection with each \( X_\alpha \) is open in \( X_\alpha \).

Notation. Just like \( X_1 \times \cdots \times X_n = \prod_{i=1}^n X_i \), we have \( X_1 \sqcup \cdots \sqcup X_n = \bigcup_{i=1}^n X_i \).

Remark. This will give coproducts in the category of topological spaces.

Selections from Categories and Functors in Chapter 7

Def. A category \( C \) consists of:
- a class of objects \( \text{Ob}(C) \)
- a class of morphisms \( \text{Hom}(C) \)
- for each \( X, Y, Z \in \text{Ob}(C) \), a composition map \( \text{Hom}_C(X, Y) \times \text{Hom}_C(Y, Z) \rightarrow \text{Hom}_C(X, Z) \)

such that:
- \((f \circ g) \circ h = f \circ (g \circ h)\)
- for all \( X \in \text{Ob}(C) \) \( \exists \text{Id}_X \in \text{Hom}_C(X, X) \) s.t. for every morphism \( f \in \text{Hom}_C(A, B) \), \( \text{Id}_B \circ f = f = f \circ \text{Id}_A \).
Example categories

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<td>rings</td>
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<tr>
<td>vector spaces</td>
<td>( \text{r-linear maps} )</td>
<td>( \text{vector space} )</td>
</tr>
<tr>
<td>over field ( \mathbb{R} )</td>
<td>( \text{module homomorphisms} )</td>
<td>( \text{module} )</td>
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<td>( \text{homomorphisms} )</td>
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<tr>
<td>( (\mathbb{R}, \leq) )</td>
<td>( a \leq b )</td>
<td>( a \leq a )</td>
</tr>
</tbody>
</table>

**Remark**

\( f \in \text{Hom}_c(X, Y) \) is often written \( f : X \rightarrow Y \), even though \( f \) need not be a function.

**Example**

\( (\mathbb{R}, \leq) \) can be thought of as a category with a single morphism \( a \rightarrow b \) for all \( a, b \in \mathbb{R} \) with \( a \leq b \).

![Diagram](attachment:image.png)

**Definition**

A morphism \( f \in \text{Hom}_c(X, Y) \) is an isomorphism if \( \exists g \in \text{Hom}_c(Y, X) \) s.t. \( g \circ f = \text{Id}_X \) and \( f \circ g = \text{Id}_Y \).

![Diagram](attachment:image.png)
**Def** A **product** of the family of objects \((X_{\alpha})_{\alpha \in A}\) in a **category** is an object \(P\) together with morphisms \(\pi_{\alpha} : P \to X_{\alpha}\) s.t. given any object \(W\) and morphisms \(f_{\alpha} : W \to X_{\alpha}\), \(\exists!\) morphism \(f : W \to P\) s.t. the following commutes \(\forall \alpha\):

- **Pic** Case \(A = \{1, \ldots, n\}\)

![Diagram](image)

In the category of sets, take \(P = X_1 \times X_2 \times \cdots \times X_n\). Given \(W\) and the \(s_i\), define \(s : W \to P\) by \(s = (s_1, s_2, \ldots, s_n)\).

<table>
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<tr>
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<td>Abelian groups</td>
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<td>Direct sum</td>
</tr>
<tr>
<td>((\mathbb{R}, \leq))</td>
<td>May not exist</td>
<td>May not exist</td>
</tr>
</tbody>
</table>

**Rmk** Products and coproducts need not exist: In \((\mathbb{R}, \leq)\) you can't take the product or coproduct of \(-2, 1, 0, 1, 2, \ldots\).

**Rmk** Note that \((\mathbb{R}^\mathbb{N}, \text{box})\) is not the product of \(\mathbb{N}\) copies of \(\mathbb{R}\) in the category of topological spaces. Indeed, take \(W = (\mathbb{R}^\mathbb{N}, \text{product})\) and note no such continuous \(f\) exists since \((0,1)^\mathbb{N}\) is open in \((\mathbb{R}^\mathbb{N}, \text{box})\) but not \((\mathbb{R}^\mathbb{N}, \text{product})\).
If a product exists in a category, then it is unique up to isomorphism.

**Proof:** Suppose \((P', (\pi_\alpha))\) and \((P'', (\pi''_\alpha))\) are products.
Take \(P = P''\) and \(W = P'\) to get \(f: P' \to P''\).
Take \(P = P'\) and \(W = P''\) to get \(g: P'' \to P'\).
Take \(P = W = P'\) to get \(g \circ f = \text{Id}_{P'}\) via uniqueness.
Take \(P = W = P''\) to get \(f \circ g = \text{Id}_{P''}\) via uniqueness.

**Definition:** A coproduct of the family of objects \((X_\alpha)_{\alpha \in A}\) in a category is an object \(S\) together with morphisms \(\{w_\alpha\}: X_\alpha \to S\) s.t. given any object \(W\) and morphisms \(\{s_\alpha\}: X_\alpha \to W\), \(\exists!\) morphism \(f: S \to W\) s.t. the following commutes: \(\{w_\alpha\}\):

\[
\begin{array}{ccc}
W & \xleftarrow{s_\alpha} & X_\alpha \\
& \searrow^{w_\alpha} & \\
S & \xrightarrow{s} &
\end{array}
\]

**Proof:** Case \(A = \{1, \ldots, n\}\):

Products have universal maps to the \(X_\alpha\)’s.
Coproducts have universal maps from the \(X_\alpha\)’s.
Quotient Spaces

Def: Let $X$ be a topological space, $Y$ be a set, and $q: X \to Y$ be surjective. The quotient topology on $Y$ declares a set $U \subseteq Y$ to be open when $q^{-1}(U)$ is open in $X$.

Pic:

$X = [0,10] \times [0,1]$  
$Y = X/\sim = \text{Möbius band}$

Let $\sim$ be the equivalence relation $(0,t) \sim (10,1-t)$ and define $q: X \to Y = X/\sim$ by sending a point in $X$ to its equivalence class in $X/\sim$.

Ex 3.47

$I = [0,1]$  
$S^1 \cong I/\sim$ where $0 \sim 1$

Ex 3.48

$B^2$  
$S^2 \cong \overline{B^2}/\sim$ where $(x,y) \sim (-x,y)$ for all $(x,y) \in \partial \overline{B^2}$

Ex 3.49

$I \times I$  
$\text{Torus} \cong (I \times I)/\sim$ where

$(x,0) \sim (x,1)$  
$(0,y) \sim (1,y)$
**Klein bottle** \( \cong (I \times I) / \sim \) where 
\[(x, 0) \sim (x, 1) \]
\[(0, y) \sim (1, 1-y) \]

**Projective space** \( \mathbb{R}P^n \) or \( \mathbb{P}^n \) is 
\( \mathbb{R}P^n \cong S^n / \sim \) where \( x \sim -x \)

Also \( \mathbb{R}P^n \cong (\mathbb{R}^{n+1} \setminus \{0\}) / \sim \) where \( x \sim \lambda x \) for any \( \lambda \neq 0 \).

Also \( \mathbb{R}P^n \cong B^n / \sim \) where \( x \sim -x \) for \( x \in \partial B^n \)

9/1/2017 **Def** For \( X \) a topological space and \( A \subseteq X \), let \( X/A \) denote the quotient topological space \( X/\sim \) where \( a \sim a' \) for all \( a, a' \in A \).
Ex 3.52 \[ \overline{B^n} = \{ x \in \mathbb{R}^n | ||x|| \leq 1 \} \quad \overline{S^n} = \overline{B^n} / \partial \overline{B^n} \]

Ex 3.53 For \( X \) a topological space, the cone on \( X \) is \( C_X = (X \times I) / (X \times \{0\}) \). \[ X = S^1 \quad X \times I \quad C_X \]

Rmk For any topological space \( X \), \( C_X \) is "contractible," i.e., "homotopy equivalent" to a point.

Ex 3.54 Let \( X_1, \ldots, X_n \) be nonempty topological spaces and \( p_i \in X_i \). The wedge sum is \( X_1 \vee \ldots \vee X_n = (X_1 \sqcup \ldots \sqcup X_n) / \sim p_1, \ldots, p_n \).

\[ S^1 \vee S^1 \quad S^1 \vee S^2 \quad S^1 \vee S^2 \vee \text{torus} \]

Rmk Wedge sums are coproducts in the category of pointed topological spaces (see Problem 7-17).

Rmk We're skipping the "Adjunction Spaces" and "Topological Groups and Group Actions" subsections for now.
Chapter 4: Connectedness and Compactness

Connectedness

Def: A topological space $X$ is connected if it cannot be written as the disjoint union of two nonempty open sets.

Pic: $X$ not connected

$X = U \cup V$

Prop 4.1: Equivalently, $X$ is connected $\iff$ the only subsets of $X$ that are both open and closed are $\emptyset$ and $X$.

Prop 4.2: Suppose $X \neq \emptyset$ is connected and $Y$ is discrete. Then any continuous $f: X \to Y$ is a constant map.

Ps: Let $x \in X$ and $y = f(x)$. Since $\exists y \exists$ is open and closed in $Y$, it follows that $f^{-1}(y) \neq \emptyset$ is open and closed in $X$. $X$ connected $\implies f^{-1}(y) = X$, so $f$ is constant.

Thm 4.7: Let $f: X \to Y$ be continuous with $X$ connected. Then $f(X)$ is connected (or equivalently, if $f$ is surjective then $Y$ is connected).

Ps: We'll prove the formulation where $f$ is surjective.

If $Y$ were not connected, we could write $Y = U \cup V$ with $U, V$ open and nonempty, hence $X = f^{-1}(U) \cup f^{-1}(V)$ is not connected.

Draw a picture!

Corollary 4.8: (Invariance of connectedness.) Every space homeomorphic to a connected space is connected.

Ps: $X \xrightarrow{f} Y$
Prop 4.11 A nonempty subset of IR is connected \iff it is a singleton or an interval.

Rmk Proof omitted. Note IR \setminus \{0\} is not connected.

Thm 4.12 (Intermediate Value Theorem) Let X be connected and \( f: X \to Y \) be continuous. If \( p, q \in X \), then \( f \) attains every value between \( f(p) \) and \( f(q) \).

Proof By Thm 4.7 \( f(X) \) is connected, then use Prop 4.11.

Path Connectedness

Rmk This is a different notion of connectedness.
X path connected \implies X connected, but not vice versa.
For "nice" X, the two notions coincide.

Def A path in space X from \( p \in X \) to \( q \in X \) is a continuous map \( f: [0, 1] \to X \) with \( f(0) = p \), \( f(1) = q \).

Def A space X is path-connected if \( \forall p, q \in X \), \exists a path \( \gamma \) in \( X \) from \( p \) to \( q \).
Thm 4.15

X path-connected \implies X connected

(Different proof than book.) Let A \subseteq X be nonempty, open, and closed. Suppose for a contradiction \( A \neq X \). Choose \( a \in A, x \in X \setminus A \) and let \( f : I \to X \) be a path with \( f(0) = a, f(1) = x \). Then \( f^{-1}(A) \) is a nonempty proper subset of \( I \) which is both open and closed, contradicting the fact that \( I \) is connected. Hence \( A = X \) and \( X \) is connected.

Pic

Ex 4.17

The topologist's sine curve is
\[ \{(x, y) \mid -1 \leq y \leq 1, \exists x \in [0, 2\pi] \mid 0 < x \leq \frac{2\pi}{3}, x \in \mathbb{R}^2 \}. \]

It is connected but not path-connected.

Rmk

A connected component of space \( X \) is a maximal nonempty connected subset of \( X \). Similarly for a path-connected component.
A space $X$ is **locally connected** if $\forall p \in X$ and neighborhoods $U$ of $p$, there is a connected neighborhood of $p$ contained in $U$.

Similar definition for **locally path-connected**.

Prop 4.26 (e) shows for $X$ locally path-connected, $X$ connected $\iff X$ path-connected.

**Compactness**

An analogy: A topological space being compact is analogous to a set being finite.

More generally: For $A \subseteq X$ and $U$ a collection of sets in $X$, we say $U$ **covers** $A$ when $A \subseteq \bigcup_{U \in U} U$.

Let $X$ be a topological space.

**Def** An open cover of $X$ is a collection $U$ of open sets in $X$ whose union is $X$, i.e., $\bigcup_{U \in U} U = X$.

**Def** Space $X$ is compact if every open cover $U$ of $X$ has a finite subcover $U' \subseteq U$, i.e., if there is a finite collection $U_1, \ldots, U_k \subseteq U$ such that $U_1 \cup \cdots \cup U_k = X$.

**Ex** $X = \mathbb{R}^2$, $U = \ell$ all open balls with integral centers and radius $1/3$.

Note this $U$ shows $\mathbb{R}^2$ is not compact.
Ex. \( X = (0, 1], \ U = \{ \frac{1}{n}, 1 \} \mid n = 2, 3, 4, \ldots \ 3 \)

Note \( U \) is an open cover with no finite subcover, so \((0, 1]\) is not compact.

Ex 4.30 (a) Every finite space is compact (regardless of its topology).
(b) Every space with the trivial topology is compact.
(c) A discrete space is compact \( \iff \) it is finite.

Rank. We'll see a subset \( A \subseteq \mathbb{R}^n \) is compact \( \iff \) it is closed and bounded.

Lemma 4.27. A subset \( A \subseteq X \) is compact \( \iff \) every cover of \( A \) by open sets in \( X \) has a finite subcover.

Thm 4.32. \( X \) compact and \( f : X \to Y \) continuous \( \implies f(X) \) compact.
Let $U$ be a cover of $\mathcal{S}(X)$ by open subsets of $X$.

Note $\bigcap_{i \leq b^2} \mathcal{S}^{-1}(U_i) \subseteq \mathcal{S}(X)$ is an open cover of $X$.

