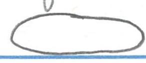
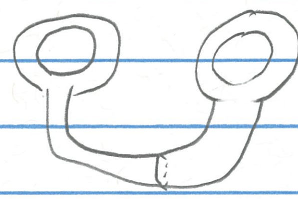
Math 570: Topology I

8/24/17 <u>Motivating question</u> When are two spaces equivalent up to stretching or bending, not allowing ripping, tearing, or gluing?

That is, when are two spaces homeomorphic ($\cong$), or homotopy equivalent ($\simeq$), or ambient isotopic?

<u>Ex</u>  $\cong$     $\cong$    <u>Ex</u> $(\cong) \Rightarrow (\simeq)$

<u>Ex</u>  $\simeq$  but  $\not\cong$    <u>Ex</u>  $\not\simeq$ 

<u>Ex</u> If surfaces  and 

were made of very flexible rubber, how do you deform one to get the other? This is an ambient isotopy

<u>Point-set topology</u> This is a generalization of metric spaces that allows us to define spaces and continuity without a notion of distance.

Given spaces X and Y, one can often show $X \cong Y$ or $X \simeq Y$ by giving an explicit deformation between them. But how would you show $X \not\cong Y$ or $X \not\simeq Y$? It's hard to consider / rule out all possible deformations.

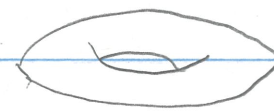
Algebraic topology

Given a space X, we'll associate to X families of algebraic invariants: the homotopy groups $\pi_i(X)$ and homology groups $H_i(X)$.

- Roughly speaking, $\pi_i(X)$ and $H_i(X)$ "count the # of i-dimensional holes in X".

"3 1-d holes"

"1 2-d hole"



"2 1-d holes"
"1 2-d hole via H_2"
"No 2-d holes via π_2"

- If $X \simeq Y$, then $\pi_i(X) \underset{\substack{\uparrow \\ \text{group isomorphism}}}{\cong} \pi_i(Y)$ and $H_i(X) \cong H_i(Y)$ for all i.

  Hence if $\pi_i(X) \not\cong \pi_i(Y)$ or $H_i(X) \not\cong H_i(Y)$ for some i, then $X \not\simeq Y$.

- Homotopy groups are easier to define and harder to compute. Homology groups are harder to define and easier to compute.
- These are examples of categories and functors.
- An example non-negative result is <u>Whitehead's Theorem</u>: If a continuous map $f : X \to Y$ between CW complexes induces an isomorphism $f_* : \pi_i(X) \to \pi_i(Y)$ for all i, then $X \simeq Y$.

8/23/17 Chapter 2: Topological Spaces

Def A *topology* T on a set X is a collection of subsets of X (called open sets) such that

(i) X and $\emptyset$ are open

(ii) A finite intersection of open sets is open

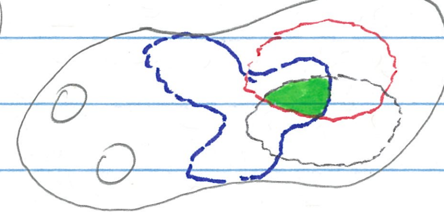
(iii) An arbitrary union of open sets is open.

The resulting topological space is denoted (X,T) or simply X.

Ex Every metric space is a topological space (where the open sets are unions of open balls)

(iii)

(ii)



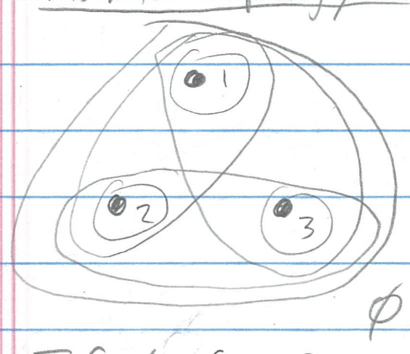
Rmk $\bigcap_{n=1}^{\infty} (-1/n, 1/n) = \{0\}$ is not open in $\mathbb{R}$

Def The *discrete topology* on X is $T = P(X)$

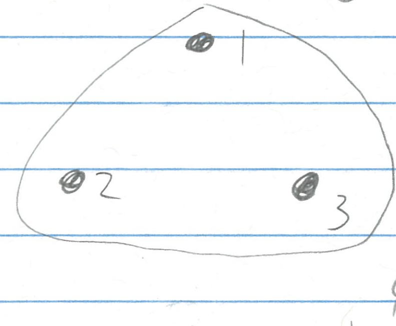
The *trivial* or *indiscrete topology* on X is $T = \{\emptyset, X\}$

Ex $X = \{1,2,3\}$

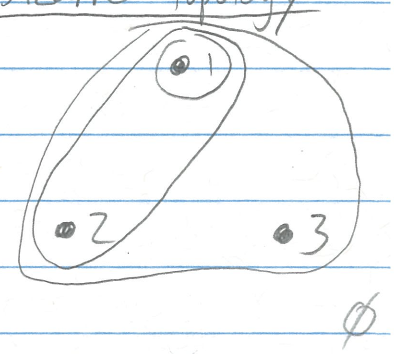
Discrete topology



Indiscrete topology



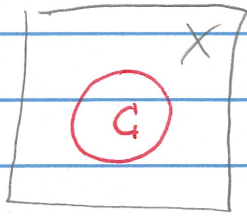
Bizarre topology

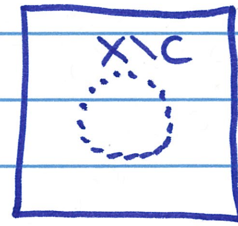


Question If $X = \{1,2,3\}$ were a metric space, which topology must it have?

Rmk Any subset X of $\mathbb{R}^n$ (with the Euclidean metric) is a topological space.

Def A subset C of topological space X is <u>closed</u> if its complement $X \setminus C$ is open.





← open in X!

(Not open in $\mathbb{R}^2$ but that's irrelevant)

Properties If X is a topological space, then

(i) $\emptyset$ and X are closed [A set can be both open and closed]

(ii) A finite union of closed sets is closed

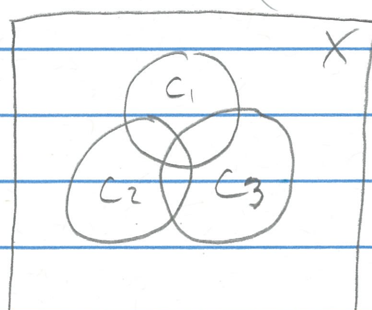
(iii) An arbitrary intersection of closed sets is closed

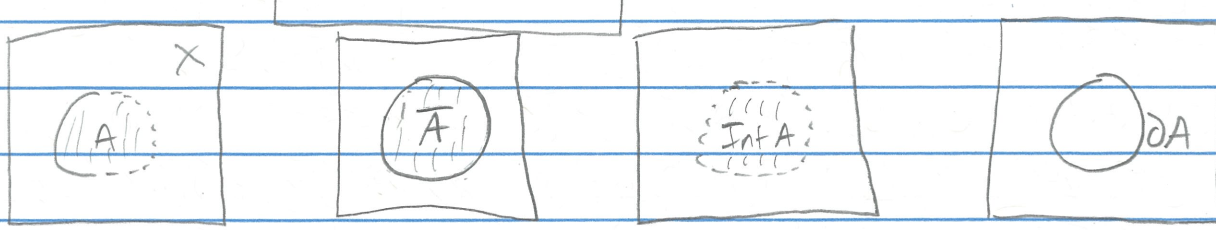
Pf (i) $X \setminus \emptyset = X$ is open ; $X \setminus X = \emptyset$ is open.

Pf (ii) $C_1, \ldots, C_n$ closed $\Rightarrow X \setminus C_1, \ldots, X \setminus C_n$ open

$\Rightarrow \bigcap_{i=1}^{n} (X \setminus C_i)$ is open

and $\bigcup_{i=1}^{n} C_i = X \setminus \left(\bigcap_{i=1}^{n} (X \setminus C_i) \right)$





Def Let X be a topological space and $A \subseteq X$.
The <u>closure of A in X</u> is the smallest closed set containing A:

$$\overline{A} = \bigcap \{ C \subseteq X \text{ closed and } A \subseteq C \}.$$

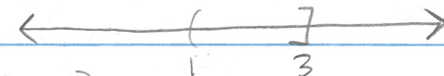
The <u>interior of A in X</u> is the largest open set contained in A:

$$\operatorname{Int} A = \bigcup \{ U \subseteq X \mid U \subseteq X \text{ open and } U \subseteq A \}$$

The <u>boundary of A in X</u> is $\partial A = \overline{A} \cap \overline{(X \setminus A)}$,

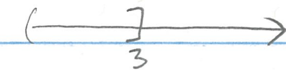
or equivalently, $\partial A = X \setminus (\operatorname{Int} A \cup (X \setminus \overline{A}))$.

<u>Ex</u> Let $A = (1,3]$ and $X = \mathbb{R}$.



Then $\overline{A} = [1,3]$, $\operatorname{Int} A = (1,3)$, $\partial A = \{1,3\}$

<u>Ex</u> Let $A = (1,3]$ and $X = (1,\infty)$.



Then $\overline{A} = (1,3]$, $\operatorname{Int} A = (1,3)$, $\partial A = \{3\}$

Save for homework?

<u>Convergence and Continuity</u>

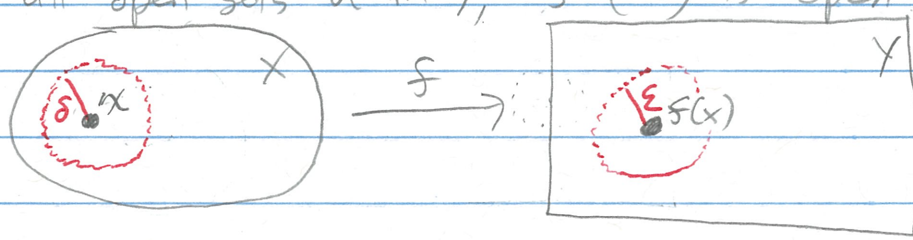
<u>Recall</u> If X and Y are metric spaces, then $f: X \to Y$ is continuous if:

For all $x \in X$ and $\varepsilon > 0$ $\exists \delta > 0$ s.t. $d(x,x') < \delta \Rightarrow d(f(x), f(x')) < \varepsilon$.