$X$ compact $\Rightarrow$ $\exists U_1, \ldots, U_n$ s.t. $\bigcap_{i \leq b^2} \mathcal{S}^{-1}(U_i)$ covers $X$.

Hence $\bigcap_{i \leq b^2} U_i$ covers $\mathcal{S}(X)$.

**Corollary 4.33** Every space homeomorphic to a compact space is compact.

**Prop 4.36**

(a) Every closed subset of a compact space is compact.

(b) Every compact subset of a Hausdorff space is closed.

(c) Every compact subset of a metric space is bounded.

(d) A finite product of compact spaces is compact.

(e) A quotient of a compact space is compact.

**PF sketch**

(a) A closed in $X$.

Let $U$ be a cover of $A$ by open sets in $X$.

Note $U \cup (X \setminus A)$ is an open cover of $X$.

$X$ compact $\Rightarrow$ there's a finite subcover $\exists U_1, \ldots, U_n, X \setminus A$ or $\exists U_1, \ldots, U_n, X \setminus A$ giving a finite subcover $\exists U_1, \ldots, U_n, X \setminus A$ of $A$. 

$X \leftarrow \mathcal{S}^{-1} \rightarrow Y$
(c) Let \( A \) be a compact subset of a metric space. Pick \( a \in A \), and consider the open cover of open balls \( B(a, r_n) \) with center \( a \) and radius \( r_n \in \mathbb{N}^\infty \). 

\[ \exists \text{ finite subcover } \implies A \text{ contained in single ball } \implies A \text{ bounded} \]

(e) If \( Y \) is a quotient space of \( X \), then we have \( q : X \rightarrow Y \) with \( q \) surjective and continuous. By Thm 4.32, \( X \) compact \( \implies q(X) = Y \) compact.

Thm 4.39: A closed and bounded interval in \( \mathbb{R} \) is compact.

Ps Sketch: Let \( U \) be an open cover of \( [a, b] \). Consider \( X = \{ x \in (a, b] \mid [a, x] \text{ is covered by finitely many sets of } U \} \). Show \( b \in X \) "by hand".
Thm 4.40 \( A \subseteq \mathbb{R}^n \) compact \( \iff \) it's closed and bounded.

**Proof Sketch** \((\Rightarrow)\) follows from Prop 4.36 (b), (c).

For \((\Leftarrow)\), note \( A \) bounded means \( A \subseteq [-R,R]^n \) for some \( R > 0 \).

Note \([-R,R]^n\) is compact by Thm 4.39 and Prop 4.36 (d).

Hence \( A \) is compact by Prop 4.36(a).

Thm 4.41 \( X \) compact and \( f : X \to \mathbb{R} \) continuous

\( \Rightarrow \) \( f \) is bounded and attains its maximum and minimum values.

**Proof** By Theorems 4.32 and 4.40, \( f(X) \) is closed and bounded.

**Remark** Sequential compactness (every sequence has a convergent subsequence) para-compactness are related notions. Para-compact Hausdorff spaces admit partitions of unity.

**Remark** Skipping Chapter 5 (Cell complexes) and 6 (Compact Surfaces) for now.
Chapter 7: Homotopy and the Fundamental Group

Recall

In topology, by "map" we mean "continuous map".

Spaces $X, Y$ are homeomorphic $(X \cong Y)$ if $\exists$ $f : X \rightarrow Y$ (with $f, f^{-1}$ continuous).

Note $f^{-1} \circ f = \text{id}_X$ and $f \circ f^{-1} = \text{id}_Y$.

Spaces $X, Y$ are homotopy equivalent $(X \simeq Y)$ if $\exists$ $f : X \rightarrow Y$ s.t. $g \circ f \cong \text{id}_X$ and $f \circ g \cong \text{id}_Y$.

Need to define homotopy equivalences b/w maps

Ex. $O \cong O$ since $f : O \rightarrow O$ defined via $f(e^{i\theta}) = e^{i\theta}$ and $g(re^{i\theta}) = e^{i\theta}$.

Note $g \circ f = \text{id}_{\text{circle}}$.

We'll see $f \circ g \cong \text{id}_{\text{annulus}}$.

Def. Let $f, g : X \rightarrow Y$. A homotopy from $f$ to $g$ is a continuous $H : X \times I \rightarrow Y$ with $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$.

If such an $H$ exists, we say $f$ and $g$ are homotopic ($f \simeq g$).

Ex. For $X, Y, X \times I$, $H$ homotopy from $f$ to $g$.

Diagram:

- $O \cong O$ via $f$ and $g$.
- $f \circ g \cong \text{id}_{\text{annulus}}$.
- $H$ shows $f$ to $g$ homotopy.
Ex. In the annulus $\mathbb{A}$ above, show $f \circ g = \text{id}_{\text{annulus}}$.

\[ \text{Annulus} \times I \to \text{Annulus} \]

$H\left(re^{i\theta}, t\right) = (1-t)e^{i\theta} + t e^{i\theta}$

$H$ is an example of a linear homotopy since $H(x,t) = (1-t)f \circ g(x) + t \text{id}_{\text{annulus}}(x)$.

Rmk. A homotopy gives a 1-parameter family of continuous maps $H_t : X \to Y$ defined by $H_t(x) = H(x,t)$.

Prop 7.1. Homotopy is an equivalence relation on the set of all continuous maps from $X$ to $Y$.

Pf. Sketch. If $f \simeq g$ via $H(x,t)$, then $g \simeq f$ via $H(x,1-t)$.

If $f \simeq g$ via $F$ and $g \simeq h$ via $G$, then $f \simeq h$ via

$H(x,t) = \begin{cases} F(x,2t) & 0 \leq t \leq \frac{1}{2} \\ G(x,2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$
Prop 7.2

The homotopy relation is preserved by composition.

\[ X \xrightarrow{\mathcal{F}} Y \xrightarrow{\mathcal{G}} Z \]

If \( f_0 \simeq f_1 \) via \( \mathcal{F} \) and \( g_0 \simeq g_1 \) via \( \mathcal{G} \),
then \( g_0 \circ f_0 \simeq g_1 \circ f_1 \) via \( H : X \times I \to Z \)
where \( H(x,t) = G(F(x,t), t) \).

Note \( H(x,0) = G(f_0(x), 0) = g_0(f_0(x)) \)
and \( H(x,1) = G(f_1(x), 1) = g_1(f_1(x)) \).

Rmk

Our book writes \( F : f_0 \simeq f_1 \) for \( f_0 \simeq f_1 \) via \( F \).
I prefer the latter or \( f_0 \simeq f_1 \), but whatever is fine.

Ex 7.4

If \( f, g : X \to B \subseteq \mathbb{R}^n \) and the straight line between \( f(x) \) and \( g(x) \) is in \( B \) \( \forall x \in X \), then the linear homotopy \( f \simeq g \) via \( H \) is defined by
\[ H(x,t) = (1-t)f(x) + tg(x). \]

[Always possible if \( B \) convex]
Def For \( f, g : X \to Y \) and \( A \subseteq X \), we say \( f \) and \( g \) are homotopic relative \( A \) (or rel \( A \)) if
\[
H(x, t) = f(x) = g(x) \quad \forall x \in A, \quad t \in I.
\]
Def Paths \( f, g : I \to Y \) are path homotopic \((f \sim g)\) if they are homotopic relative \( \{0, 1\} \subseteq I \).

Pic

\[\begin{array}{ccc}
0 & \longrightarrow & 1 \\
\end{array}\]

\[\begin{array}{ccc}
0 & \longrightarrow & 1 \\
\end{array}\]

Rmk Homotopy equivalence rel \( A \) is still an equivalence relation.

Def For \( X \) a space and \( p \in X \), the fundamental group \( \pi_1(X, p) \) is the group of all path homotopy classes of loops based at \( p \).

For a loop \( s : I \to X \) with \( s(0) = s(1) = p \), we write
\[ [s] \in \pi_1(X, p). \]

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Rmk Often denoted \( \pi_1(X) \) since \( X \) path-connected
\[ \Rightarrow \pi_1(X, p) \cong \pi_1(X, q) \] for all \( p, q \in X \).

(Thm 7.13)

iso morphism of groups

Rmk The \( \pi_i(X) \) for \( i \geq 1 \) will be the homotopy groups.
Rmk \( \pi_i \) is a "functor" from the category of spaces to groups.
Rmk \( X \simeq Y \Rightarrow \pi_i(X) \cong \pi_i(Y) \) for all \( i \).
Group operation on $\pi_1(X, p)$

For $f, g : I \to X$ loops based at $p$, their product $f \cdot g : I \to X$ is

$$f \cdot g(s) = \begin{cases} f(2s) & 0 \leq s \leq \frac{1}{2} \\ g(2s-1) & \frac{1}{2} \leq s \leq 1 \end{cases}$$

Note $f \cdot g$ is continuous since $f(1) = g(0) = p$, and is a loop based at $p$ since $f(0) = g(1) = p$.

For $[f], [g] \in \pi_1(X, p)$, we define their product by $[f][g] = [f \cdot g]$.

To see $[f \cdot g]$ is well-defined, note...

Prop 7.10

If $f_0, f_1, g_0, g_1 : I \to X$ are loops based at $p$ with $f_0 \sim f_1$ and $g_0 \sim g_1$, then $f_0 \cdot g_0 \sim f_1 \cdot g_1$.

Proof

If $f_0 \sim f_1$ via $F$ and $g_0 \sim g_1$ via $H$, then $f_0 \cdot g_0 \sim f_1 \cdot g_1$ via

$$H(s, t) = \begin{cases} F(2s, t) & 0 \leq s \leq \frac{1}{2}, 0 \leq t \leq 1 \\ G(2s-1, t) & \frac{1}{2} \leq s \leq 1, 0 \leq t \leq 1 \end{cases}$$
Indeed, note \[ H(s,0) = \begin{cases} \gamma_0(2s) & 0 \leq s < \frac{1}{2} \\ \gamma_0(2s-1) & 0 \leq s \geq \frac{1}{2} \end{cases} = (\gamma_0 \cdot g_0)(s), \]

and similarly \[ H(s,1) = (\gamma_1 \cdot g_1)(s) \]

Ex. \[ \pi_1(S^1) \cong \mathbb{Z} \cong \langle a \rangle, \] the free group on one generator.

\[ \text{group isomorphism} \]

\[ -2 \quad -1 \quad 0 \quad 1 \quad 2 \]

12/20/17 Identity and inverses in \( \pi_1(X, x) \)

Let \( c_p : I \to X \) be the constant loop defined by \( c_p(s) = p \) \( \forall s \in I \).

Given \( \gamma : I \to X \), let \( \bar{\gamma} : I \to X \) be defined by \( \bar{\gamma}(s) = \gamma(1-s) \).

Thm 7.11. For \( \gamma, \theta, h : I \to X \) loops in \( X \) based at \( p \),

(a) \( [c_p] \cdot [\gamma] = [\bar{\gamma}] = [\gamma] \cdot [c_p] \) and hence \([c_p] \) is the identity in \( \pi_1(X, p) \).

(b) \( [\gamma] \cdot [\bar{\gamma}] = [c_p] = [\gamma] \cdot [\bar{\gamma}] \) and hence the inverse of \([\gamma] \) in \( \pi_1(X, p) \) is \([\bar{\gamma}] \).

(c) \( [\gamma] \cdot (\theta \cdot h) = (\gamma) \cdot [\theta \cdot h] \) and hence the multiplication in \( \pi_1(X, p) \) is associative.

Corollary 7.12. Hence \( \pi_1(X, p) \) is a group.
Pf of Thm 7.11 (a)

Note $H : I \times I \to X$ defined by

$$H(s,t) = \begin{cases} \frac{1}{2} s \left( \frac{2s-t}{2-s} \right) & t \leq 2s \\ \frac{1}{2} \left( 2s-t \right) & t \geq 2s \end{cases}$$

is a homotopy from $H(s,0) = \overline{c}_p(s)$ to $H(s,1) = (c_p \cdot \overline{f})(s)$

(b)

The blue curves are "iso curves" of $H$.

Note $H : I \times I \to X$ defined by

$$H(s,t) = \begin{cases} \frac{1}{2} s \left( \frac{2s-t}{2-s} \right) & 0 \leq s \leq \frac{1}{2} \\ \frac{1}{2} \left( 2s-t \right) & \frac{1}{2} \leq s \leq 1 - \frac{1}{2} \\ \frac{1}{2} \left( 2s-2s \right) & 1 - \frac{1}{2} \leq s \leq 1 \end{cases}$$

Note $s(2-2s) = s(1 -(2s-1)) = \overline{s}(2s-1)$,

is a homotopy from $H(s,0) = \overline{c}_p(s)$ to $H(s,1) = \overline{f} \cdot \overline{f}(s)$.
Writing $H$ down explicitly is possible but not very instructive.

More Examples

$H_{15}(S^1 \times S^1) \cong \langle a, b \rangle$

Here $\langle a, b \rangle$ is the free group on two generators, with elements the finite strings in $a, b, a^{-1}, b^{-1}$, modulo "combinatorial terms":

$$abba^{-1}a^{-1}a = a^{-1}b^{-1}a^{-1}a^{-1}a = ab^{-2}a^{-3}a^{-1}a = ab^{-2}.$$  

Multiplication is defined via

$$(ab^2a^{-2})(a^2b^{-2}ab^5a^{-1}) = ab^2a^{-2}a^2b^{-2}ab^5a^{-1}$$

$$= ab^2b^{-2}ab^5a^{-1}$$

$$= a^2b^5a^{-1}.$$  

Note this group is not commutative:

$$(a^2b^{-2}ab^5a^{-1})(ab^2a^{-2}) = a^2b^{-2}ab^5a^{-1}a^{-1}a^{-1}ab^2a^{-2}$$

$$= ab^{-7}ab^7a^{-2}$$

which is different.