$\Leftrightarrow$ For all $B(f(x), \varepsilon) \subseteq Y$ $\exists \delta > 0$ s.t. $B(x,\delta) \subseteq f^{-1}(B(f(x),\varepsilon))$

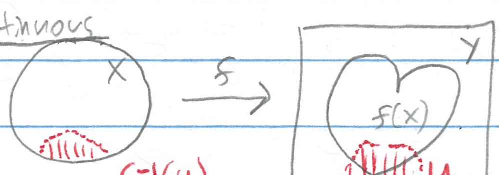
$\Leftrightarrow$ For all open balls B in Y, $f^{-1}(B)$ is open in X

$\Leftrightarrow$ For all <u>open sets</u> U in Y, $f^{-1}(U)$ is open in X.

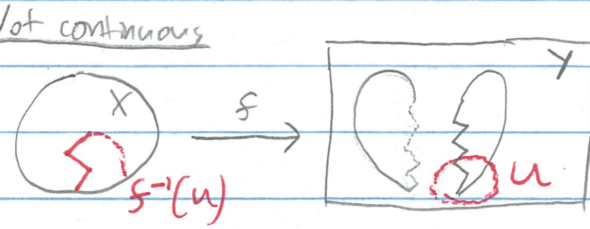


<u>Def</u> If X and Y are topological spaces, then $f: X \to Y$ is continuous if for all open sets U in Y, $f^{-1}(U)$ is open in X.

<u>Continuous</u>

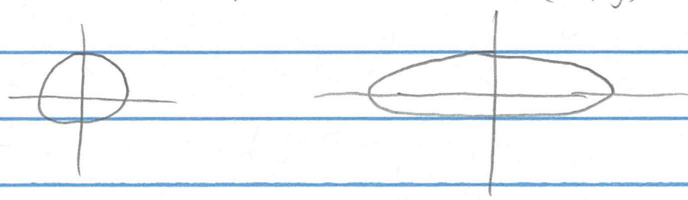


<u>Not continuous</u>

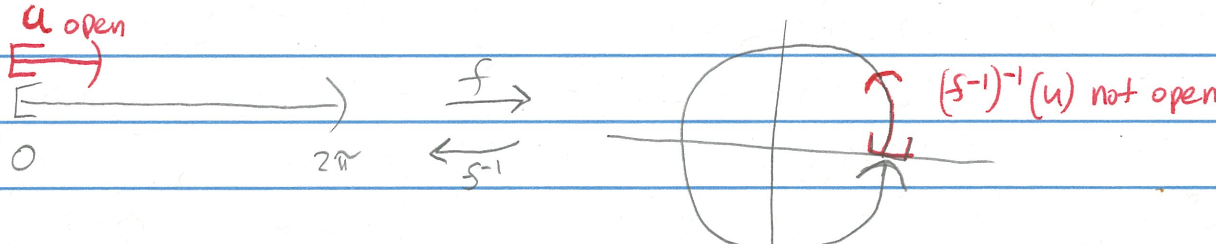


8/25/17 **Def** A <u>homeomorphism</u> between topological spaces X and Y is a bijection $f: X \to Y$ s.t. f and f^{-1} are continuous. If such a map exists, we say X and Y are <u>homeomorphic</u> $(X \cong Y)$.

Ex $S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$ and $E = \{(x,y) \in \mathbb{R}^2 \mid (x/3)^2+y^2=1\}$ are homeomorphic via $(x,y) \longmapsto (3x,y)$



Rmk The hypothesis that f^{-1} is continuous is necessary. Note $f: [0,2\pi) \to S^1$ via $f(t) = (\cos t, \sin t)$ is a continuous bijection that is not a homeomorphism since f^{-1} is not continuous.

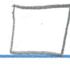


Ex Let $B^n = \{x \in \mathbb{R}^n \mid \|x\| < 1\}$. Can you find a homeomorphism $f: B^n \to \mathbb{R}^n$?

Ans $f(x) = \frac{x}{1-\|x\|}$. $\qquad f^{-1}(y) = \frac{y}{1+\|y\|}$.

Check $f^{-1} \circ f = id_X$ and $f \circ f^{-1} = id_Y$

Hence "boundedness" is not a "topological property".

Ex  $\cong$  and  $\cong$ 

"Corners" or "smoothness" is not a topological property.

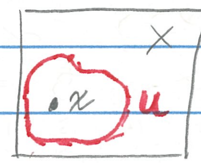
Rmk For $f: X \to Y$ a homeomorphism, note

U is open in $X \iff f(U)$ is open in Y.

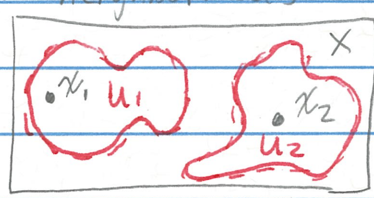
Since arbitrary topological spaces can be wild (for instance a convergent sequence may have multiple limit points), we now study nicer spaces.

Hausdorff Spaces

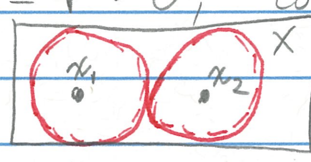
Def A *neighborhood of point x* in topological space X is an open set U with $x \in U$.



Def A topological space X is *Hausdorff* if given any distinct $x_1, x_2 \in X$, there exist disjoint open neighborhoods $U_1 \ni x_1$ and $U_2 \ni x_2$.



Ex A metric space X is Hausdorff: given $x_1, x_2 \in X$ with $d(x_1, x_2) = r > 0$, consider the open balls $B(x_1, r/2)$ and $B(x_2, r/2)$.



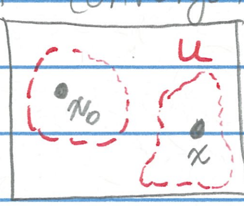
Ex A discrete space X is Hausdorff: given $x_1 \neq x_2$ in X, consider the disjoint open neighborhoods $\{x_1\}$ and $\{x_2\}$.

Prop 2.37 For X a Hausdorff space,

(a) Every finite subset is closed

(b) A convergent sequence has a unique limit

Pf (a)



We first show a single point $\{x_0\}$ is closed. Let $x \in X \setminus \{x_0\}$; since X Hausdorff we can find an open neighborhood $U \ni x$ contained in $X \setminus \{x_0\}$. Hence $X \setminus \{x_0\}$ is open and $\{x_0\}$ is closed.

Finally, note any finite set is a finite union of closed points and hence closed.

Bases and Countability

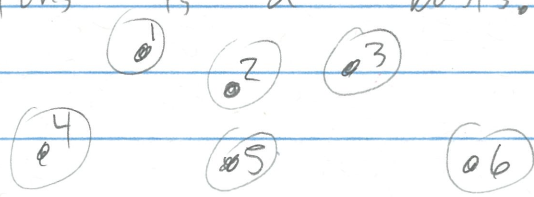
Let X be a topological space. Instead of writing down all the open subsets of X, a more convenient way to describe the topology is to give a basis.

Def A collection $\mathcal{B}$ of open subsets of X is a <u>basis for X</u> if every open set in X is a union of sets in $\mathcal{B}$.

Ex M a metric space.

The collection of open balls is a basis for M since every open set is a union of open balls.

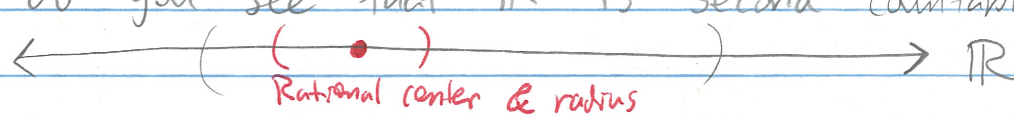
Ex In a discrete topological space, the collection of singletons is a basis.



8/30/2017

Def A topological space is <u>second countable</u> if it has a countable basis.

Ex How do you see that $\mathbb{R}$ is second countable?

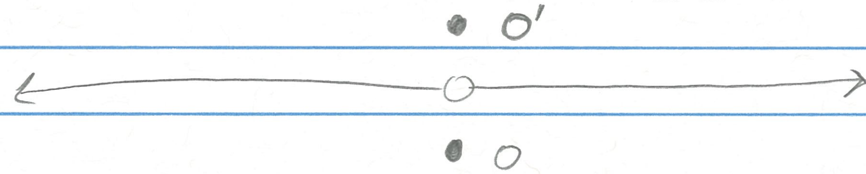


Consider the collection of all balls with rational centers and radii

Ex $\mathbb{R}^n$ is second countable, by the same argument!

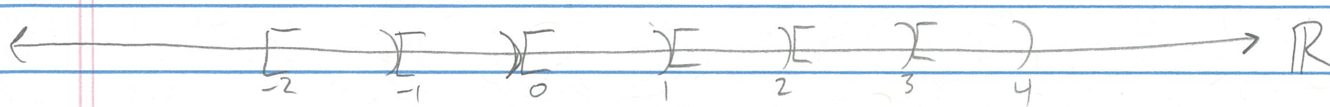
Rmk We'll define a manifold to be a second countable Hausdorff space that is "locally Euclidean".

Hausdorff will rule out the <u>line with two origins</u>:



Open sets consist of all open sets U in $\mathbb{R}$, along with $(U \setminus \{0\}) \cup \{0'\}$ if $0 \in U$.
Not Hausdorff since any open sets about $0, 0'$ intersect.

Second countable will rule out the <u>long line</u>:
Whereas $\mathbb{R}$ is a countable union of half-open intervals $[0,1)$ laid end-to-end, the long line is an uncountable such union.