The identity element is the empty word.
\[ \pi_1(S^1 \times S^1) \cong \langle a, b \mid aba^{-1}b^{-1} = 1 \rangle \cong \mathbb{Z} \oplus \mathbb{Z} \]

2 generators
1 relation

This space can be formed by starting with \( S^1 \times S^1 \) and then gluing a disk in along its boundary via \( aba^{-1}b^{-1} \), giving the relation \( aba^{-1}b^{-1} = 1 \).

This group is commutative:

\[
\begin{align*}
aba^{-1}b^{-1} &= 1 \\
\Rightarrow aba^{-1} &= b^{-1} \\
\Rightarrow ab &= ba
\end{align*}
\]

\[ \pi_1(\mathbb{R}P^2) \cong \langle a \mid a^2 = 1 \rangle \cong \mathbb{Z}/2\mathbb{Z} \]

This space can be formed by starting with \( S^1 = \mathbb{R} \mathbb{P} \) and then gluing a disk in along its boundary via \( a^2 \) (wrapping around twice).

Disk (that self-intersects in \( \mathbb{R}^3 \))
\( \pi_1 (\text{Klein bottle}) \cong \langle a, b \mid abab^{-1} = 1 \rangle \)

\( \pi_1 (\text{Genus } 2 \text{ torus}) \cong \langle a, b, c, d \mid abab^{-1}bcdc^{-1}d^{-1} \rangle \)

See the YouTube video "From an octagon to a genus 2 surface - MathLapse":
www.youtube.com/watch?v=61yy5PSHgqw

\( \pi_1 (\text{Genus } n \text{ torus}) \cong \langle a_1, b_1, a_2, b_2, \ldots, a_n, b_n \mid a_1b_1a_1^{-1}b_1^{-1} \ldots a_nb_n a_n^{-1}b_n^{-1} \rangle \)

\( \pi_1 (\text{circle with disk wrapped around } n \text{ times}) \cong \langle a \mid a^n \rangle \cong \mathbb{Z}/n\mathbb{Z} \)

\( n=5 \)

Disk (that self-intersects in \( \mathbb{R}^3 \))
More generally, let $X$ be a finite connected cell complex with a single vertex $v$. This is not a big restriction:

$$R^2 \xrightarrow{\rightarrow}$$

Minimal spanning tree

Theorem 10.15 will say

$$\pi_1(X,v) \cong \langle \beta_1, \ldots, \beta_n \mid \delta_1, \ldots, \delta_m \rangle$$

generator for each 1-cell relation for each 2-cell

For $X$ drawn above we have

$$\pi_1(X,v) \cong \langle \beta_1, \beta_2, \beta_3 \mid \beta_1 \beta_2, \beta_3 \beta_1 \rangle$$

$$\cong \langle \beta_2, \beta_3 \mid \beta_3^2 \rangle$$

$$\cong \mathbb{Z} \ast \mathbb{Z}/2\mathbb{Z}$$

(free product of groups which we'll see is the coproduct in the category of groups.)

10/2/2017

Thm 7.13

Our justification for writing $\pi_1(X)$ instead of $\pi_1(X,p)$ is:
Suppose $X$ is a path-connected space with $p, q \in X$ and $g$ a path from $p$ to $q$. Then $\Phi_g : \pi_1(X,p) \rightarrow \pi_1(X,q)$ defined by $\Phi_g([f]) = [g] \cdot [f] \cdot [g]^{-1}$ is a group isomorphism, with inverse $\Phi_{g^{-1}}$.
To see that $\phi g$ is a group homomorphism, note

\[
\phi_g[f_1] \cdot \phi_g[f_2] = [g][f_1][g][f_2][g] = [g][f_1][c_0][f_2][g] = [g][f_1][f_2][g] = \phi g([f_1] \cdot [f_2])
\]

Fundamental groups of spheres

**Thm 7.20** For $n \geq 2$, the $n$-dimensional sphere $S^n$ is "simply connected" ($S^n$ is path-connected and $\pi_1(S^n)$ is the trivial group).

**Lemma 7.19** If $M$ is a manifold of dimension $n \geq 2$, $f$ is a loop in $M$ based at $p$, and $g \neq p$, then $f$ is homotopic to a loop not passing through $g$.

**Proof** Analysis. Not obvious due to space-filling curves (there exist surjective continuous maps $I \rightarrow I^n \forall n \geq 1$).
PS sketch of Thm 7.20

Let $\mathcal{F}$ be a loop in $S^n$ based at $p \neq (0,0,1)$. By Lemma 7.19, we can assume $\mathcal{F}$ doesn't pass through $(0,0,1)$. Stereographic projection gives a homeomorphism $S^n \setminus \{(0,0,1)\} \xrightarrow{\sim} \mathbb{R}^n$; see HW2 #3. In $\mathbb{R}^n$ any loop is homotopic to the constant loop, and hence $\mathcal{F}$ is homotopic to the constant loop in $S^n \setminus \{(0,0,1)\}$. This shows $\pi_1(S^n)$ is the trivial group.

**Rmk** Since $S^n$ is a cell complex (CW complex) with one 0-cell and one $n$-cell, Thm 10.15 says $\pi_1(S^n) \cong$ trivial group for $n \geq 2$.

**Thm 7.21** The fundamental group of a manifold is countable.

**PF** Analysis, de-emphasized.
Prop 7.22
Homomorphisms induced by continuous maps
If \( s_0, s_1 : I \rightarrow X \) are path-homotopic and \( p : X \rightarrow Y \) is continuous, then \( p \circ s_0, p \circ s_1 : I \rightarrow Y \) are path-homotopic.

**Def**
Given \( p : X \rightarrow Y \), let \( p_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0) \) be defined by \( p_*[\gamma] = [p \circ \gamma] \).
This is well-defined by Prop 7.22.

Prop 7.24
\( p_* \) is a group homomorphism

**PS**
Here \( p_* : \pi_1(X) \rightarrow \pi_1(Y) \) via
\[
(p_*[a]) = c \quad (\text{or } 1) , \quad (p_*[b]) = \{3\} \quad (\text{or } 0) \\
(p_*[\gamma] \cdot [\eta]) = (p_*([\gamma \cdot \eta]) = [(p \circ (\gamma \cdot \eta))] = [(p \circ \gamma) \cdot (p \circ \eta)] = [(p \circ \gamma)] \\
= (p_*[\gamma]) \cdot (p_*[\eta]) .
\]

(Indeed, note we have \( (p \circ (\gamma \cdot \eta)) = (p \circ \gamma) \cdot (p \circ \eta) \)) on the nose.
Corollary 7.26 If \( (p: X \to Y) \) is a homeomorphism, then \( p_\ast: \pi_1(X) \to \pi_1(Y) \) is an isomorphism.

Prove:

\[ (p^{-1})_\ast \circ p_\ast = (p^{-1} \circ p)_\ast = (\text{Id}_X)_\ast = \text{Id}_{\pi_1(X)} \]

and

\[ p_\ast \circ (p^{-1})_\ast = \text{Id}_{\pi_1(Y)} \]

So \( p_\ast \) has an inverse homomorphism \( (p^{-1})_\ast = (p_\ast)^{-1} \).
Warning: \( p : X \to Y \) injective need not imply \( p_* : \pi_1(X) \to \pi_1(Y) \) injective.

\[
X = S^1 \quad \quad \quad Y = B^2 = \{ x \in \mathbb{R}^2 \mid \|x\| \leq 1 \}
\]

\[
\begin{array}{ccc}
\circ & \xrightarrow{4p} & \bigcirc \\
\text{inclusion} & &
\end{array}
\]

Note \( p_* : \pi_1(X) \to \pi_1(Y) \) is not injective.

Another example is

\[
\begin{array}{ccc}
\bigcirc & \xrightarrow{c} & \bigcirc \\
\text{a \& b} & &\text{c}
\end{array}
\]

Note \( p_* : \pi_1(X) \to \pi_1(Y) \) via \( a \mapsto c \) and \( b \mapsto \text{cd} \) is not injective.

Warning: \( p : X \to Y \) surjective need not imply \( p_* : \pi_1(X) \to \pi_1(Y) \) surjective.

\[
X = \mathbb{R} \quad \quad \quad Y = S^1 \\
p : \mathbb{R} \to S^1 \quad \text{via} \quad p(t) = e^{2\pi i t}
\]

Note \( p_* : \pi_1(\mathbb{R}) \to \pi_1(S^1) \) is not surjective.
Def. For $X$ a space and $A \subseteq X$, a retraction is a continuous map $r : X \to A$ such that $r \circ i_A = \text{Id}_A$.
Equivalently, $r \circ i_A = \text{Id}_A$ where $i_A : A \to X$ is the inclusion map.

Ex. $r : \mathbb{R}^n \setminus \{0\} \to S^n$ via $r(x) = \frac{x}{\|x\|}$
$X = \mathbb{R}^n \setminus \{0\}$
$A = S^n$

Ex. $r : S^1 \times S^1 \to S^1$ via $r(z, w) = (z, 1)$
$X = S^1 \times S^1$
$A = S^1$

Non-Ex. There is no retract from $B^{n+1}$ onto its boundary sphere $S^n$. A proof (for $n=1$) is...

Prop 7.28. If $r : X \to A$ is a retraction, then $r_* : \pi_1(X) \to \pi_1(A)$ is surjective and $(i_A)_* : \pi_1(A) \to \pi_1(X)$ is injective.

Ps. Since $r \circ i_A = \text{Id}_A$, by Prop 7.25 we have
$r_* \circ (i_A)_* = (\text{Id}_A)_* = \text{Id}_{\pi_1(A)}$
and hence $(i_A)_*$ is injective and $r_*$ is surjective.
Def 10/9/2017

A (covariant) functor from category $C$ to $D$ is a collection of mappings $F: \text{Ob}(C) \rightarrow \text{Ob}(D)$ such that for each $X, Y \in \text{Ob}(C)$, there is a mapping $F: \text{Hom}_C(X, Y) \rightarrow \text{Hom}_D(F(X), F(Y))$ such that

- $F(g \circ h) = F(g) \circ F(h)$
- $F(\text{Id}_X) = \text{Id}_F(X)$

(We denote the entire functor by $F: C \rightarrow D$)

Ex

The fundamental group $\pi_1$ is a functor from the category of pointed topological spaces to the category of groups. For $(p: X \rightarrow Y)$ a pointed map of spaces, we denote $\pi_1(p): \pi_1(X) \rightarrow \pi_1(Y)$ by $p_*$. Note

- $\pi_1(g \circ h) = \pi_1(g) \circ \pi_1(h)$
- $\pi_1(\text{Id}_X) = \text{Id}_{\pi_1(X)}$

Follow from Prop 7.25.

"Land" of topological spaces  "Land" of groups

Only a subcollection of objects/morphisms are drawn! Blue loops are identity morphisms.
The forgetful functor $F : \text{Top} \to \text{Set}$ assigns to each topological space its underlying set, and to each continuous map its underlying function. There is a similar forgetful functor $F : \text{Group} \to \text{Set}$. 

Fundamental groups of product spaces

For topological spaces $X_1, X_2, \ldots, X_n$, the map

$$P : \pi_1(X_1 \times \cdots \times X_n, (x_1, \ldots, x_n)) \to \pi_1(X_1, x_1) \times \cdots \times \pi_1(X_n, x_n)$$

defined by $P[S] = (p_{i*}[S], \ldots, p_n* [S])$

where $p_i : X_1 \times \cdots \times X_n \to X_i$ is the canonical projection map, is a group isomorphism.

\[ (1,3) \]

\[ p_{i*}([S]) \to 1 \]

\[ \pi_1(S') \cong \mathbb{Z} \]

\[ \pi_1(S') \cong \mathbb{Z} \]

\[ S' \times S' \]

\[ \pi_1(S' \times S') \cong \mathbb{Z} \times \mathbb{Z} \]

\[ [S] \leftrightarrow (1,3) \]

\[ P \text{ is a group homomorphism since each } p_i \text{ is:} \]

\[ P([S] \cdot [g]) = P([S] \cdot g) = (p_{1*}[S] \cdot g, \ldots, p_n* [S] \cdot g) \]

\[ = (p_{1*}[S], \ldots, p_n* [S]) \cdot (p_{1*} [g], \ldots, p_n* [g]) \]

\[ = P([S]) \cdot P([g]). \]
$P$ is surjective since if $[\xi_i] \in \pi_1(X_i, x_i)$, then we can define a continuous loop $f: I \to X_1 \times \cdots \times X_n$ by $f(s) = (f_1(s), \ldots, f_n(s))$.

Note $P[f] = (p_1 \circ f, \ldots, p_n \circ f) = ([p_1 \circ f], \ldots, [p_n \circ f]) = ([f_1], \ldots, [f_n])$.

---

$P$ is injective since if $P[f] = \text{identity in } \pi_1(X_1, x_1) \times \cdots \times \pi_1(X_n, x_n)$, where $f(s) = (f_1(s), \ldots, f_n(s))$, then $[\xi_i] = p_i \circ f = [p_i \circ f] = [f_i]$ for all $i$, so $f_i \sim_c x_i$ via $H_i: I \times I \to X_i$ for all $i$.

Define $H: I \times I \to X_1 \times \cdots \times X_n$ via $H(s, t) = (H_1(s, t), \ldots, H_n(s, t))$.

This is a continuous homotopy from $f$ to $c_{(x_1, \ldots, x_n)}$.

Hence $[f] = \text{identity in } \pi_1(X_1 \times \cdots \times X_n)$.
Recall that spaces $X, Y$ are homotopy equivalent ($X \approx Y$) if there exist continuous maps $X \xrightarrow{g} Y$ and $Y \xrightarrow{f} X$ such that $g \circ f \approx \text{Id}_X$ and $f \circ g \approx \text{Id}_Y$.