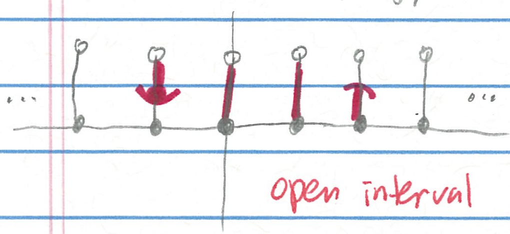


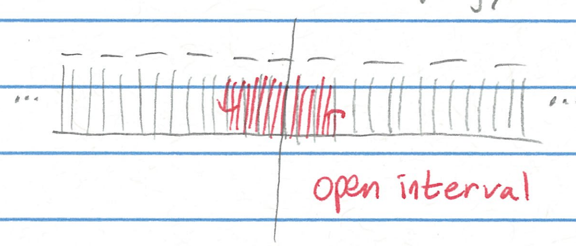
Homework A second countable space has a countable dense subset (i.e. is "separable").

Rmk It's easy to believe that the long line is not separable.

Picture $\mathbb{R} = \mathbb{Z} \times [0,1)$ with order topology

Long ray $= \mathbb{R} \times [0,1)$ with order topology

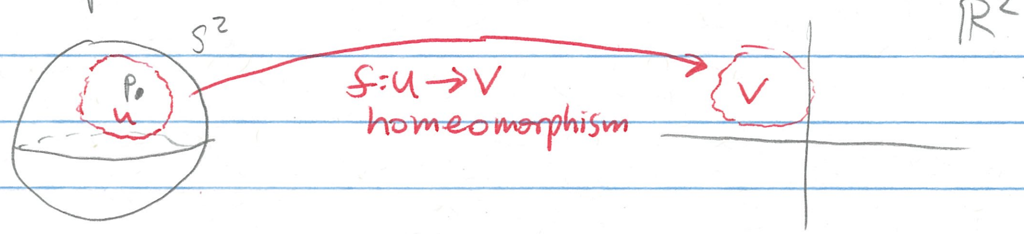


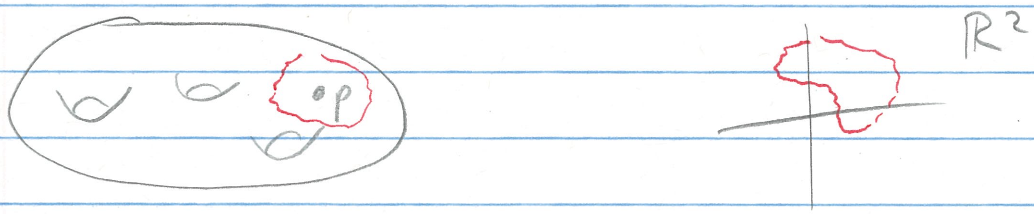


Manifolds

Def A space M is <u>locally Euclidean of dimension n</u> if every $p \in M$ has a neighborhood homeomorphic to an open subset of $\mathbb{R}^n$.

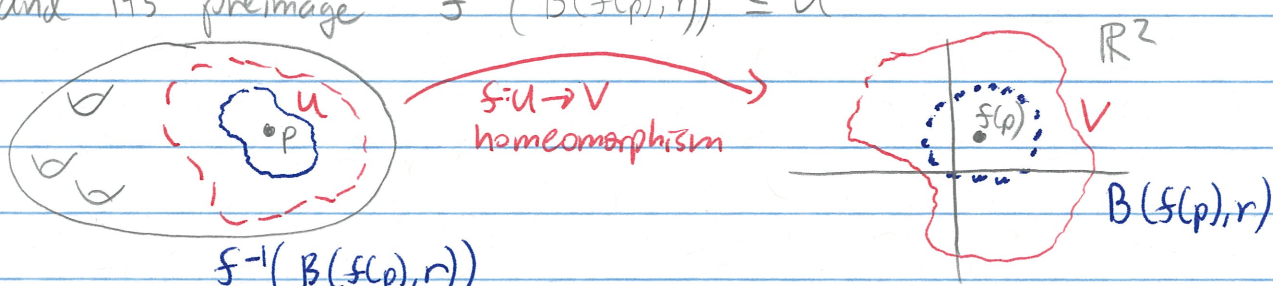
Ex





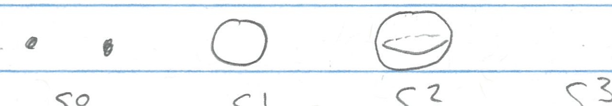
Lemma 2.52 Could equivalently replace (i) "open subset of $\mathbb{R}^n$" with (ii) "open ball of $\mathbb{R}^n$" or (iii) "$\mathbb{R}^n$".

Pf (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (i) clear.
(ii) $\Leftrightarrow$ (iii) since $B(x,r) \cong \mathbb{R}^n$.
To see (i) $\Rightarrow$ (ii), consider some $B(f(p),r) \subseteq V$ and its preimage $f^{-1}(B(f(p),r)) \subseteq U$

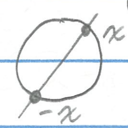


Def An n-dimensional manifold is a second countable Hausdorff space that is locally Euclidean of dimension n.

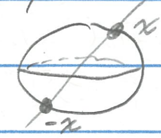
Ex $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$
n-sphere



Ex Projective space $\mathbb{R}P^n = S^n / x \sim -x$

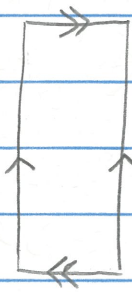


$\mathbb{R}P^1 \cong S^1$



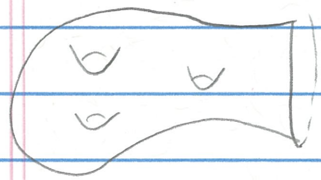
$\mathbb{R}P^2$

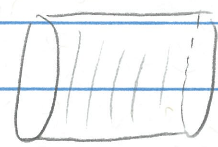
Ex Klein bottle



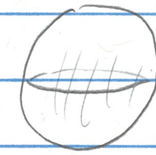
9/1/2017 <u>Manifolds with boundary</u>

Ex





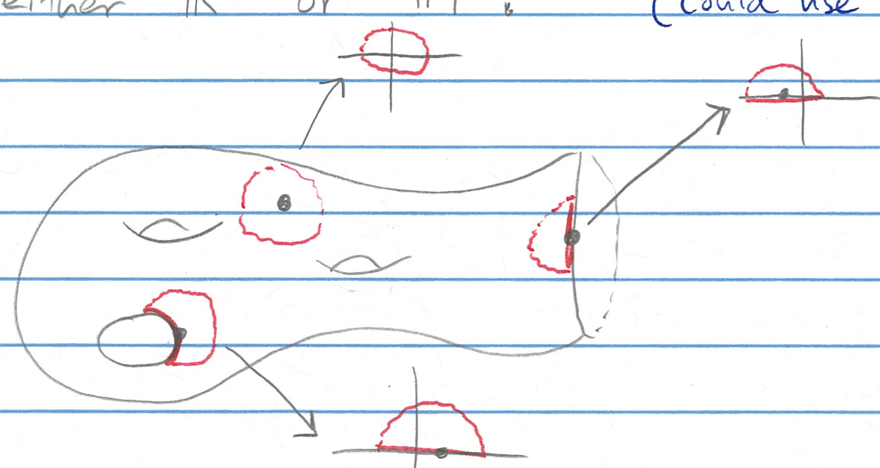
$S^1 \times I$



$\overline{B^3} = \{x \in \mathbb{R}^3 \mid \|x\| \leq 1\}$

Def Upper half-space $\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$.

Def An <u>n-dimensional manifold with boundary</u> is a second countable Hausdorff space in which every point has a neighborhood homeomorphic to an open subset of either $\mathbb{R}^n$ or $\mathbb{H}^n$. (Could use only $\mathbb{H}^n$)



Def The *boundary of $\mathbb{H}^n$* is $\partial\mathbb{H}^n = \{(x_1,\dots,x_n)\in\mathbb{H}^n \mid x_n = 0\}$

Def For M an n-dimensional manifold with boundary, the *boundary of M* (denoted ∂M) is the set of all $p\in M$ s.t. some (hence all) homeomorphism f from a neighborhood of p to $\mathbb{H}^n$ has $f(p)\in\partial\mathbb{H}^n$.

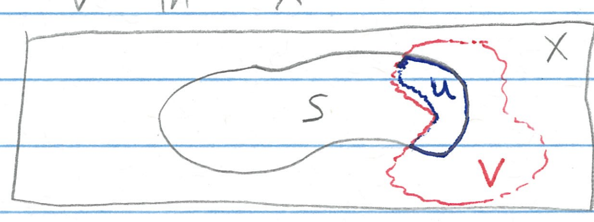
Warning The boundary of a manifold and the boundary of some subset $A\subseteq X$ (with X a topological space) mean different things.

Chapter 3 New spaces from old

– subspaces, product spaces, disjoint unions, quotients

Subspaces

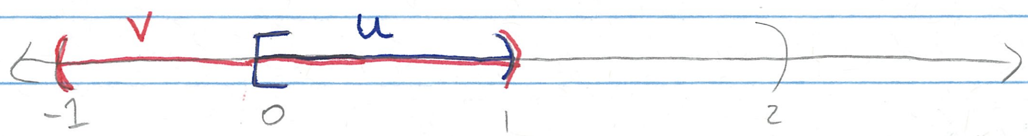
Def Given a topological space X and a subset $S\subseteq X$, the *subspace topology on S* has as its open sets all $U\subseteq S$ such that $U = S\cap V$ for some open set V in X



Example open sets in X

Example open sets in S

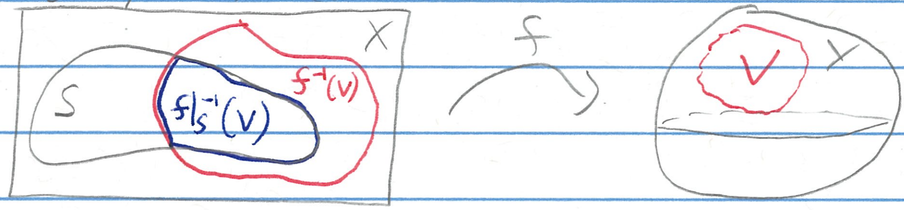
Ex $[0,1)$ is not open in $\mathbb{R}$, but $[0,1)$ is open in $[0,2)$ since $\underbrace{[0,1)}_{U} = \underbrace{[0,2)}_{S} \cap \underbrace{(-1,1)}_{V,\text{ open in }\mathbb{R}}$



$X=\mathbb{R}$
$S=[0,1)$

Rmk For $S\subseteq\mathbb{R}^n$, the Euclidean and subspace topologies on S agree.

Corollary 3.10(a) If X, Y are spaces, $f: X \to Y$ is continuous, and $S \subseteq X$ has the subspace topology, then $f|_S : S \to Y$ is continuous.



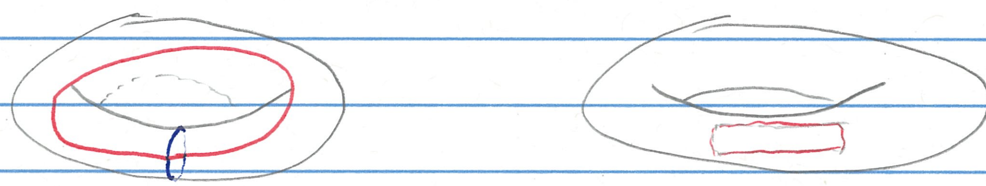
Pf V open in $Y \Rightarrow f^{-1}(V)$ open in X (since f continuous).
Hence $f|_S^{-1}(V) = S \cap f^{-1}(V)$ is open in S.

Product Spaces

Def Given topological spaces $X_1, \ldots, X_n$, a basis for the product topology on $X_1 \times \cdots \times X_n = \{(x_1, \ldots, x_n) \mid x_i \in X_i \ \forall i\}$ is $\mathcal{B} = \{U_1 \times \cdots \times U_n \mid U_i \text{ is open in } X_i \ \forall i\}$

"for all"

Pic



$S^1 \times S^1$ open basis element

Rmk The Euclidean and product topologies on $\mathbb{R}^n$ agree!
More generally,

Def Given an indexed family of topological spaces $\{X_\alpha\}_{\alpha \in A}$, a basis for the product topology on $\prod_{\alpha \in A} X_\alpha$ is
$$\mathcal{B} = \left\{ \prod U_\alpha \;\middle|\; \begin{array}{l} U_\alpha \text{ is open in } X_\alpha \ \forall \alpha \\ \text{and } U_\alpha = X_\alpha \text{ for all but finitely many } \alpha \end{array} \right\}$$

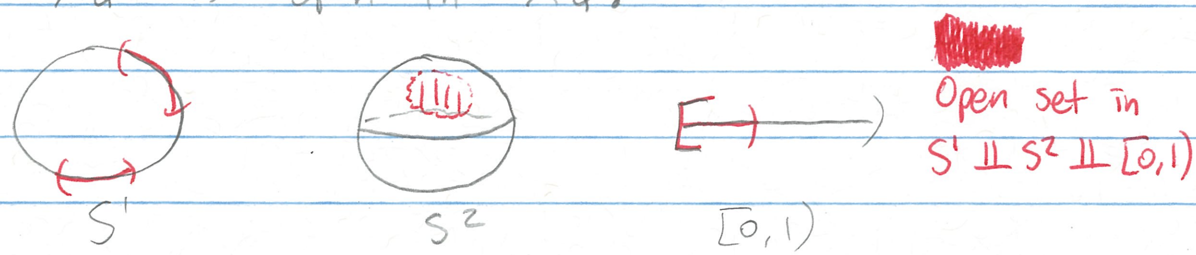
Rmk Removing the "$U_\alpha = X_\alpha$ for all but finitely many α" gives the box topology, which is different for infinite products.
Note $(0,1)^{\mathbb{N}}$ is open in $(\mathbb{R}^{\mathbb{N}}, \text{box})$ but not $(\mathbb{R}^{\mathbb{N}}, \text{product topology})$.

Rmk The product topology is the "right one" since it gives the product in the category of topological spaces.

9/6/2017

Disjoint Unions

Def Given an indexed family of topological spaces $\{X_\alpha\}_{\alpha \in A}$, the <u>disjoint union topology</u> on $\coprod_{\alpha \in A} X_\alpha$ declares a set to be open if its intersection with each X_α is open in X_α.



Notation Just like $X_1 \times \dots \times X_n = \prod_{i \in \{1,\dots,n\}} X_i$, we have $X_1 \amalg \dots \amalg X_n = \coprod_{i \in \{1,\dots,n\}} X_i$.

Rmk This will give coproducts in the category of topological spaces.

Selections from <u>Categories and Functors</u> in Chapter 7

Def A <u>category</u> $\mathcal{C}$ consists of

- a class of objects $Ob(\mathcal{C})$
- a class of morphisms $Hom(\mathcal{C})$
- for each $X, Y, Z \in Ob(\mathcal{C})$, a composition map

$$\begin{aligned} Hom_{\mathcal{C}}(X,Y) \times Hom_{\mathcal{C}}(Y,Z) &\longrightarrow Hom_{\mathcal{C}}(X,Z) \\ (f, g) &\longmapsto g \circ f \end{aligned}$$

such that

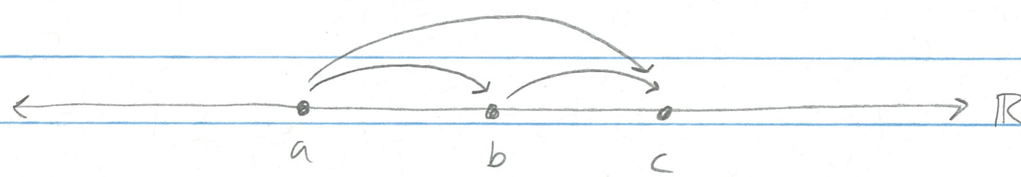
- $(f \circ g) \circ h = f \circ (g \circ h)$
- for all $X \in Ob(\mathcal{C})$ $\exists$ $Id_X \in Hom_{\mathcal{C}}(X,X)$ s.t. for every morphism $f \in Hom_{\mathcal{C}}(A,B)$, $Id_B \circ f = f = f \circ Id_A$.

Example categories

Objects	Morphisms	Isomorphisms
sets	functions	bijections
groups	group homomorphisms	group isomorphisms
rings	ring homomorphisms	ring "
vector spaces over field k	k-linear maps	vector space "
modules over ring R	module homomorphisms	module "
topological spaces	continuous maps	homeomorphisms
differentiable manifolds	differentiable maps	diffeomorphisms
$(\mathbb{R}, \leq)$	$a \leq b$	$a \leq a$

Rmk $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ is often written $f: X \to Y$, even though f need not be a function.

Ex $(\mathbb{R}, \leq)$ can be thought of as a category with a single morphism $a \to b$ for all $a, b \in \mathbb{R}$ with $a \leq b$.



Def A morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ is an isomorphism if $\exists g \in \mathrm{Hom}_{\mathcal{C}}(Y,X)$ s.t. $g \circ f = \mathrm{Id}_X$ and $f \circ g = \mathrm{Id}_Y$.

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd}$$

Def A <u>product of the family of objects $(X_\alpha)_{\alpha \in A}$</u> in a category is an object P together with morphisms $\pi_\alpha : P \to X_\alpha$ s.t. given any object W and morphisms $f_\alpha : W \to X_\alpha$, $\exists!$ morphism $f : W \to P$ s.t. the following commutes $\forall \alpha$:

$$\begin{tikzcd} & P \arrow[d, "\pi_\alpha"] \\ W \arrow[ur, dashed, "f"] \arrow[r, "f_\alpha"'] & X_\alpha \end{tikzcd}$$

Definition via a "universal property"

Pic Case $A = \{1, \dots, n\}$

$$\begin{tikzcd} & & W \arrow[d, dashed, "f"] \arrow[dll, "f_1"'] \arrow[dl, "f_2"] \arrow[drr, "f_{n-1}"] \arrow[drrr, "f_n"] & & & \\ X_1 & X_2 & P \arrow[ll, "\pi_1"] \arrow[l, "\pi_2"] \arrow[rr, "\pi_{n-1}"] \arrow[rrr, "\pi_n"] & \cdots & X_{n-1} & X_n \end{tikzcd}$$

In the category of sets, take $P = X_1 \times X_2 \times \dots \times X_n$.
Given W and the f_i, define $f : W \to P$ by $f = (f_1, f_2, \dots, f_n)$.

Category	products	coproducts
Sets	Cartesian product	disjoint union
Topological spaces	product topology	disjoint union topology
Groups	direct product	free product
Abelian groups	direct product	direct sum ← only finitely many nonzero coordinates
$(\mathbb{R}, \leq)$	May not exist	May not exist

Rmk Products and coproducts need not exist: In $(\mathbb{R}, \leq)$ you can't take the product or coproduct of $\dots -2, -1, 0, 1, 2, \dots$

Rmk Note that $(\mathbb{R}^{\mathbb{N}}, \text{box})$ is not the product of $\mathbb{N}$ copies of $\mathbb{R}$ in the category of topological spaces. Indeed, take $W = (\mathbb{R}^{\mathbb{N}}, \text{product})$ and note no such continuous f exists since $(0,1)^{\mathbb{N}}$ is open in $(\mathbb{R}^{\mathbb{N}}, \text{box})$ but not $(\mathbb{R}^{\mathbb{N}}, \text{product})$.

9/8/2017

Thm 7.54 If a product exists in a category, then it is unique up to isomorphism.

Pf Suppose $(P', (\pi'_\alpha))$ and $(P'', (\pi''_\alpha))$ are products.
Take $P = P''$ and $W = P'$ to get $f: P' \to P''$.
Take $P = P'$ and $W = P''$ to get $g: P'' \to P'$.
Take $P = W = P'$ to get $g \circ f = Id_{P'}$ via uniqueness.
Take $P = W = P''$ to get $f \circ g = Id_{P''}$ via uniqueness.