**Example:**

![Diagram of a circle and annulus with maps]

(Here $g \circ f \approx \text{Id}_{\text{circle}}$ whereas $f \circ g \approx \text{Id}_{\text{annulus}}$)

**Thm 7.40** If $\phi: X \to Y$ is a homotopy equivalence, then $\phi_*: \pi_1(X) \to \pi_1(Y)$ is an isomorphism.

**Rmk** Indeed, $\phi_*: \pi_1(X, p) \to \pi_1(Y, \phi(p))$ is an isomorphism for all $p \in X$.

**Rmk** If we could ignore the subtleties of basepoints, then our proof would be analogous to Corollary 7.26 for homeomorphisms.

**Simplified** **Lemma 7.45** Suppose $\psi, \psi: X \to Y$ with $\psi(p) = \psi(p)$ for some $p \in X$, and furthermore $p \approx \psi$ rel $\partial X$.

Then $\psi_* = \psi_*: \pi_1(X, p) \to \pi_1(Y, \psi(p))$.

![Diagram of a circle, mapping to another circle, and a square]

$\phi(p) = \psi(p)$
Ps Let $f$ be any loop in $X$ based at $p_0$. We have $[\psi \circ f] = [\psi \circ f] = [\psi \circ \delta_1] = \psi \circ \delta_2$.

The middle step is since $(\psi \circ f) \sim (\psi \circ \delta_1)$ (i.e. $\psi \circ f \sim \psi \circ \delta_1$ rel $x_0, x_1$), which we only have since we made the extra assumption $\psi \sim \Psi$ rel $\Sigma_p^3$.

Rmk In the actual Lemma 7.45 we don't get $\psi_\ast = \Psi_\ast$ but only $\Phi_{\psi} \circ \psi_\ast = \Psi_\ast$ for some path $h$ in $X$ from $\psi(p)$ to $\Psi(p)$.

Ps of Thm 7.40 under the simplified assumption that homotopy equivalence $(\psi : X \to Y)$ has a homotopy inverse $\psi : Y \to X$ satisfying

- $\Psi (\psi(p)) = p$ for some $p \in X$
- $\psi \circ \Psi = \text{Id}_X$ rel $\Sigma_p^3$
- $\psi \circ \Psi = \text{Id}_Y$ rel $\Sigma_{\psi(p)}^3$
Note \( \psi^* \circ \varphi^* = (\text{Id}_X)^* \) by Simplified Lemma 7.45
and \( \varphi^* \circ \psi^* = (\text{Id}_Y)^* \) by Simplified Lemma 7.45

Higher homotopy groups

For \( X \) a space and \( p \in X \), fundamental group \( \pi_1(X, p) \) can be recast as the collection of based maps \( S^1 \to X \), up to homotopy equivalence rel basepoint.

More generally, homotopy group \( \pi_n(X, p) \) is the collection of based maps \( S^n \to X \), up to homotopy equivalence rel basepoint.
$\pi_0(X,p)$ is a set that "measures the path-connected components of $X$".

$\pi_1(X,p)$ is a group that "measures the 1-dimensional holes of $X$".

$\pi_n(X,p)$ for $n \geq 2$ is an abelian group that "measures the $n$-dimensional holes of $X$".

What is the group operation in $\pi_1(X)$ for $n \geq 1$?

\[\pi_1 \xrightarrow{a} X \xrightarrow{b} X\]

\[\pi_2 \xrightarrow{a \cdot b} \text{Diagram of two spheres connected by a loop} \xrightarrow{b} X\]
Why is $\pi_n(X)$ abelian for $n \geq 2$?

Data $\Theta \xrightarrow{a} X$ is the same as $\begin{array}{c} a \\ \downarrow \\ \leftarrow X \end{array}$

Data $\Theta \xrightarrow{a \cdot b} X$ is the same as $\begin{array}{c} a \\ \downarrow \\ \leftarrow X \end{array}$

Note

\[
\begin{array}{ccc}
\begin{array}{c}
\ast \\
\downarrow \\
\leftarrow a
\end{array} & \simeq & \\
\begin{array}{c}
\ast \\
\downarrow \\
\leftarrow b
\end{array} & \simeq & \\
\begin{array}{c}
\ast \\
\downarrow \\
\leftarrow a
\end{array}
\end{array}
\]

Remarks

Let $n \geq 1$. Then

- $\pi_n$ is a functor from pointed topological spaces to groups (or abelian groups for $n \geq 2$).
- $\pi_n(X_1 \times \cdots \times X_k) \cong \pi_n(X_1) \times \cdots \times \pi_n(X_k)$ by an analogous proof.
- $\pi_n(S^n) \cong \mathbb{Z}$
- $\pi_n(S^k) \cong \mathbb{Z} \cdot \text{id}_k^n$ for $1 \leq n \leq k$
- $\pi_n(S^k)$ is notoriously hard to compute for $n > k$.

For example, $\pi_3(S^2) \cong \mathbb{Z}$ via the Hopf map. See the table at the Wikipedia page "Homotopy groups of spheres".
The Hopf Fibration (not tested on)

- The Hopf fibration is a map \( p: S^3 \rightarrow S^2 \) that is not homotopy equivalent to the constant map.
- This shows \( \pi_3(S^2) \neq \text{Id} \).
- Indeed, \( \pi_3(S^2) = \mathbb{Z} = \langle p \rangle \) is generated by the Hopf fibration.

See the Youtube video "Hopf fibration - fibers and base" www.youtube.com/watch?v=AKotMP6FJYk

The preimage \( p^{-1}(x^2) \) of each point \( x \in S^2 \) is a circle \( S^1 \) in \( S^3 \).

Fiber Total space Base

\[
\begin{align*}
S^0 & \longrightarrow S^1 & S^1 & \text{Real version} \\
S^1 & \longrightarrow S^3 & S^2 & \text{Defined using complex numbers} \\
S^3 & \longrightarrow S^7 & S^4 & \text{Quaternion version} \\
S^7 & \longrightarrow S^{15} & S^8 & \text{Octonion version}
\end{align*}
\]

J.F. Adams proved such fibrations b/w spheres occur only in these dimensions.

This is related to his proof that \( IR^n \) is an algebra in which division (except by zero) is allowed only for \( n=1 \) (reals), \( n=2 \) (complex), \( n=4 \) (quaternions), and \( n=8 \) (octonions).

- Note \( \pi_3(S^2) \cong \mathbb{Z} \)
- \( \pi_7(S^4) \cong \mathbb{Z} \times \mathbb{Z}/12 \mathbb{Z} \)
- \( \pi_{15}(S^8) \cong \mathbb{Z} \times \mathbb{Z}/120 \mathbb{Z} \)

are not finite groups.
Chapter 5  Cell complexes

We'll study two types of cell complexes:
 CW complexes and simplicial complexes.
- CW complexes remove many wild phenomena in
  point-set topology.
- Simplicial complexes have even more restrictions
  (every simplicial complex is a CW complex)
  that allow them to be defined purely combinatorially.

\[ \text{CW complexes} \]

\[ \text{Ex} \]

\[ \text{Torus} \]

\[ \text{RP}^2 \]

\[ \text{Pinched torus with a disk glued in.} \]

\[ \text{Def (See Prop 5.18, Thm 5.20 instead of the definition on pg 132):} \]
A CW complex is a topological space constructed as follows:
(i) Start with a nonempty discrete set \( X_0 \).
(ii) Inductively, form \( n \)-skeleton \( X_n \) from \( X_{n-1} \) by attaching
  \( n \)-cells along their boundary spheres.
(iii) Equip \( X = \bigcup X_n \) with the topology such that \( A \subseteq X \) is declared to
  be open (resp. closed) iff \( A \cap X_n \) is open (resp. closed) in \( X_n \) for all \( n \).
Exercise $\mathbb{R}P^2$

0-skeleton

1-skeleton

2-skeleton

- Each $X_n$ has the quotient space (or "adjunction space") topology $X_n = \left( X_{n-1} \amalg_{\alpha \in A} D^n_{\alpha} \right) / \sim$
- where each $n$-cell $D^n_{\alpha}$ is a closed $n$-ball $(\overline{B^n})$,
- where $\nu_{\alpha} : \partial D^n_{\alpha} \to X_{n-1}$ are the attaching maps,
- and where $\nu_{\alpha}(x) \sim x$ for all $x \in \partial D^n_{\alpha}$.
- When $X$ is finite dimensional ($X = X_n$ for some $n$), then (iii) is superfluous.

Exercise $S^\infty$

$X_0 = S^0$  $X_1 = S^1$  $X_2 = S^2$  $X_3 = S^3$

\[ \cdots \]

This CW complex has two $n$-cells in each dimension $n$, and $X_n = S^n$ for all $n$. Surprisingly, $S^\infty$ is contractible.
CW complexes have nice point-set topological properties:
- They're Hausdorff
- They're connected $\iff$ they're path-connected (Thm 5.11)
- Etc

**Simplicial Complexes**
A simplicial complex is a cell complex that can be defined purely combinatorially.

**Def** An abstract simplicial complex on a vertex set $V$ is a collection $K$ of finite subsets of $V$ such that if $\emptyset \in K$ and $I \subseteq \emptyset$, then $I \in K$.

**Ex** $V = \{v_0, v_1, v_2, v_3, v_4\}$
$K = \{\emptyset, \{v_1, v_2, v_3, v_4\}, \{v_0, v_1, v_2\}, \{v_0, v_2, v_3\}, \{v_1, v_2, v_3\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}, \{v_0, v_3\}, \{v_0, v_4\}, \{v_1, v_4\}\}$

**Ex** $V = \{v_0, v_1, v_2, v_3\}$
$K = \text{the set of all subsets of } V$

This is a (solid) tetrahedron, or 3-simplex.
The elements of $K$ are called simplices or faces. A $k$-dimensional simplex or $k$-simplex has $k+1$ vertices, often indexed from 0 to $k$.

<table>
<thead>
<tr>
<th>0-simplex</th>
<th>1-simplex</th>
<th>2-simplex</th>
<th>3-simplex</th>
<th>4-simplex</th>
</tr>
</thead>
<tbody>
<tr>
<td>vertex</td>
<td>edge</td>
<td>triangle</td>
<td>tetrahedron</td>
<td></td>
</tr>
</tbody>
</table>

**Non-Ex**

$V = \{ v_0, v_1, v_2, v_3, v_4 \}$

$\exists v_1, v_2, v_3 \in K$ but $\exists v_2, v_3 \notin K$

But how is this a topological space? What are the open sets?

Every abstract simplicial complex $K$ has an associated geometric realization $\tilde{K}$ which is a topological space.

(By an abuse of notation we denote the abstract simplicial complex and its geometric realization by the same symbol $K$.)

Can't have a triangle with a missing edge!
For vertex set $V = \{v_1, \ldots, v_n\}$ finite, assign each vertex $v_i$ to the standard basis vector $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)^\top$ (i-th coordinate).

Assign each simplex $\{v_0, \ldots, v_k\}$ to the convex hull of $\{e_0, \ldots, e_k\}$.

$K = \{v_1, v_3, 2v_3, 2v_2, 2v_1, 2v_3, 2v_2, 2v_1\}$, $K = \{v_1, v_3, v_3, v_2, v_3, v_2, v_1, v_3\}$.

The geometric realization of $K$ is then the union of these convex hulls in $\mathbb{R}^n$, equipped with the (Euclidean) subspace topology.

More generally ($V$ is possibly infinite), the geometric realization of $K$ as a set is

$$\left\{ \sum_{i=0}^{k} \lambda_i x_i \mid k \in \mathbb{N}, \lambda_i \geq 0, \sum_{i=0}^{k} \lambda_i = 1, \{x_0, \ldots, x_n\} \in K^2 \right\}$$

[Diagrams and equations related to barycentric coordinates and convex combinations]
A subset of the geometric realization of $K$ is declared to be open (resp. closed) if its intersection with (the geometric realization of) each (finite) simplex is open (resp. closed).
10/23/2017 We are momentarily skipping some very important aspects of the fundamental group (the Seifert-Van Kampen Theorem in Chapter 10, covering spaces in Chapters 11-12) to cover...

**Chapter 13 Homology**

- Much like the homotopy group $\pi_p(X)$, the homology group $H_p(X)$ roughly speaking "measures the # of $p$-dimensional holes in $X" •
- Unlike $\pi_p(X)$, homology group $H_p(X)$ is hard to define but easy to compute (linear algebra).
- There are many homology theories.

We'll start with simplicial homology, which is only defined for simplicial complexes.
We'll then move to singular homology, defined for all topological spaces and covered in our book. These two homology theories (simplicial and singular) agree on simplicial complexes.
Introduction to simplicial homology

Let $X$ be a simplicial complex.

**Def.** Let $C_p(X)$ be the free abelian group on the set of all oriented $p$-simplices in $X$.

I.e., $C_p(X)$ is the set of all formal sums of $p$-simplices in $X$ with coefficients in $\mathbb{Z}$. Its elements are called $p$-chains.

**Ex.**

\[ X = \begin{array}{cc}
0 & 3 \\
1 & 2 \\
\end{array} \]

\[ C_0(X) = \{ a[0] + b[1] + c[2] + d[3] \mid a, b, c, d \in \mathbb{Z} \} \cong \mathbb{Z}^4 \]

Group operation:

\[(a[0]+...+d[3]) + (a'[0]+...+d'[3]) = (a+a')[0] + ... + (d+d')[3].\]

So \((0)[0] + [2] + [3]) + ([0] - [2] + 4[3]) = 2[0] + 5[3].\]

\[ C_1(X) = \{ a[0,1] + b[0,3] + c[1,2] + d[1,3] + e[2,3] \mid a, b, c, d, e \in \mathbb{Z} \} \cong \mathbb{Z}^5 \]

We write a simplex as $[0,3]$ instead of $\{0,3\}$ to denote that it is oriented: $[0,3] = -[3,0]$.