Def A coproduct of the family of objects $(X_\alpha)_{\alpha \in A}$ in a category is an object S together with morphisms $\iota_\alpha : X_\alpha \to S$ s.t. given any object W and morphisms $f_\alpha : X_\alpha \to W$, $\exists!$ morphism $f: S \to W$ s.t. the following commutes $\forall \alpha$:

$$\begin{tikzcd}
 & S \arrow[dl, dashed, "f"'] \\
W & X_\alpha \arrow[l, "f_\alpha"] \arrow[u, "\iota_\alpha"']
\end{tikzcd}$$

Pic Case $A = \{1, \dots, n\}$

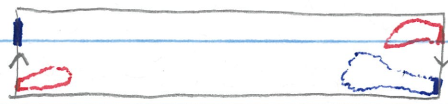
$$\begin{tikzcd}
 & & W & & \\
 & & S \arrow[u, dashed, "f"] & & \\
X_1 \arrow[urr, "\iota_1"] \arrow[uurr, "f_1"] & X_2 \arrow[ur, "\iota_2"] \arrow[uur, "f_2"'] & \cdots & X_{n-1} \arrow[ul, "\iota_{n-1}"'] \arrow[uul, "f_{n-1}"] & X_n \arrow[ull, "\iota_n"'] \arrow[uull, "f_n"']
\end{tikzcd}$$

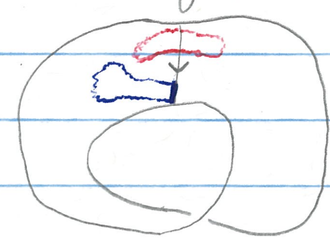
Rmk Products have universal maps to the X_α's.
Coproducts have universal maps from the X_α's.

Quotient Spaces

Def Let X be a topological space, Y be a set, and $q: X \to Y$ be surjective. The <u>quotient topology</u> on Y declares a set $U \subseteq Y$ to be open when $q^{-1}(U)$ is open in X.

Pic





$X = [0,10] \times [0,1]$ $\qquad$ $Y = X/\sim \, =$ Möbius band

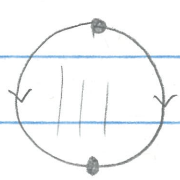
Let $\sim$ be the equivalence relation $(0,t) \sim (10, 1-t)$ and define $q: X \to Y = X/\sim$ by sending a point in X to its equivalence class in $X/\sim$.

Ex 3.47

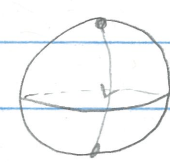
$I = [0,1]$

$S^1 \cong I/\sim$ where $0 \sim 1$

Ex 3.48

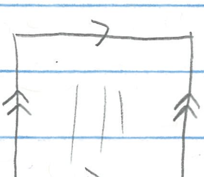


$\overline{B^2}$

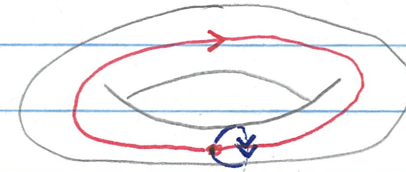


$S^2 \cong \overline{B^2}/\sim$ where $(x,y) \sim (-x,y)$ for all $(x,y) \in \partial \overline{B^2}$

Ex 3.49



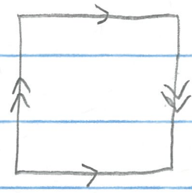
$I \times I$



Torus $\cong (I \times I)/\sim$ where

$(x,0) \sim (x,1)$
$(0,y) \sim (1,y)$

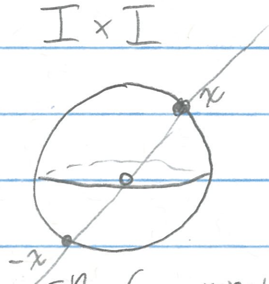
Ex



$I \times I$

Klein bottle $\cong (I \times I)/\sim$ where
$(x,0) \sim (x,1)$
$(0,y) \sim (1,1-y)$

Ex 3.51

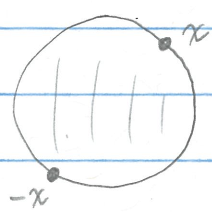


$S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$

me — our book

Projective space $\mathbb{R}P^n$ or $\mathbb{P}^n$ is
$\mathbb{R}P^n \cong S^n/\sim$ where $x \sim -x$

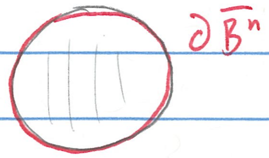
Also $\mathbb{R}P^n \cong (\mathbb{R}^{n+1} \setminus \{\vec{0}\})/\sim$ where $x \sim \lambda x$ for any $\lambda \neq 0$.
Also $\mathbb{R}P^n \cong \overline{B}^n/\sim$ where $x \sim -x$ for $x \in \partial \overline{B}^n$

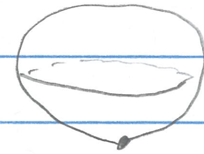


$\overline{B}^n$

9/11/2017 Def For X a topological space and $A \subseteq X$, let X/A denote the quotient topological space $X/\sim$ where $a \sim a'$ for all $a, a' \in A$.

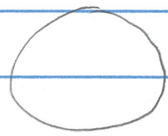
Ex 3.52



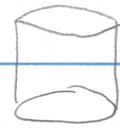


$\overline{B}^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ $\qquad$ $S^n = \overline{B}^n / \partial \overline{B}^n$

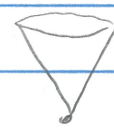
Ex 3.53 For X a topological space, the <u>cone on X</u> is $CX = (X \times I) / (X \times \{0\})$



$X = S^1$



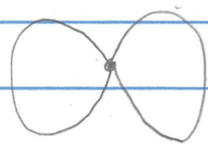
$X \times I$



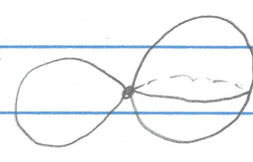
CX

Rmk For any topological space X, CX is "contractible", i.e. "homotopy equivalent" to a point.

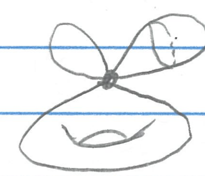
Ex 3.54 Let $X_1, \ldots, X_n$ be nonempty topological spaces and $p_i \in X_i$. The <u>wedge sum</u> is $X_1 \vee \cdots \vee X_n = (X_1 \amalg \cdots \amalg X_n) / \{p_1, \ldots, p_n\}$



$S^1 \vee S^1$



$S^1 \vee S^2$



$S^1 \vee S^2 \vee$ torus

Rmk Wedge sums are coproducts in the category of pointed topological spaces (see Problem 7-17)

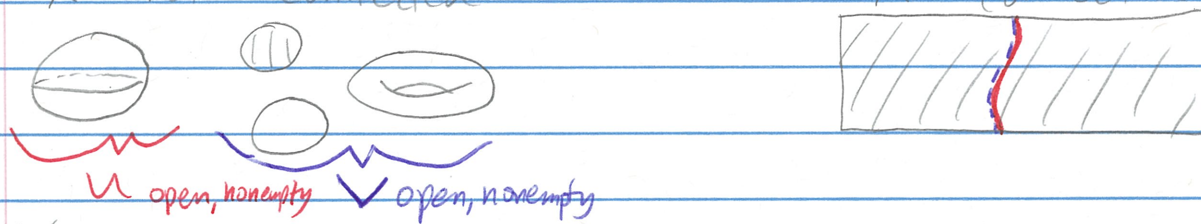
Rmk We're skipping the "Adjunction Spaces" and "Topological Groups and Group Actions" subsections for now.

Chapter 4: Connectedness and Compactness

Connectedness

Def A topological space X is <u>connected</u> if it cannot be written as the disjoint union of two nonempty open sets.

Pic X not connected $\qquad\qquad$ X connected



$X = U \amalg V$

Prop 4.1 Equivalently, X is connected $\Longleftrightarrow$ the only subsets of X that are both open and closed are $\emptyset$ and X.

Prop 4.2 Suppose $X \neq \emptyset$ is connected and Y is discrete. Then any continuous $f: X \to Y$ is a constant map.

Pf Let $x \in X$ and $y = f(x)$. Since $\{y\}$ is open and closed in Y, it follows that $f^{-1}(y) \neq \emptyset$ is open and closed in X. X connected $\Rightarrow f^{-1}(y) = X$, so f is constant.

Thm 4.7 Let $f: X \to Y$ be continuous with X connected. Then $f(X)$ is connected (or equivalently, if f is surjective then Y is connected).

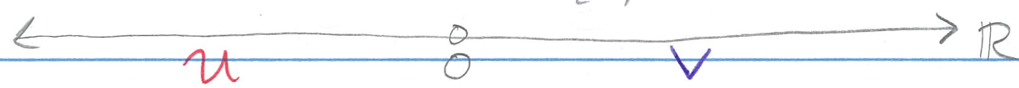
Pf We'll prove the formulation where f is surjective. If Y were not connected, we could write $Y = U \amalg V$ with U, V open and nonempty, hence $X = f^{-1}(U) \amalg f^{-1}(V)$ is not connected. $\boxed{\text{Draw a picture!}}$

Corollary 4.8 (Invariance of connectedness) Every space homeomorphic to a connected space is connected.

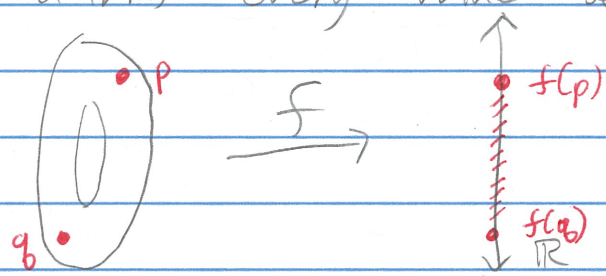
Pf $X \underset{g}{\overset{f}{\rightleftarrows}} Y$

Prop 4.11 A nonempty subset of $\mathbb{R}$ is connected $\iff$ it is a singleton or an interval.

Rmk Proof omitted. Note $\mathbb{R} \setminus \{0\}$ is not connected.



Thm 4.12 (Intermediate Value Theorem) Let X be connected and $f: X \to Y$ be continuous. If $p, q \in X$, then f attains every value between $f(p)$ and $f(q)$.