The group operation is analogous:

\[(1,2)[2,3] + ([1,3] + [2,3]) = [1,2] + [2,3] + 2[2,3].\]

\[(1,2)[2,3] + ([1,3] + [3,2]) = ([1,2] + [2,3]) + ([1,3] - [2,3]) = [1,2] + [1,3].\]

\[ C_2 = \{ a[1,2,3] \mid a \in \mathbb{Z} \} \cong \mathbb{Z} \]

\[ [1,2,3] = [2,3,1] = [3,1,2] \quad (\text{differ from } [1,2,3] \text{ by an even permutation}) \]

\[ (-[1,2,3]) = [1,3,2] = [3,2,1] = [2,1,3] \quad (\text{differ from } [1,2,3] \text{ by an odd permutation}) \]
Define the boundary operator \( \partial_p : C_p(X) \rightarrow C_{p-1}(X) \) (or \( \partial : C_p(X) \rightarrow C_{p-1}(X) \)) by setting

\[
\partial([x_0, \ldots, x_p]) = \sum_{i=0}^{p} (-1)^i [x_0, \ldots, \hat{x}_i, \ldots, x_p]
\]

and extending linearly

(where the hat \( \hat{x}_i \) means \( x_i \) is omitted).

\[ \partial_2 : C_2(X) \rightarrow C_1(X) \text{ via} \]

\[
\partial_2([1,2,3]) = [2,3] - [1,3] + [1,2] = [1,2] + [2,3] + [3,1] 
\]

So \( \partial_2(5[1,2,3]) = 5[1,2] + 5[2,3] + 5[3,1] \).

\[ \partial_1 : C_1(X) \rightarrow C_0(X) \text{ via} \]

\[
\partial_1([x_0, x_1]) = [x_1] - [x_0]
\]

So \( \partial_1([0,1]) + 3[1,3] = \partial_1([0,1]) + 3 \partial_1([1,3]) 
\]

\[ = [1]-[0] + 3([-3] - [-1]) 
\]

\]

\[ \partial_0 : C_0(X) \rightarrow 0 \text{ sends any 0-chain to zero.} \]
Def Let $Z_p(X) = \ker(\partial_p)$ be the set of $p$-cycles.
Let $B_p(X) = \text{im}(\partial_{p+1})$ be the set of $p$-boundaries.
Let $H_p(X) = Z_p(X) / B_p(X)$ be the $p$-dimensional simplicial homology group.

Example $Z_1(X) = \ker(\partial_1) = \{ a([0,1] + [1,3] + [3,0]) + b([1,2] + [2,3] + [3,1]) | a, b \in \mathbb{Z} \} \cong \mathbb{Z}^2$

Note $[0,1] + [1,2] + [2,3] + [3,0] \in Z_1(X)$ by letting $a = b = 1$.

$B_1(X) = \text{im}(\partial_2) = \{ \frac{1}{3} a([1,2] + [2,3] + [3,1]) | a \in \mathbb{Z} \} \cong \mathbb{Z}$

$H_1(X) = Z_1(X) / B_1(X) \cong \mathbb{Z}$

"$X$ has a single 1-dimensional hole"

$C_2(X) \cong \mathbb{Z}$
$C_1(X) \cong \mathbb{Z}^5$
$C_0(X) \cong \mathbb{Z}^4$

$Z_2(X) = 0$
$Z_1(X) \cong \mathbb{Z}^2$
$Z_0(X) \cong \mathbb{Z}^4$

$B_2(X) = 0$
$B_1(X) \cong \mathbb{Z}$
$B_0(X) \cong \mathbb{Z}^3$

$H_2(X) = 0$
$H_1(X) \cong \mathbb{Z}$
$H_0(X) \cong \mathbb{Z}$

"$X$ has a single
2-dimensional hole."
"$X$ has a single connected component."
$\mathbb{Z}_2(X) = \ker (\partial_2) = 0$ since $\partial_2 (1,1,2,3) \neq 0$.

$B_2 (X) = \text{im} (\partial_3) = 0$ since $c_3 (x) = 0$.

$H_2 (X) = \frac{\mathbb{Z}_2 (x)}{B_2 (x)} \cong 0$.

"$X$ has no 2-dimensional holes."

\[ Z_0 (X) = \ker (\partial_0) = c_0 (x) = \{ a[0] + b[1] + c[2] + d[3] \mid a,b,c,d \in \mathbb{Z} \} \cong \mathbb{Z}^4. \]

\[ B_0 (X) = \text{im} (\partial_1) \]

\[ = \{ a[1,1] + b[1,3] + c[1,2] + d[1,3] + e[2,3] \mid a,b,c,d,e \in \mathbb{Z} \} \cong \mathbb{Z}^3 \]

\[ = \{ a[1,1] + b[1,3] + c[1,2] \mid a,b,c \in \mathbb{Z} \} \cong \mathbb{Z}^3 \]

since $2[1,3] = [3,1] - [-1,0] + [1,0] = -[1,0] + [1,0] = 0$.

And since $2[2,3] = [3,2] - [-1,1] + [1,0] = -[1,1] + [1,0] = -[1,1]$.

(Note using the 3 edges $[0,1], [0,3], [1,2]$ we can walk from any vertex to any other vertex.)

\[ H_0 (X) = \frac{Z_0 (x)}{B_0 (x)} \cong \mathbb{Z}. \]

To see this, note we can "change basis" on $Z_0$ so that $3$ of the $4$ generators are $[1,1], [1,3], [1,2]$. and $[2,3] - [1,1]$.

"$X$ has a single connected component."
In the prior example, how might we algorithmically compute $\mathbb{Z}_1(x) \cong \mathbb{Z}^2$ and $\mathbb{B}_0(x) \cong \mathbb{Z}^3$, while also finding generators?

$4 \times 5$ Matrix representing $\partial_1 : C_1(x) \to C_0(x)$

\[
\begin{bmatrix}
[0,1] & [0,3] & [1,2] & [1,3] & [2,3] \\
[0] & -1 & -1 & 0 & 0 & 0 & [1] & 1 & 0 & -1 & -1 & 0 \\
[1] & 1 & 0 & -1 & -1 & 0 & [3] & 0 & 1 & 0 & 1 & 1 \\
[3] & 0 & 1 & 0 & 1 & 1 & [0] & -1 & -1 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Add third row to first

\[
\begin{bmatrix}
[0,1] & [0,3] & [1,2] & [1,3] & [2,3] \\
[1] & 1 & 0 & 0 & -1 & -1 & [1]-[0] & 1 & 0 & 0 & -1 & -1 \\
[3] & 0 & 1 & 0 & 1 & 1 & [3]-[0] & 0 & 1 & 0 & 1 & 1 \\
[2]-[1] & 0 & 0 & 1 & 0 & -1 & [2]-[1] & 0 & 0 & 1 & 0 & -1 \\
[0] & -1 & -1 & 0 & 0 & 0 & [0] & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Add col 1 - col 2 to col 4
Add col 2 - col 2 + col 3 to col 5

\[
\begin{bmatrix}
[1]-[0] & 1 & 0 & 0 & 0 & 0 \\
[3]-[0] & 0 & 1 & 0 & 0 & 0 \\
[2]-[1] & 0 & 0 & 1 & 0 & 0 \\
[0] & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

This matrix has rank 3 and nullity 2.

The first three rows give a generating set for $\mathbb{B}_0(x) = \text{im}(\partial_1)$.

The last two columns give a generating set for $\mathbb{Z}_1(x) = \text{ker}(\partial_1)$.
Simplicial homology: spanning trees and \( \partial_1 : C_1(X) \to C_0(X) \)

\[
X = \begin{array}{c}
\text{1} \\
\text{2} \\
\text{3} \\
\text{4} \\
\text{5} \\
\text{6} \\
\text{7} \\
\text{8}
\end{array}
\]

Pick a spanning tree for each connected component. This can help you identify generating sets for \( B_0(X) = \text{im}(\partial_1) \) and \( Z_1(X) = \ker(\partial_1) \).

\[
\begin{aligned}
B_0(X) &= \{a([0,1]) + b([0,3]) + \ldots + \ell a_i([7,8]) \mid a, b, \ldots, \ell, a_i \in \mathbb{Z}\} \\
&= \{a([0,1]) + b([0,3]) + \ldots + g a_i([6,8]) \mid a, b, \ldots, g, a_i \in \mathbb{Z}\} \cong \mathbb{Z}^7
\end{aligned}
\]

12 letters for 12 edges

7 letters for the 7 edges in the spanning trees

\[
Z_1(X) = \left\{ \begin{array}{l}
a([0,3] + [3,1] + [1,0]) + b([1,7] + [3,1] + [1,2]) + c([4,6] + [6,5] + [5,4]) \\
+d([4,7] + [7,6] + [6,5] + [5,4]) + e([7,8] + [8,6] + [6,7])
\end{array} \right\} \cong \mathbb{Z}^5
\]

Such that \( a, b, c, d, e \in \mathbb{Z} \)

Note \( \dim(Z_1(X)) + \dim(B_0(X)) = \dim(Z_1(X)) + \dim(Z_2(X)) = 5 + 7 = 12 \)

\( \# \) edges = \( \dim(\mathcal{C}_1(X)) \)

More generally,

\( \dim(Z_p(X)) + \dim(B_{p-1}(X)) = \dim(\ker \partial_p) + \dim(\text{im} \partial_p) = \dim(C_p(X)) \)

(\( \# p\)-simplices)

\[
C_p(X) \xrightarrow{\partial_p} C_{p-1}(X)
\]
In the definition $H_p(x) = \mathbb{Z}_p(x)/\mathbb{B}_p(x)$ why do we know $\mathbb{B}_p(x) = \text{im}(\partial_p+1)$ is a subgroup of $\mathbb{Z}_p(x) = \text{ker}(\partial_p)$?

The reason is that $\partial_p \circ \partial_p+1 = 0$.
(This is often written $\partial \circ \partial = 0$ or $\partial^2 = 0$.)

Pic of $\partial_1 \circ \partial_2 = 0$

$\partial_1(\partial_2(0)) = \partial_1(0) = 0$

$\partial_1(\partial_2([0,1,2])) = \partial_1(\begin{bmatrix} [1,2]-[0,2] + [0,1], \\
[2,3]-[1,3] - [2,3]-[0,3] + [1,1]-[0,3] 
\end{bmatrix}) = 0$.  

Pic of $\partial_2 \circ \partial_3 = 0$

$\partial_2(\partial_3(0)) = \partial_2(0) = 0$

$\partial_2(\partial_3([0,1,2,3])) = \partial_2(\begin{bmatrix} [1,2,3]-[0,2,3]+[0,1,3]-[0,1,2], \\
[2,3]-[1,3]+[2,3]-[0,3]+[1,1]-[0,3] 
\end{bmatrix}) = 0$.  

Proof of $\partial_p \circ \partial_{p+1} = 0$

Since the boundary operators are linear, it suffices to show that applying $\partial_p \circ \partial_{p+1}$ to a single $(p+1)$-simplex gives zero. Note

$\partial_p (\partial_{p+1} (\{x_0, \ldots, x_{p+1}\}))$

$= \partial_p (\sum_{i=0}^{p+1} (-1)^i \{x_0, \ldots, \hat{x}_i, \ldots, x_{p+1}\})$

$= \sum_{j=0}^{p+1} (-1)^{p+1-j} \{x_0, \ldots, \hat{x}_j, \ldots, x_{p+1}\}$

$+ \sum_{i<j} (-1)^{i+j-1} \{x_0, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{p+1}\}$

$= 0$ (by symmetry — terms cancel in pairs)

Singular homology

• This is in our book

• Singular homology is more complicated than simplicial homology, but the upshot is that given a homotopy equivalence $X \simeq Y$, it is easier to prove with singular homology that $H_p(X) \cong H_p(Y)$.

Let $\Delta_p$ denote both an abstract $p$-simplex $[e_0, \ldots, e_p]$ and its geometric realization.

Rmk

You can think of the geometric realization of $\Delta_2$ as the 2-dimensional convex hull of 3 points in Euclidean space.
Let $X$ be a topological space.

A singular $p$-simplex in $X$ is a continuous map $\sigma : \Delta_p \to X$

Here we're using the geometric realization

**Example**

$X = \bigcirc$

**Singular 0-simplices**

$\sigma', \sigma'' : \Delta_0 \to X$

$17\sigma + 5\sigma' - 3\sigma''$

is a singular 0-chain

**Singular 1-simplices**

$\sigma', \sigma'' : \Delta_1 \to X$

$-3\sigma + 5\sigma' - 9\sigma''$

is a singular 1-chain

**Singular 2-simplices**

$\sigma, \sigma', \sigma'' : \Delta_2 \to X$

$-\sigma - 8\sigma' + 3\sigma''$

is a singular 2-chain.

Non-injectivity is allowed; hence the name singular!

**Definition**

Let $C_p(X)$ be the free abelian group on the set of all singular $p$-simplices in $X$.

An element of $C_p(X)$, which is called a singular $p$-chain, is a formal sum of singular $p$-simplices with coefficients in $\mathbb{Z}$. 

**Note:** using same symbol as in simplicial homology
What's the group operation on $C_p(X)$?
If $\sigma, \sigma', \sigma'', \sigma''' \in C_p$ are singular $p$-simplices, then
\[
(5\sigma + 7\sigma' - 3\sigma'') + (7\sigma' - 2\sigma'' + \sigma''') = 5\sigma + 14\sigma' - 5\sigma'' + \sigma'''
\]

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**Def**

For $i = 0, 1, \ldots, p$, define $F_i : \Delta_{p-1} \to \Delta_p$ to be the affine map taking $\Delta_{p-1}$ homeomorphically onto the face of $\Delta_p$ opposite the $i$-th vertex:

\[
\begin{align*}
\Delta_{p-1} &= \{ e_0, \ldots, e_{p-1} \} \\
\Delta_p &= \{ e_0, \ldots, e_p \}
\end{align*}
\]

This is the $i$-th face map in dimension $p$.