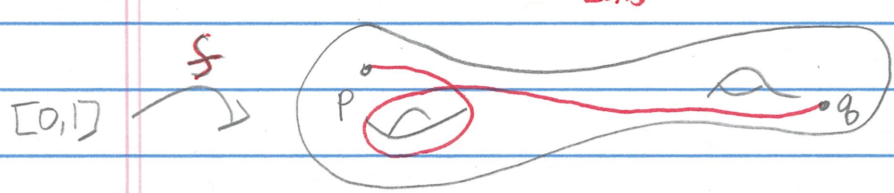


Pf By Thm 4.7 $f(X)$ is connected, then use Prop 4.11.

<u>Path Connectedness</u>

Rmk This is a different notion of connectedness
X path connected $\Longrightarrow$ X connected, but not vice versa.
For "nice" X, the two notions coincide.

Def A <u>path in space X from $p \in X$ to $q \in X$</u> is a continuous map $f: \underset{[0,1]}{\overset{\shortparallel}{I}} \to X$ with $f(0) = p$, $f(1) = q$.

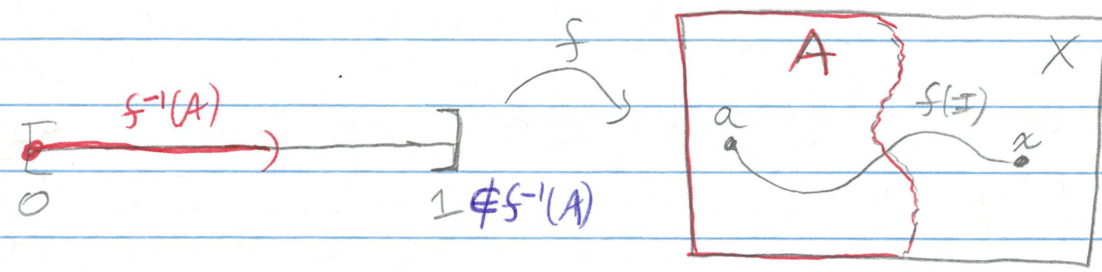


Def A space X is <u>path-connected</u> if $\forall p, q \in X$, $\exists$ a path in X from p to q.

Thm 4.15 X path-connected $\Rightarrow$ X connected

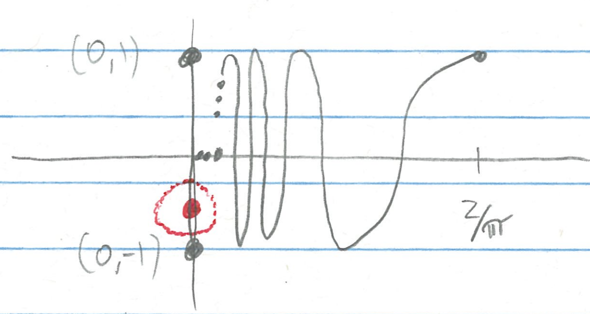
Pf (Different proof than book.) Let $A \subseteq X$ be nonempty, open, and closed. Suppose for a contradiction $A \neq X$. Choose $a \in A$, $x \in X \setminus A$ and let $f: I \to X$ be a path with $f(0)=a$, $f(1)=x$. Then $f^{-1}(A)$ is a nonempty proper subset of I which is both open and closed, contradicting the fact that I is connected. Hence $A = X$ and X is connected.

Pic

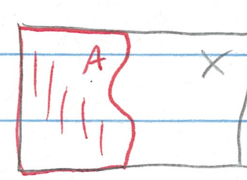


9/13/2017

Ex 4.17 The topologist's sine curve is
$$\{(0,y) \mid -1 \le y \le 1\} \cup \{(x, \sin(1/x)) \mid 0 < x \le 2/\pi\} \subseteq \mathbb{R}^2.$$
It is connected but not path-connected



Rmk A <u>connected component</u> of space X is a maximal nonempty connected subset of X. Similarly for a <u>path-connected component</u>.

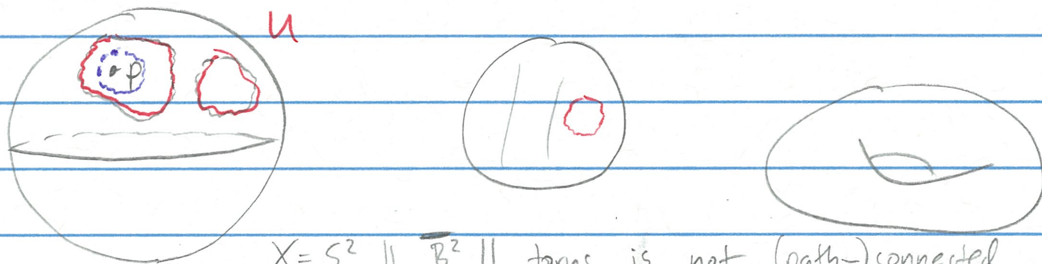


A is connected but not a connected component.

Rmk A space X is <u>locally connected</u> if $\forall p \in X$ and neighborhoods $\mathcal{U}$ of p, there is a connected neighborhood of p contained in $\mathcal{U}$.

Rmk Similar definition for <u>locally path-connected</u>

Pic



$X = S^2 \amalg \overline{B}^2 \amalg$ torus is not (path-)connected but is locally (path-) connected.

Rmk Prop 4.26 (e) shows for X locally path-connected, X connected $\iff$ X path-connected.

Compactness

Analogy A topological space being compact is analogous to a set being finite.

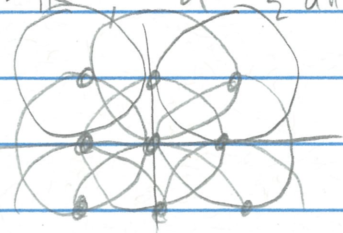
> <u>More generally</u>: For $A \subseteq X$ and $\mathcal{U}$ a collection of sets in X, we say $\mathcal{U}$ covers A when $A \subseteq \bigcup_{u \in \mathcal{U}} u$.

Let X be a topological space.

Def An <u>open cover of X</u> is a collection $\mathcal{U}$ of open sets in X whose union is X, i.e. $\bigcup_{u \in \mathcal{U}} u = X$

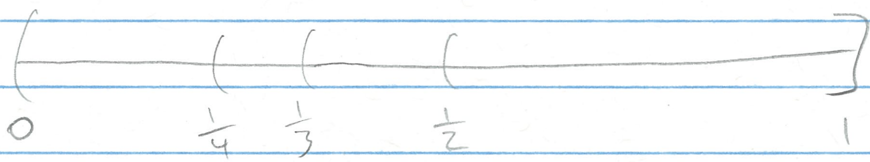
Def Space X is compact if every open cover $\mathcal{U}$ of X has a finite subcover $\mathcal{U}' \subseteq \mathcal{U}$, i.e. if there is a finite collection $u_1, \ldots, u_k \in \mathcal{U}$ s.t. $u_1 \cup \ldots \cup u_k = X$.

Ex $X = \mathbb{R}^2$, $\mathcal{U} = \{$all open balls with integral centers and radius $1\}$.



Note this $\mathcal{U}$ shows $\mathbb{R}^2$ is not compact!

Ex $X = (0,1]$, $\mathcal{U} = \{(\frac{1}{n}, 1] \mid n = 2,3,4,\dots\}$



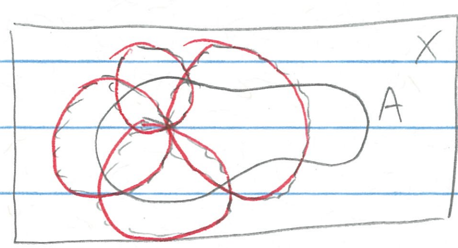
Note $\mathcal{U}$ is an open cover with no finite subcover, so $(0,1]$ is not compact.

Ex 4.30 (a) Every finite space is compact (regardless of its topology)
(b) Every space with the trivial topology is compact.
(c) A discrete space is compact $\iff$ its finite.

Rmk We'll see a subset $A \subseteq \mathbb{R}^n$ is compact $\iff$ its closed and bounded.

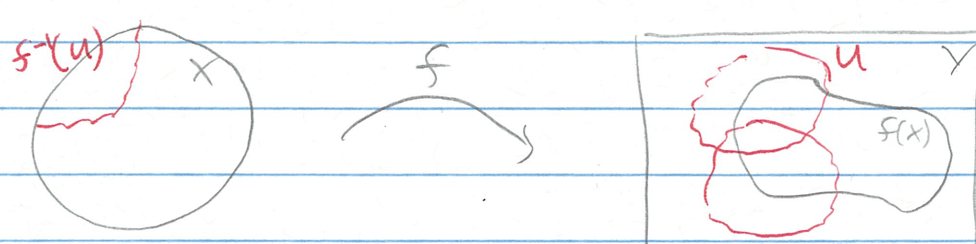
(A has the subspace topology)

Lemma 4.27 A subset $A \subseteq X$ is compact $\iff$ every cover of A by open sets in X has a finite subcover



Open sets in X that cover A (but not X)

Thm 4.32 X compact and $f: X \to Y$ continuous $\Rightarrow$ $f(X)$ compact.



Pf Let $\mathcal{U}$ be a cover of $f(X)$ by open subsets of Y.
Note $\{f^{-1}(U) \mid U \in \mathcal{U}\}$ is an open cover of X.
X compact $\Rightarrow \exists\ U_1, \dots, U_k$ s.t. $\{f^{-1}(U_i) \mid 1 \le i \le k\}$ covers X.
Hence $\{U_1, \dots, U_k\}$ covers $f(X)$.

Corollary 4.33 Every space homeomorphic to a compact space is compact

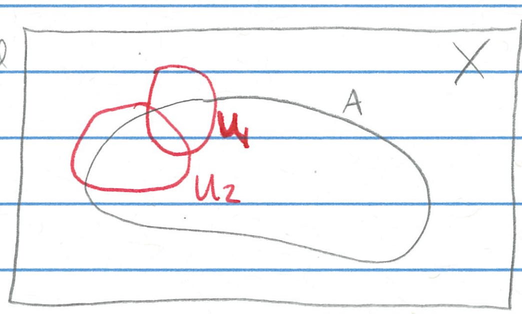
Pf

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

Prop 4.36 (a) Every closed subset of a compact space is compact.
(b) Every compact subset of a Hausdorff space is closed.
(c) Every compact subset of a metric space is bounded.
(d) A finite product of compact spaces is compact.
(e) A quotient of a compact space is compact.