**Ex $p=1$**

\[
\begin{array}{c}
\text{0} \overset{F_0,1}{\longrightarrow} \text{1} \\
\text{0} \overset{F_1,1}{\longrightarrow} \text{0}
\end{array}
\]

**Ex $p=2$**

\[
\begin{array}{c}
F_0,2 \\
F_1,2
\end{array}
\]

**Ex $p=3$**

\[
\begin{array}{c}
F_0,3 \\
F_1,3 \\
F_2,3
\end{array}
\]
Define the (singular) boundary operator $\partial_p : C_p(X) \to C_{p-1}(X)$ by setting

$$\partial \sigma = \sum_{i=0}^p (-i) \sigma \circ F_{i,p}$$

for any singular $p$-simplex $\sigma : \Delta^p \to X$, and then extending linearly to $p$-chains.

**Example $p=1$**

$$\partial(\sigma + \sigma') = \partial \sigma + \partial \sigma'$$

$$= \sigma \circ F_{0,1} - \sigma' \circ F_{1,1} + \sigma' \circ F_{0,1} - \sigma \circ F_{1,1}$$

Note $\sigma \circ F_{0,1} = \sigma' \circ F_{1,1}$.

**Example $p=2$**

$$\partial \sigma = \sigma \circ F_{0,2} - \sigma \circ F_{1,2} + \sigma \circ F_{2,2}$$
Minor remark. In simplicial homology, $C_p(X)$ was generated by all oriented $p$-simplices. For example, we had $[x_0, x_1] = -[x_1, x_0]$ on the nose (i.e. in $C_1(X)$).

$$\partial [x_0, x_1, x_2] = [x_1, x_2] - [x_0, x_2] + [x_0, x_1]$$

$$= [x_0, x_1] + [x_1, x_2] + [x_2, x_0]$$

In singular homology, $C_p(X)$ is generated by all singular $p$-simplices (no orientation data).

For example, in $C_1(X)$ we have

$$\sigma \neq -\sigma$$

However, we will have that $\sigma$ and $-\sigma$ differ by an element of $B_1(X)$. 
Lemma 13.1 If \( c \in C_p(X) \) is a singular \( p \)-chain, then \( \partial_{p-1}(\partial_p(c)) = 0 \).

In other words, \( \partial_{p-1} \partial_p = 0 \), or \( \partial \circ \partial = 0 \).

Pic of \( \partial_1 \circ \partial_2 = 0 \)

\[
\partial_1 (\partial_2 (\sigma \rightarrow X)) = \partial_1 \left( \begin{array}{c}
\begin{array}{c}
+1 \\
0 \\
-1
\end{array} \\
\begin{array}{c}
+1 \\
0 \\
-1
\end{array}
\end{array} \rightarrow X \right) = 0
\]

\[
\partial_1 (\partial_2 (\sigma)) = \partial_1 (\sigma \circ F_{0,2} - \sigma \circ F_{1,2} + \sigma \circ F_{2,2})
\]

\[
= \partial_1 (\sigma \circ F_{0,2} - \partial_1 (\sigma \circ F_{1,2} + \sigma \circ F_{2,2})
\]

\[
= (\sigma \circ F_{0,2} - \sigma \circ F_{1,2} - \sigma \circ F_{2,2}) + (\sigma \circ F_{1,2} + \sigma \circ F_{2,2})
\]

\[
= 0 \quad \text{since} \quad F_{1,2} \circ F_{0,1} = F_{0,2} \circ F_{0,1} \quad \text{and} \quad F_{2,2} \circ F_{0,1} = F_{0,2} \circ F_{1,1} \quad \text{and} \quad F_{2,2} \circ F_{1,1} = F_{1,2} \circ F_{1,1}
\]
These are examples of the commutation relation
$F_{i,p} \circ F_{j,p-1} = F_{j,p} \circ F_{i-1,p-1}$ when $i > j$.

This is equation (13.1) in our book.
Proof of Lemma 13.1 Since the boundary operators are linear and $C_\rho(x)$ is generated by singular $p$-simplices, it suffices to show $\partial_p^{-1}(\partial_p(\sigma)) = 0$ when $\sigma$ is a singular $p$-simplex.

We compute
\[
\partial_p^{-1}(\partial_p(\sigma)) = \partial_p^{-1}\left(\sum_{i=0}^{\rho} (-1)^i \sigma \circ F_{i,p}\right)
= \sum_{i=0}^{\rho-1} \sum_{j=0}^{\rho} (-1)^{i+j} \sigma \circ F_{i,p} \circ F_{j,p-1}
= \sum_{0 \leq j \leq i \leq \rho} (-1)^{i+j} \sigma \circ F_{i,p} \circ F_{j,p-1} + \sum_{0 \leq i \leq j \leq \rho-1} (-1)^{i+j-1} \sigma \circ F_{j,p} \circ F_{i,p-1}
\]

In this sum replace $i$ with $j$ and $j$ with $i-1$.

\[
= \sum_{0 \leq j < i \leq \rho} (-1)^{i+j} \sigma \circ F_{i,p} \circ F_{j,p-1} + \sum_{0 \leq j < i \leq \rho} (-1)^{i+j-1} \sigma \circ F_{j,p} \circ F_{i,p-1}
= 0 \ \text{since} \ F_{i,p} \circ F_{j,p-1} = F_{j,p} \circ F_{i,p-1} \ .
\]

We have \[ \cdots \to C_{p+1}(X) \xrightarrow{\partial_{p+1}} C_p(X) \xrightarrow{\partial_p} C_{p-1}(X) \to \cdots \]

Define the $p$-cycles by \[ Z_p(X) = \ker(\partial_p) \leq C_p(X) \],

Define the $p$-boundaries by \[ B_p(X) = \ker(\partial_{p+1}) \leq C_p(X) \].

By Lemma 13.1 we have
\[ B_p(X) = \ker(\partial_{p+1}) \leq \ker(\partial_p) \leq Z_p(X) \].

Hence we can define the $p$-th singular homology group by
\[ H_p(X) = \frac{Z_p(x)}{B_p(X)} = \frac{\ker(\partial_p)}{\ker(\partial_{p+1})} \]
Singular homology is a functor.

Let \( f : X \to Y \) be a continuous map of topological spaces. This induces a homomorphism \( f^\# : C_p(X) \to C_p(Y) \) of chain groups via \( f^\#(\delta) = f \circ \delta \) (and hence \( f^\#(17 \sigma - 3 \sigma' + 4 \sigma'') = 17 f \circ \sigma - 3 f \circ \sigma' + 4 f \circ \sigma'' \)).

Key fact: \( f^\circ \circ f^\# = f^\circ \circ f^\# \) as maps from \( C_p(X) \) to \( C_{p-1}(Y) \).

\[
\begin{align*}
C_p(X) & \xrightarrow{\partial} C_{p-1}(X) \\
\downarrow f^\# & \quad \downarrow f^\# \\
C_p(Y) & \xrightarrow{\partial} C_{p-1}(Y)
\end{align*}
\]

**Proof**
\[
f^\#(\partial \delta) = f^\#(\sum_{i=0}^p \delta \circ F_{i, p}) = \sum_{i=0}^p f^\#(\delta \circ F_{i, p}) = \sum_{i=0}^p (f \circ \delta) \circ F_{i, p} = \partial(f \circ \delta) = \partial(f^\# \circ \delta)
\]

**Key consequences**
- \( f^\# : Z_p(X) \to Z_p(Y) \) since \( c \in Z_p(X) \Rightarrow \partial c = 0 \)
  \( \Rightarrow \partial(f^\# c) = f^\#(\partial c) = f^\#(0) = 0 \Rightarrow f^\# c \in Z_p(Y) \).
- \( f^\# : B_p(X) \to B_p(Y) \) since \( c \in B_p(X) \Rightarrow c = \partial d \) for some \( d \in C_{p+1}(X) \)
  \( \Rightarrow f^\# c = f^\#(\partial d) = \partial(f^\# d) \Rightarrow f^\# c \in B_p(Y) \).
We therefore get an induced homomorphism $s_*: \hat{H}_p(X) \to \hat{H}_p(Y)$ on the quotient, defined by $s_*(c + B_p(X)) = s_# c + B_p(Y)$ for $c \in \mathbb{Z}_p(X)$. (Recall $\hat{H}_p(X) = \frac{Z_p(X)}{B_p(X)}$ and $H_p(Y) = \frac{Z_p(Y)}{B_p(Y)}$.)

**Prop 13.2** Singular $p$-dimensional homology is a functor from the category of topological spaces to the category of (abelian) groups.

A space $X$ is assigned to $\hat{H}_p(X)$.

A map $f: X \to Y$ is assigned to $f_*: \hat{H}_p(X) \to \hat{H}_p(Y)$.

**Our book's proof** It is easy to check that the necessary properties hold already for $s_*: C_p(X) \to C_p(Y)$.

**More detailed proof** Let $X$ be a space.

Given $c + B_p(X) \in \hat{H}_p(X)$, note $\text{Id}_*(c + B_p(X)) = \text{Id}_# c + B_p(Y) = c + B_p(Y)$.

Hence $\text{Id}_* = \text{Id}_{\hat{H}_p(X)}$, as required.
Let $f : X \to Y$ and $g : Y \to Z$.

Given $c + B_p(x) \in H_p(x)$, note
\[
g \circ (f \circ (c + B_p(x))) = g \circ (f \circ c + B_p(y)) = g \circ (f \circ c) + B_p(z) = (g \circ f) \circ (c + B_p(x))
\]

Hence $(g \circ f)_* = g \circ f_* : H_p(x) \to H_p(z)$.

**Corollary 13.3** If $f : X \to Y$ is a homeomorphism then $f_* : H_p(X) \to H_p(Y)$ is an isomorphism.

**Proof** Functors take isomorphisms to isomorphisms.

**Corollary 13.4** If $A \subseteq X$ is a retract of $X$
(meaning there exists $r : X \to A$ with $r \circ u_A = id_X$),
then $r_* : H_p(X) \to H_p(A)$ is surjective and $(u_A)_* : H_p(A) \to H_p(X)$ is injective.

[Diagram]

\[
\begin{array}{ccc}
A & \xrightarrow{u_A} & X \\
& \searrow & \swarrow \quad r \\
& X & \xrightarrow{f} & A \\
& \downarrow & \downarrow \\
& H_p(A) & \xrightarrow{(u_A)_*} & H_p(X) \\
& \downarrow & \downarrow & \downarrow \\
& H_p(A) & \xrightarrow{r_*} & H_p(A)
\end{array}
\]

$r \circ u_A = id_X$

$r_* \circ (u_A)_* = id_{H_p(A)}$
Singular homology is a functor

Let $f: X \rightarrow Y$ be a continuous map of topological spaces. This induces a homomorphism $f_\#: C_p(X) \rightarrow C_p(Y)$ of chain groups via $f_\#(\sigma) = f \circ \sigma$ (and hence $f_\#(1 + \sigma - 3\sigma' + 4\sigma'') = 17f \circ \sigma - 3f \circ \sigma' + 4f \circ \sigma''$).

Key fact $f_\# \circ \partial = \partial \circ f_\#$ as maps from $C_p(X)$ to $C_{p-1}(Y)$.

\[
\begin{align*}
C_p(X) & \xrightarrow{\partial_p} C_{p-1}(X) \\
\downarrow f_\# & \quad \downarrow f_\# \\
C_p(Y) & \xrightarrow{\partial_p} C_{p-1}(Y)
\end{align*}
\]

Proof $f_\#(\partial \sigma) = f_\# \left( \sum_{i=0}^{p} (-1)^i \partial_i \circ F_{i,p} \right)$

\[
= \sum_{i=0}^{p} (-1)^i f_\#(\partial_i \circ F_{i,p})
\]

\[
= \sum_{i=0}^{p} (-1)^i \partial_i (f_\# \circ \sigma)
\]

\[
= \partial (f_\# \sigma)
\]

\[
= \partial (f_\# \sigma)
\]

Key consequences

1. $f_\#: Z_p(X) \rightarrow Z_p(Y)$ since $c \in Z_p(X) \Rightarrow \partial c = 0$.

2. $f_\#: B_p(X) \rightarrow B_p(Y)$ since $c \in B_p(X) \Rightarrow c = \partial d$ for some $d \in C_{p+1}(X)$.

\[
\Rightarrow f_\# c = f_\# (\partial d) = \partial (f_\# d) \Rightarrow f_\# c \in B_p(Y).
\]
We therefore get an induced homomorphism \( s_\ast : H_p(x) \to H_p(y) \) on the quotient, defined by \( s_\ast (c + B_p(x)) = s_\# c + B_p(y) \) for \( c \in \mathbb{Z}_p(x) \).

(Recall \( H_p(x) = \frac{\mathbb{Z}_p(x)}{B_p(x)} \) and \( H_p(y) = \frac{\mathbb{Z}_p(y)}{B_p(x)} \)).

Prop 13.2: Singular \( p \)-dimensional homology is a functor from the category of topological spaces to the category of abelian groups.

A space \( X \) is assigned to \( H_p(X) \).

A map \( f : X \to Y \) is assigned to \( f_\ast : H_p(X) \to H_p(Y) \).

**Our book's proof** It is easy to check that the necessary properties hold already for \( s_\# : C_p(x) \to C_p(y) \).

**More detailed proof** Let \( X \) be a space.

Given \( c + B_p(x) \in H_p(x) \), note

\[ \text{Id}_\ast (c + B_p(x)) = \text{Id}_\# c + B_p(x) = c + B_p(x). \]

Hence \( \text{Id}_\ast = \text{Id}_H_p(x) \), as required.
Let \( f : X \to Y \) and \( g : Y \to Z \).