Pf sketch

(a) A closed in X

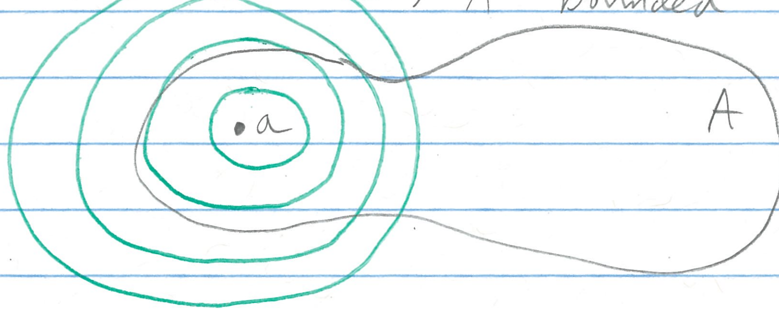


Let $\mathcal{U}$ be a cover of A by open sets in X.
Note $\mathcal{U} \cup \{X \setminus A\}$ is an open cover of X.
X compact $\Rightarrow$ there's a finite subcover
$\{U_1, \dots, U_k, X \setminus A\}$ or $\{U_1, \dots, U_k\} \subseteq \mathcal{U} \cup \{X \setminus A\}$,
giving a finite subcover $\{U_1, \dots, U_k\} \subseteq \mathcal{U}$ of A.

9/15/2017 (b) HW?

(c) Let A be a compact subset of a metric space. Pick $a \in A$, and consider the open cover
$\{$open ball $B(a,n)$ with center a and radius $n \in \mathbb{N}\}$.
Could be open balls in X or A by Lemma 4.27

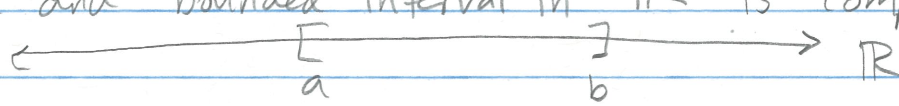
$\exists$ of finite subcover $\Rightarrow$ A contained in single ball
$\Rightarrow$ A bounded



(e) If Y is a quotient space of X, then we have $q: X \to Y$ with q surjective and continuous. By Thm 4.32, X compact $\Rightarrow$ $q(X) = Y$ compact.

Thm 4.39 A closed and bounded interval in $\mathbb{R}$ is compact.

Pf Sketch



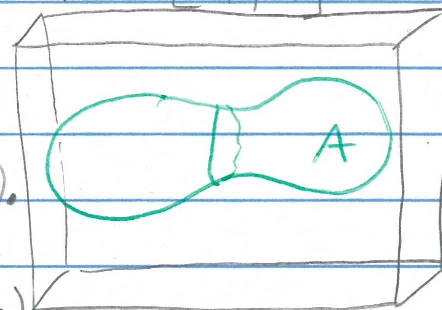
Let $\mathcal{U}$ be an open cover of $[a,b]$.
Consider $X = \{x \in (a,b] \mid [a,x]$ is covered by finitely many sets of $\mathcal{U}\}$.
Show $b \in X$ "by hand".

Thm 4.40 $A \subseteq \mathbb{R}^n$ compact $\Leftrightarrow$ its closed and bounded.

Pf Sketch ($\Rightarrow$) follows from Prop 4.36 (b), (c).

For ($\Leftarrow$), note A bounded means $A \subseteq [-R,R]^n$ for some $R > 0$.

Note $[-R,R]^n$ is compact by Thm 4.39 and Prop 4.36 (d).

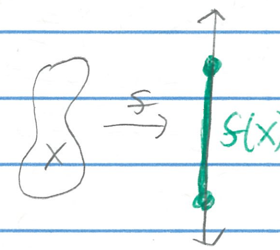


Hence A is compact by Prop 4.36(a).

Thm 4.41 X compact and $f: X \to \mathbb{R}$ continuous

$\Rightarrow$ f is bounded and attains its maximum and minimum values.

Pf By Theorems 4.32 and 4.40, $f(X)$ is closed and bounded.



Rmk Sequential compactness (every sequence has a convergent subsequence) paracompactness are related notions. Paracompact Hausdorff spaces admit partitions of unity.

Rmk Skipping Chapter 5 (Cell Complexes) and 6 (Compact Surfaces) for now.

Chapter 7 Homotopy and the Fundamental Group

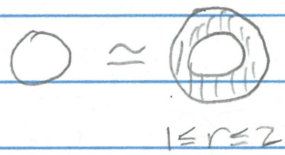
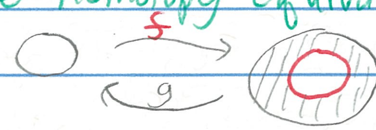
In topology, by "map" we mean "continuous map".

Recall Spaces X, Y are homeomorphic ($X \cong Y$) if $\exists$
$X \underset{f^{-1}}{\overset{f}{\rightleftarrows}} Y$ (with f, f^{-1} continuous).

Note $f^{-1} \circ f = id_X$ and $f \circ f^{-1} = id_Y$.

Def Spaces X, Y are homotopy equivalent ($X \simeq Y$) if $\exists$
$X \underset{g}{\overset{f}{\rightleftarrows}} Y$ s.t. $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$.

Need to define homotopy equivalences b/w maps

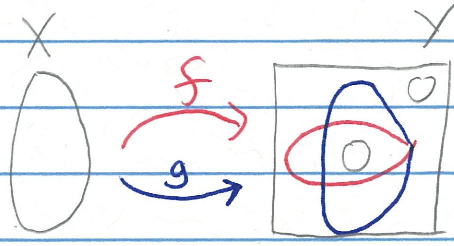
Ex  since $\exists$  defined via

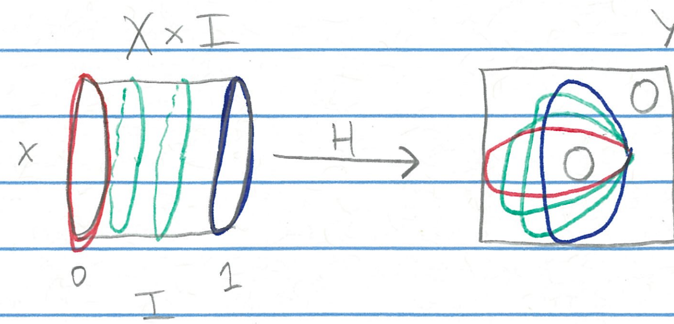
$f(e^{i\theta}) = e^{i\theta}$ and $g(re^{i\theta}) = e^{i\theta}$.

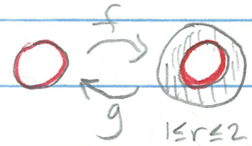
Note $g \circ f = id_{circle}$

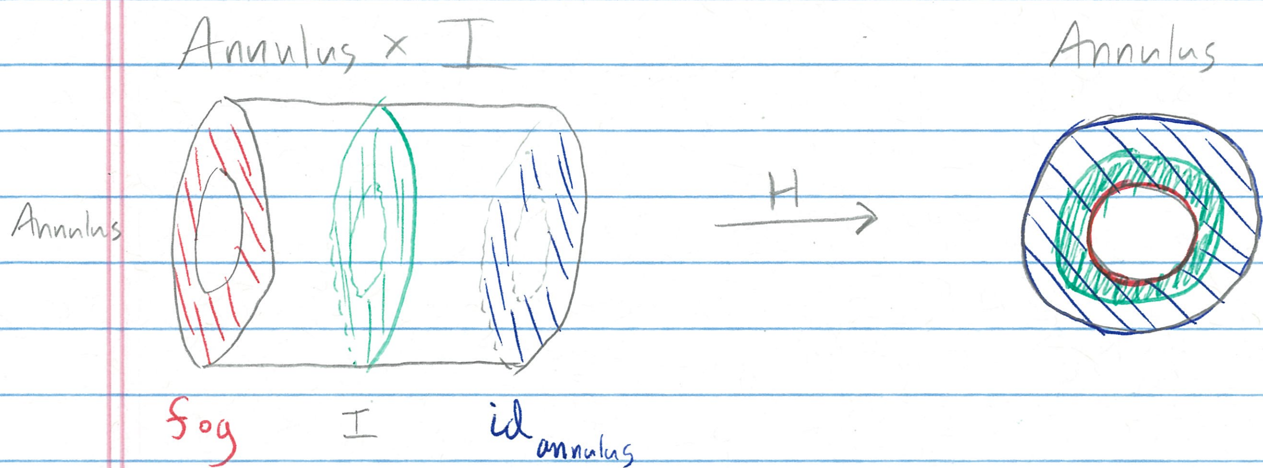
We'll see $f \circ g \simeq id_{annulus}$.

Def Let $f, g : X \to Y$. A homotopy from f to g is a continuous $H : X \times I \to Y$ with
$H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$.
If such an H exists, we say f and g are homotopic ($f \simeq g$).

Ex 



Ex In  above, show $f \circ g \simeq id_{annulus}$.



$$H(re^{i\theta}, t) = (1-t)e^{i\theta} + tre^{i\theta}$$

[H is an example of a linear homotopy since
$H(x,t) = (1-t)\, f\circ g(x) + t\, id_{annulus}(x)$.]

Rmk A homotopy gives a 1-parameter family of continuous maps $H_t : X \to Y$ defined by $H_t(x) = H(x,t)$.

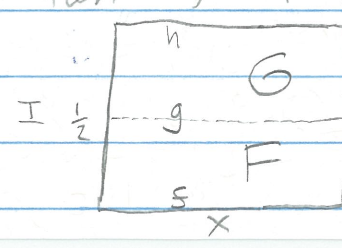
Prop 7.1 Homotopy is an equivalence relation on the set of all continuous maps from X to Y.