Given \( c + Bp(x) \in H_p(x) \), note
\[
g^* (f^* (c + Bp(x))) = g^* (f^* c + Bp(y)) = g^* (f^* c) + Bp(z) = (g \circ f)^* (c + Bp(x))
\]

Hence \( (g \circ f)^* = g^* \circ f^* : H_p(x) \to H_p(z) \).

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**Corollary 13.3** If \( f : X \to Y \) is a homeomorphism then \( f^* : H_p(X) \to H_p(Y) \) is an isomorphism.

**Proof** Functors take isomorphisms to isomorphisms.

**Corollary 13.4** If \( A \leq X \) is a retract of \( X \) (meaning there exists \( r : X \to A \) with \( r \circ f_A = \text{id}_A \)),
then \( r^* : H_p(X) \to H_p(A) \) is surjective
and \( (f_A)^* : H_p(A) \to H_p(X) \) is injective.

**Picture**

\[
\begin{array}{ccc}
A & \xrightarrow{f_A} & X \xrightarrow{r} A \\
\text{r} \circ f_A = \text{id}_X & & \text{r}^* \circ (f_A)^* = \text{id}_{H_p(A)}
\end{array}
\]
Now that I've played a bit with simplicial and singular homology, how should I think about the homology groups?

Let \( X = S^2 \times I \) be a hollow coconut. Recall \( \pi_2(X) \cong \pi_2(S^2) \cong \mathbb{Z} \).

The red and blue pointed maps \( S^2 \rightarrow X \) represent the same element of \( \pi_2(X) \) because they are homotopy equivalent (relative basepoints).

It turns out that \( H_2(X) \cong H_2(S^2 \times I) \cong \mathbb{Z} \).

The red and blue (simplicial or singular) 2-cycles represent the same element of \( H_2(X) \) because they are homologous, i.e. their difference is a 2-boundary.

From the perspective of either \( \pi_2 \) or \( H_2 \), \( X \) has a "single 2-dimensional hole".
Here's a picture of a triangulation of the "hollow coconut" $X = s^2 \times I$ we considered on the previous page of notes.
Homotopy and homology groups often can't come to an agreement whether they think there's a hole or not:

\[ \pi_2(S' \times S') \cong \pi_2(S') \times \pi_2(S') = \text{trivial group} \]

Homotopy doesn't think there's a 2-dimensional hole in the torus since any pointed map \( S^2 \to S' \times S' \) is homotopy equivalent to a constant map.

\[ H_2(S' \times S') \cong \mathbb{Z} \] since \( S' \times S' \) is a compact connected orientable 2-manifold without boundary.

The blue 2-cycle above is not the boundary of any 3-chain, so it represents a nontrivial element of \( H_2(S' \times S') \), and in fact generates \( H_2(S' \times S') \cong \mathbb{Z} \).

**Fact** More generally, if \( M \) is a compact connected orientable \( n \)-manifold with boundary, then \( H_n(M) \cong \mathbb{Z} \).
The loop drawn above is a nontrivial element of $\pi_1(X) \cong \langle a, b, c, d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} \rangle$ since it's not homotopy equivalent to the constant loop. This loop corresponds to the element $aba^{-1}b^{-1}$ or $dc^{-1}d^{-1}$.

The red 1-cycle drawn above is a trivial element of $H_1(X) \cong \mathbb{Z}$ since it's the boundary of the blue 2-chain.

The projective plane example $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}/2\mathbb{Z}$ and $H_1(\mathbb{R}P^2) \cong \mathbb{Z}/2\mathbb{Z}$ show that the "number of i-dimensional holes" is an imprecise notion that isn't made fully precise by either homotopy groups or homology groups.
Recall from HW II #1(b) that a sequence of abelian groups and group homomorphisms
\[ \ldots \rightarrow G_{p+1} \xrightarrow{\alpha_{p+1}} G_p \xrightarrow{\alpha_p} G_{p-1} \rightarrow \ldots \]
is exact if \( \text{im}(\alpha_{p+1}) = \ker(\alpha_p) \) for all \( p \).

**Thm 13.16 (Mayer-Vietoris Theorem)** Let \( X \) be a topological space, and let \( U, V \) be open subsets of \( X \) with \( X = U \cup V \).

Then \( H_p \) there is a homomorphism \( d_* : H_p(X) \rightarrow H_{p-1}(U \cap V) \) such that the following sequence is exact:

\[
\begin{align*}
&\xrightarrow{d_*} H_p(U \cap V) \xrightarrow{i_* \otimes j_*} H_p(U) \oplus H_p(V) \xrightarrow{k_* - l_*} H_p(X) \\
&\xrightarrow{d_*} H_{p-1}(U \cap V) \xrightarrow{i_* \otimes j_*} H_{p-1}(U) \oplus H_{p-1}(V) \xrightarrow{k_* - l_*} H_{p-1}(X) \\
&\xrightarrow{d_*} H_{p-2}(U \cap V) \xrightarrow{i_* \otimes j_*} \\
\end{align*}
\]

\( d_* : H_p(X) \rightarrow H_{p-1}(U \cap V) \) is called the connecting homomorphism.

The other maps are defined:

\( [i_* \otimes j_*][c] = [i_*[c], j_*[c]] \) for \([c] \in H_p(U \cap V)\) and
\( (k_* - l_*)[c, [c']] = k_*[c] - l_*[c'] \) for \([c] \in H_p(U)\) and \([c'] \in H_p(V)\).
Remark: This theorem allows us to understand the homology of $X = U \cup V$ from the homologies of $U$, of $V$, and of $U \cap V$.

Example:
Let $X = S^2$ with north and south poles $N, S \in S^2$.
Let $U = S^2 \setminus \{N\} \simeq \ast$ (U is homotopy equivalent to a point)
Let $V = S^2 \setminus \{S\} \simeq \ast$
So $U \cap V = S^2 \setminus \{N, S\} \simeq S^1$

A homotopy equivalent picture is

The Mayer-Vietoris Theorem gives an exact sequence

$$
\begin{align*}
0 & \to H_2(U \cap V) \to H_2(U) \oplus H_2(V) \to H_2(X) \to 0 \\
\delta^2 & \to H_1(U \cap V) \to H_1(U) \oplus H_1(V) \to \cdots \\
\end{align*}
$$

Let's isolate our attention to $0 \to H_2(X) \xrightarrow{\partial_*} H_1(U \cap V) \to 0$.
Exactness at $H_2(X) \Rightarrow \partial_*$ injective
Exactness at $H_1(U \cap V) \Rightarrow \partial_*$ surjective
Hence $\partial_*$ is an isomorphism showing $H_2(S^2) = H_2(X) \cong H_1(U \cap V) \cong \mathbb{Z}$. 
Remark: We used $H_1(S^1) \cong \mathbb{Z}$ to get $H_2(S^2) \cong \mathbb{Z}$ via

One could use $H_2(S^2) \cong \mathbb{Z}$ to get $H_3(S^3) \cong \mathbb{Z}$ via

Inductively, one could use $H_{n-1}(S^{n-1}) \cong \mathbb{Z}$ to get $H_n(S^n) \cong \mathbb{Z}$ via
The Mayer-Vietoris Theorem is proven using...

**Lemma 13.17: The Zigzag (or Snake) Lemma**

Let $0 \rightarrow C_\ast \xrightarrow{f} D_\ast \xrightarrow{g} E_\ast \rightarrow 0$

be a short exact sequence of chain complexes:

\[
\begin{array}{cccccc}
0 & \rightarrow & C_{p+1} & \xrightarrow{f} & D_{p+1} & \xrightarrow{g} & E_{p+1} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & C_p & \xrightarrow{f} & D_p & \xrightarrow{g} & E_p & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & C_{p-1} & \xrightarrow{f} & D_{p-1} & \xrightarrow{g} & E_{p-1} & \rightarrow & 0 \\
\end{array}
\]

"short exact sequence of chain complexes"

(C\_\ast, D\_\ast, and E\_\ast being chain complexes means the composition of any two vertical arrows is zero)

(Short exact here means that each row is short exact, as in HW11 \#1(c)).

Then $H_p$ there is a connecting homomorphism $\partial_p : H_p(E) \rightarrow H_{p-1}(C)$ such that the following sequence is exact:

\[
\begin{array}{cccc}
\partial_p & H_p(C) & \rightarrow & H_p(D) & \rightarrow & H_p(E) \\
\partial_p & H_{p-1}(C) & \rightarrow & H_{p-1}(D) & \rightarrow & H_{p-1}(E) \\
\end{array}
\]

"long exact sequence of homology groups"

In English: A short exact sequence of chain complexes gives a long exact sequence of homology groups.
Homotopy Invariance (of Singular Homology)

If \( f_0, f_1 : X \to Y \) are homotopic maps, then for each \( p \geq 0 \) the induced homomorphisms
\((f_0)_*, (f_1)_* : H_p(X) \to H_p(Y)\) are equal.

(Recall that the corresponding result for fundamental groups,
Lemma 7.45, had a mess of basepoint issues.
We don't have that here; we have equality on the nose!)

Corollary 13.9
Suppose \( f : X \to Y \) is a homotopy equivalence. Then
\( f_* : H_p(X) \to H_p(Y) \) is an isomorphism for each \( p \geq 0 \).

"Homotopy equivalent spaces have isomorphic homology groups."

\[ \begin{align*}
\text{Pf} \quad X & \xrightarrow{g} Y \\
\text{H}_p(X) & \xrightarrow{f_*} \text{H}_p(Y)
\end{align*} \]

Note \( g \circ f \simeq \text{id}_X \) gives
\[ \text{id}_{H_p(X)} = (\text{id}_X)_* = (g \circ f)_* = g_* \circ f_* \]

\( \uparrow \text{functoriality} \quad \uparrow \text{Thm 13.8} \quad \uparrow \text{functoriality} \)

and \( f \circ g \simeq \text{id}_Y \) gives
\[ \text{id}_{H_p(Y)} = (\text{id}_Y)_* = (f \circ g)_* = f_* \circ g_* \]

Hence \( f_* \) is an isomorphism (with inverse \( g_* \)).
Pf of Thm 13.8: By the magic of functoriality, it will suffice to prove the very special case where

- \( Y = X \times I \)
- \( f_0 = l_0 : X \to X \times I \) is given by \( l_0(x) = (x, 0) \), and
- \( f_1 = l_1 : X \to X \times I \) is given by \( l_1(x) = (x, 1) \).

(Clearly \( l_0 \simeq l_1 \))

Indeed, suppose \( Y \) is arbitrary and \( f_0, f_1 : X \to Y \) satisfy \( f_0 \simeq f_1 \). Then there is a homotopy \( H : X \times I \to Y \) with \( H \circ l_0 = f_0 \) and \( H \circ l_1 = f_1 \).

Functoriality then gives

\[
(f_0)_* = (H \circ l_0)_* = H_* \circ (l_0)_* = H_* (l_1)_* = (H \circ l_1)_* = (f_1)_* \quad \text{by the very special case}
\]

So it suffices to consider this special case and prove \((l_0)_* = (l_1)_*\).

Aside: To prove \((l_0)_* = (l_1)_* : \text{H}^p(X) \to \text{H}^p(X \times I)\) it would be enough to define a map \( h : \text{Z}^p(X) \to \text{C}^{p+1}(X \times I) \) such that

\[
\partial h(c) = (l_1)_* c - (l_0)_* c \quad \text{for all } c \in \text{Z}^p(X).
\]

Indeed, we'd then have \((l_0)_* c + \text{B}^p(X \times I) = (l_1)_* c + \text{B}^p(X \times I),\)

meaning \((l_0)_*[c] = (l_1)_*[c]\) for all \([c] \in \text{H}^p(X)\),

i.e., meaning \((l_0)_* = (l_1)_* : \text{H}^p(X) \to \text{H}^p(X \times I)\).
Instead we’ll instead define a homomorphism
\[ h : C_p(X) \to C_{p+1}(X \times I) \quad \text{for all} \quad p \geq 0 \quad \text{satisfying} \]
\[ h \circ \partial + \partial \circ h = (v_1)^\# - (v_0)^\# . \]

\[ \cdots \to C_{p+1}(X) \xrightarrow{\partial} C_p(X) \xrightarrow{h} C_{p+1}(X \times I) \xrightarrow{\partial} C_p(X \times I) \xrightarrow{h} \cdots \]

\[ (h \text{ is called a chain homotopy from } (v_0)^\# \text{ to } (v_1)^\#) \]

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For \( c \in Z_p(X) \) we’ll then have
\[ (v_1)^\# c - (v_0)^\# c = h(\partial c) + \partial h(c) \]
\[ = \partial h(c) \quad \text{since } \partial c = 0. \]

For the same reason as in the aside, this gives
\[ (v_0)^\# = (v_1)^\# : H_p(X) \to H_p(X \times I). \]

Pic of \( h \) Let \( \sigma : \Delta_p \to X \) be a singular \( p \)-simplex in \( C_p(X) \).

Then \( h \sigma \) will be a singular \((p+1)\)-chain in \( C_{p+1}(X \times I) \).

\( \Delta_2 \times I \) is a sum of 3-simplices
Pic of why $h \circ \partial + \partial h = (u_1)\# - (u_0)\#$

(for a singular 2-simplex)

$\delta : \Delta_2 \to X$ in $C_2(X)$

$\delta = (-1)\# - u_0\#$

Coefficients $\pm 1$ on all six side triangles.

Coefficients $\pm 1$ on all six side triangles, and $+1$ on top, $-1$ on bottom.

Coefficient $+1$ on top and $-1$ on bottom.

One can check that in $h(\partial \sigma) + \partial h(\sigma)$, the coefficients on the six side triangles cancel, but we won't.

Pic of $\partial h(c) = (u_1)\# c - (u_0)\# c$ for $c \in Z_2(X)$

$\partial h(c) \in C_2(X \times I)$

$(u_1)\# c - (u_0)\# c \in C_2(X \times I)$
Pic of \( \partial h(c) = (u_i)_* c - (u_0)_* c \) for \( c \in Z_1(X) \)

\( c \in Z_1(X) \quad \implies h(c) \in C_2(X \times I) \quad \partial h(c) \in C_1(X \times I) \)

\( (u_i)_* c \quad \quad (u_0)_* c \)

\( +1 \quad -1 \quad +1 \quad -1 \)
Write the torus as the union of two cylinders and use the Mayer-Vietoris Theorem to deduce $H_2(\text{torus})$.

Remark
Since the torus is an orientable 2-manifold, we know the correct answer will be $H_2(\text{torus}) \cong \mathbb{Z}$.

Solution

$$U \cap V \cong S^1 \times S^1$$  \quad  $H_1(U \cap V) \cong \mathbb{Z} \oplus \mathbb{Z}$  \quad  $H_2(U \cap V) = 0$

$U \cong S^1$  \quad  $H_1(U) \cong \mathbb{Z}$  \quad  $H_2(U) = 0$

$V \cong S^1$  \quad  $H_1(V) \cong \mathbb{Z}$  \quad  $H_2(V) = 0$

$i_* : H_1(U \cap V) \to H_1(U)$ by $(a, b) \mapsto a + b$.

$j_* : H_1(U \cap V) \to H_1(V)$ by $(a, b) \mapsto a + b$. 

By Mayer–Vietoris (Thm 13.16) we have a long exact sequence

\[ \cdots \to H_2(U \cap V) \xrightarrow{i_* \otimes j_*} H_2(U) \oplus H_2(V) \xrightarrow{k_* - l_*} H_2(X) \to \cdots \]

Note \( D_* \) is injective since \( \ker(D_*) = \im(k_* - l_*) = 0 \).

So \( H_2(\text{torus}) = H_2(X) \cong \im D_* \) (by 1st isomorphism theorem)

\[ \cong \ker(i_* \otimes j_*) \] (by exactness at \( H_1(U \cap V) \))

\[ \cong \mathbb{Z} \]

Since \( i_* \otimes j_* : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \) has a 1-dimensional kernel \( \ker(i_* \otimes j_*) = \{ (a, -a) \mid a \in \mathbb{Z} \} \cong \mathbb{Z} \).

**Ex** Use this same gluing to deduce \( H_1(\text{torus}) \).

**Rmk** We know \( H_1(\text{torus}) \cong \mathbb{Z} \oplus \mathbb{Z} \) since \( H_1 \) is the abelianization of \( \pi_1 \), and \( \pi_1(\text{torus}) \cong \mathbb{Z} \oplus \mathbb{Z} \).
Solution We have a long exact sequence

\[ H_1(U \cup V) \xrightarrow{i_*} H_1(U) \oplus H_1(V) \xrightarrow{k_* - l_*} H_1(X) \]

\[ H_0(U \cup V) \xrightarrow{j_*} H_0(U) \oplus H_0(V) \]

Let \( i_* : H_0(U \cup V) \to H_0(U) \) by \( (a, b) \to a + b \)

Let \( j_* : H_0(U \cup V) \to H_0(V) \) by \( (a, b) \to a + b \)

Note \( \text{im}(i_*) = \text{ker}(i_* \oplus j_* : H_0(U \cup V) \to H_0(U) \oplus H_0(V)) \)

\[ = \{(a, -a) | a \in \mathbb{Z}^3\} \]

\[ \cong \mathbb{Z} \]

Note \( \text{ker}(j_*) = \text{im}(k_* - l_*) \)

\[ \cong (H_1(U) \oplus H_1(V)) / \text{ker}(k_* - l_*), \text{by 1st iso thm} \]

\[ \cong (\mathbb{Z} \oplus \mathbb{Z}) / \{(a, a) | a \in \mathbb{Z}^3\} \]

\[ \cong \mathbb{Z} \]

By the 1st iso thm we have \( H_1(X) / \text{ker}(j_*) \cong \text{im}(j_*) \)

with \( \text{ker}(j_*) \cong \mathbb{Z} \) and \( \text{im}(j_*) \cong \mathbb{Z} \).

In general these extension problems are hard to solve, but in this case we have \( H_1(X) \cong \mathbb{Z} \oplus \mathbb{Z} \).
Degree Theory for Spheres

Algebra Fact: Any group homomorphism $h: \mathbb{Z} \to \mathbb{Z}$ is multiplication by some fixed integer $d$.

\[
\begin{align*}
\text{Ex} & \quad \text{Ex} \\
& \quad \text{Ex} \\
& \quad \text{Ex} \\
& \quad \text{Ex} \\
& \quad \text{Ex} \\
\end{align*}
\]

\[
\begin{align*}
h(-1) &= -3 & h(-1) &= 8 \\
h(0) &= 0 & h(0) &= 0 \\
h(1) &= 3 & h(1) &= -8 \\
h(2) &= 6 & h(2) &= -16 \\
& \quad \text{ Proof } \quad \text{ Proof } \\
& \quad \text{ Proof } \quad \text{ Proof } \\
& \quad \text{ Proof } \quad \text{ Proof } \\
& \quad \text{ Proof } \quad \text{ Proof } \\
& \quad \text{ Proof } \quad \text{ Proof } \\
\end{align*}
\]

Let $d = h(1)$. Note for any $m \in \mathbb{Z}$ we have 
\[
h(m) = h(1 + \ldots + 1) = h(1) + \ldots + h(1) = m \cdot h(1).
\]

Def: Let $n \geq 1$. Recall $H_n(S^n) \cong \mathbb{Z}$.

We define the degree of a continuous map $f: S^n \to S^n$ to be the unique integer $d$ such that the induced homomorphism $f_*: H_n(S^n) \to H_n(S^n)$ is multiplication by $d$.

This integer is denoted $\text{deg}(f) = d$.

\[
\begin{align*}
\text{Ex} & \quad \text{Ex} \\
& \quad \text{Ex} \\
& \quad \text{Ex} \\
& \quad \text{Ex} \\
& \quad \text{Ex} \\
\end{align*}
\]

\[
\begin{align*}
\text{deg}(g) &= -2 & \text{deg}(g) &= -1 & \text{deg}(g) &= 0 & \text{deg}(g) &= 1 & \text{deg}(g) &= 2 \\
& \quad \text{...} & \quad \text{...} & \quad \text{...} & \quad \text{...} & \quad \text{...} & \quad \text{...} \\
\end{align*}
\]
\textbf{Ex} \quad S : S^1 \rightarrow S^1

\begin{align*}
\text{deg}(f) &= 0 \\
\text{deg}(f) &= 0
\end{align*}

Thm 13.28 will say \text{deg}(f) \neq 0 \Rightarrow f \text{ surjective.} \quad \text{This picture shows the converse is false!}

\textbf{Ex} \quad S : S^2 \rightarrow S^2

\begin{align*}
\text{deg}(f) &= 0 \\
\text{deg}(f) &= 1
\end{align*}

\begin{itemize}
\item Constant map to a point
\item Identity map
\end{itemize}

This map cuts the globe at the prime meridian (say), and then wraps the surface of the earth around twice, fixing the north and south poles.

As Alex looked up for us, Fort Collins (40.59° latitude, 105.08° longitude) and At-Bashi, Kyrgyzstan (40.59°, 75.08°) get mapped on top of each other under this map.
Prop 13.25: If \( f, g : S^n \to S^n \) are continuous, then

(a) \( \deg (f \circ g) = \deg (g) \cdot \deg (f) \).

(b) If \( f \simeq g \), then \( \deg (f) = \deg (g) \).

Proof:
Part (a) follows from \((f \circ g)_* = g_* \circ f_*\).
Indeed, for any integer \( m \in \mathbb{Z} \) we have
\[
(f \circ g)_*(m) = g_* (f_* (m)) = g_* (\deg (f) m) = \deg (g) \cdot \deg (f) m.
\]
Part (b) follows since \( f \simeq g \) implies \( f_* = g_* : H_n (S^n) \to H_n (S^n) \) (by Thm 13.8).

**Picture of (b)**: \( S^n \simeq g \)

**Rmk:** (b) is in fact an \( \Longleftrightarrow \), but the proof of the reverse direction requires more machinery.

Prop 13.27: Degrees of some common maps.

(a) The identity map \( \text{id} : S^n \to S^n \) has degree 1.

(b) A constant map \( c : S^n \to S^n \) has degree zero.

(c) A reflection map \( R : S^n \to S^n \) (about an \( n \)-dimensional hyperplane through the origin in \( \mathbb{R}^{n+1} \)) has degree -1.

(d) The antipodal map \( \alpha : S^n \to S^n \) given by \( \alpha (x) = -x \) has degree \( (-1)^{n+1} \).
(a) follows since \( \text{id}_*: \mathbb{H}_n(S^n) \to \mathbb{H}_n(S^n) \) is the identity homomorphism.

(b) follows since if \( c: S^n \to S^n \) is a constant map (meaning \( \exists \hat{p} \in S^n \) with \( c(\hat{x}) = \hat{p} \ \forall \hat{x} \in S^n \)), then \( c_*: \mathbb{H}_n(S^n) \to \mathbb{H}_n(S^n) \) maps every element to zero.

Instead of proving (c), we'll draw a picture:

Reflecting through a hyperplane "turns the sphere inside-out."

(d) follows from (c). Indeed, for \( 1 \leq i \leq n+1 \) let \( R_i: S^n \to S^n \) be the reflection given by \( R_i(x_1, \ldots, x_i, \ldots, x_{n+1}) = (x_1, \ldots, -x_i, \ldots, x_{n+1}) \).

Since \( a(x_1, \ldots, x_{n+1}) = (-x_1, \ldots, -x_{n+1}) \),

this means \( a = R_{n+1} \circ R_n \circ \ldots \circ R_2 \circ R_1 \).

Hence \( \deg(a) = \deg(R_{n+1}) \deg(R_n) \ldots \deg(R_1) \) (Prop 13.25(a))

\[ = (-1) \cdot (-1) \ldots (-1) \] by (c)

\[ = (-1)^{n+1} \].
Def: A vector field on $S^n$ is a continuous map $V : S^n \rightarrow \mathbb{R}^{n+1}$ such that $V(\vec{x})$ is tangent to $S^n$ at $\vec{x}$ (i.e., $V(\vec{x}) \cdot \vec{x} = 0$) for all $\vec{x} \in S^n$.

Thm 13.32 (The Hairy Ball Theorem): There exists a nowhere vanishing vector field on $S^n$ if and only if $n$ is odd.

$n=1$: You can comb a hairy circle.

$n=2$: You can't comb a hairy ball.

This can be rephrased as saying there is always a spot on the earth where the (tangential) wind speed is zero!
Proof

When \( n = 2k - 1 \) is odd, then \( S^n \subseteq \mathbb{R}^{2k} \).

The following vector field is tangent to the sphere and nowhere vanishing:
\[
V(x_1, x_2, \ldots, x_{2k-1}, x_{2k}) = (x_2 - x_1, x_4 - x_3, \ldots, x_{2k} - x_{2k-1}).
\]

Now let \( n \) be even. Suppose for a contradiction that we had a nowhere vanishing vector field \( V \) on \( S^n \).

We may assume \( |V(x)| = 1 \) for all \( x \in S^n \) (simply replace \( x \rightarrow V(x) \) with \( x \rightarrow V(x)/|V(x)| \)).

We use \( V \) to construct a homotopy between the identity map \( \text{id} : S^n \rightarrow S^n \) and the antipodal map \( \alpha : S^n \rightarrow S^n \) (with \( \alpha(x) = -x \)) as follows:

\[
H : S^n \times I \rightarrow S^n \quad \text{via}
\]

\[
H(x, t) = \cos(\pi t) x + \sin(\pi t) V(x).
\]

This map is well-defined (lands in \( S^n \)) since
\[
\| H(x, t) \|^2 = \cos^2(\pi t) \| x \|^2 + 2 \cos(\pi t) \sin(\pi t) \langle x, V(x) \rangle + \sin^2(\pi t) \| V(x) \|^2
\]

\[
= \cos^2(\pi t) + \sin^2(\pi t)
\]

\[
= 1 \quad \text{for all } x \in S^n \text{ and } t \in I.
\]

The continuity of this map follows from the continuity of \( V \).

Finally, note \( H(x, 0) = x = \text{id}(x) \)

and \( H(x, 1) = -x = \alpha(x) \).
However, this homotopy between the identity and antipodal maps contradicts Prop 13.25 (b) and 13.27 (d), since

\[ \text{deg}(\alpha) = (-1)^{n+1} = (-1) \neq 1 = \text{deg}(\text{id}) \]

since \( n \) is even.

Hence for \( n \) even there cannot be a nowhere vanishing vector field on \( S^n \). □

**Remark** When \( n = 2k - 1 \) is odd, the homotopy \( H \) can be combined with the vector field

\[ V(x_1, \ldots, x_{2k}) = (x_2, -x_1, x_4, -x_3, \ldots, x_{2k}, -x_{2k-1}) \]

to get a homotopy \( \text{id} \sim \alpha : S^n \to S^n \)

![Diagram](image)

**Pie n=1** \( V(x_1, x_2) = (x_2, -x_1) \)

\[ H(\bar{x}, 0) = \bar{x} \]

\[ H(\bar{x}, 1) = -\bar{x} = \alpha(\bar{x}) \]

\[ H(\bar{x}, 0) = \bar{x} = \text{id}(\bar{x}) \]

\[ H(\bar{x}, t) = V(\bar{x}) \]

This essentially gives a proof of...

**Prop 13.31** The antipodal map \( \alpha : S^n \to S^n \) is homotopic to the identity map if and only if \( n \) is odd.