Pf Sketch $f \simeq f$ via $H(x,t) = f(x)$

If $f \simeq g$ via $H(x,t)$, then $g \simeq f$ via $H(x, 1-t)$

If $f \simeq g$ via F and $g \simeq h$ via G, then $f \simeq h$ via

$$H(x,t) = \begin{cases} F(x,2t) & 0 \le t \le \frac{1}{2} \\ G(x,2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$



$$\begin{tikzcd} X \arrow[r, bend left, "h"] \arrow[r, "g"] \arrow[r, bend right, "f"'] & Y \end{tikzcd}$$

Prop 7.2 The homotopy relation is preserved by composition.

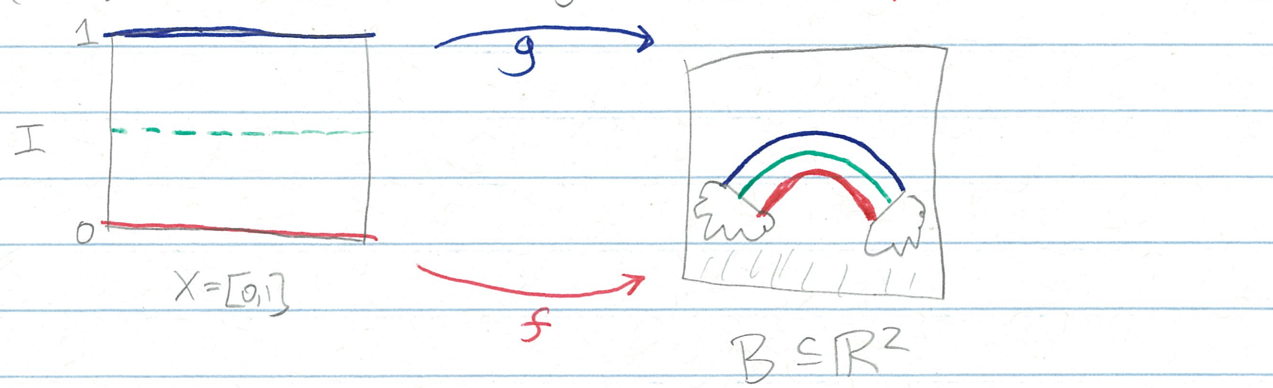
Pf

$$\begin{tikzcd} X \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_0"'] \arrow[r, phantom, "F"] & Y \arrow[r, bend left, "g_1"] \arrow[r, bend right, "g_0"'] \arrow[r, phantom, "G"] & Z \end{tikzcd}$$

If $f_0 \simeq f_1$ via F
and $g_0 \simeq g_1$ via G,
then $g_0 \circ f_0 \simeq g_1 \circ f_1$ via $H: X \times I \to Z$
where $H(x,t) = G(F(x,t),t)$.
Note $H(x,0) = G(f_0(x),0) = g_0(f_0(x))$
and $H(x,1) = G(f_1(x),1) = g_1(f_1(x))$.

Rmk Our book writes $F: f_0 \simeq f_1$ for $f_0 \simeq f_1$ via F.
I prefer the latter or $f_0 \simeq_F f_1$, but whatever is fine.

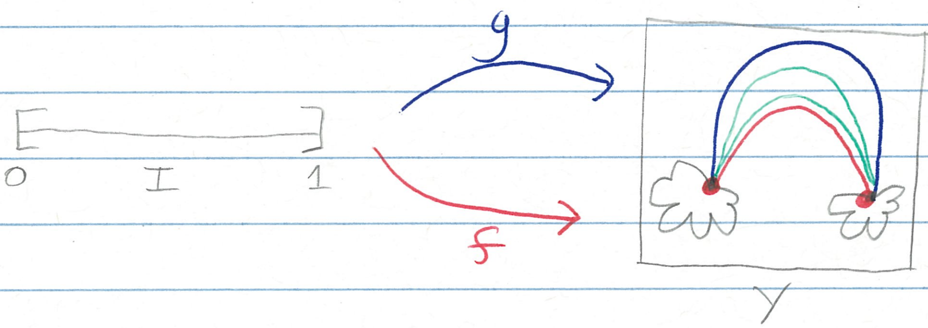
Ex 7.4 If $f,g: X \to B \subseteq \mathbb{R}^n$ and the straight line between $f(x)$ and $g(x)$ is in B $\forall x \in X$, then the <u>linear homotopy</u> $f \simeq g$ via H is defined by
$H(x,t) = (1-t)f(x) + tg(x)$. [Always possible if B convex]



Def For $f, g: X \to Y$ and $A \subseteq X$, we say f and g are <u>homotopic relative A</u> (or rel A) if $H(x,t) = f(x) = g(x)$ $\forall x \in A, t \in I$.

Def Paths $f, g: I \to Y$ are <u>path homotopic</u> $(f \sim g)$ if they are homotopic relative $\{0,1\} \subseteq I$.

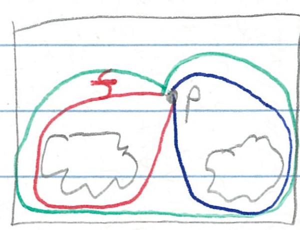
Pic



Rmk Homotopy equivalence rel A is still an equivalence relation.

Def For X a space and $p \in X$, the <u>fundamental group</u> $\pi_1(X,p)$ is the group of all path homotopy classes of loops based at p.

For a loop $f: I \to X$ with $f(0) = p = f(1)$, we write $[f] \in \pi_1(X,p)$.



9/20/2017

Rmk Often denoted $\pi_1(X)$ since X path-connected $\implies \pi_1(X,p) \cong \pi_1(X,q)$ for all $p, q \in X$.
$\uparrow$ isomorphism of groups (Thm 7.13)

Rmk The $\pi_i(X)$ for $i \geq 1$ will be the <u>homotopy groups</u>.

Rmk π_i is a "functor" from the category of spaces to groups.

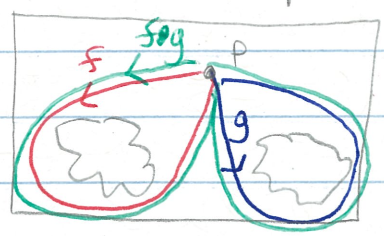
Rmk $X \simeq Y \implies \pi_i(X) \cong \pi_i(Y)$ for all i.

Group operation on $\pi_1(X,p)$

For $f,g : I \to X$ loops based at p, their product $f \cdot g : I \to X$ is

$$f \cdot g(s) = \begin{cases} f(2s) & 0 \leq s \leq \frac{1}{2} \\ g(2s-1) & \frac{1}{2} \leq s \leq 1 \end{cases}$$

Note $f \cdot g$ is continuous since $f(1) = g(0) = p$, and is a loop based at p since $f(0) = g(1) = p$.

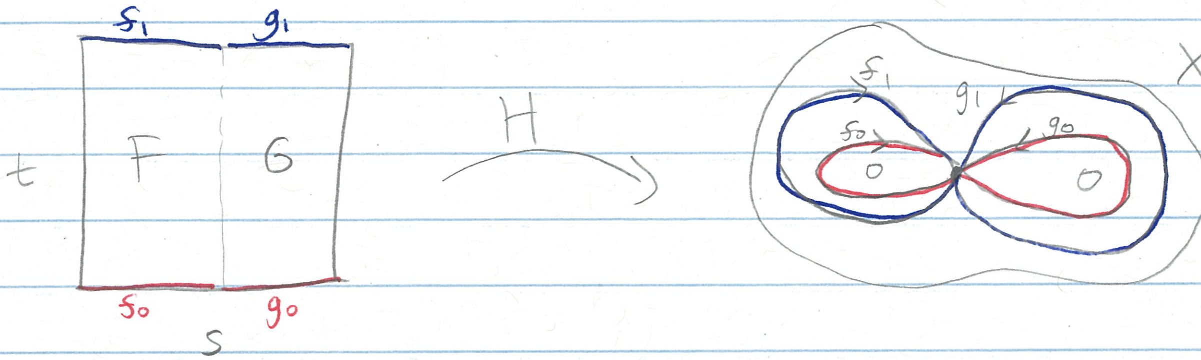


For $[f], [g] \in \pi_1(X,p)$, we define their product by $[f] \cdot [g] = [f \cdot g]$.

To see $[f \cdot g]$ is well-defined, note...

Prop 7.10 If $f_0, f_1, g_0, g_1 : I \to X$ are loops based at p with $f_0 \sim f_1$ and $g_0 \sim g_1$, then $f_0 \cdot g_0 \sim f_1 \cdot g_1$

Pf



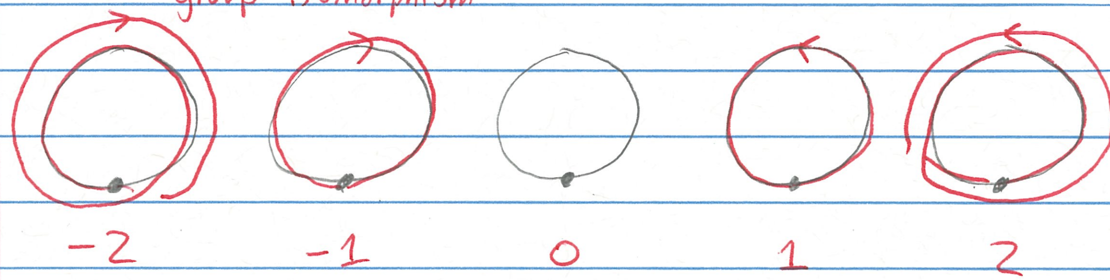
If $f_0 \sim f_1$ via F and $g_0 \circ g_1$ via H, then $f_0 \cdot g_0 \sim f_1 \cdot g_1$ via

$$H(s,t) = \begin{cases} F(2s,t) & 0 \leq s \leq \frac{1}{2} \quad 0 \leq t \leq 1 \\ G(2s-1,t) & \frac{1}{2} \leq s \leq 1 \quad 0 \leq t \leq 1 \end{cases}$$ ●

$$\left(\begin{array}{l} \text{Indeed, note } H(s,0) = \begin{cases} f_0(2s) & 0 \le s \le \frac{1}{2} \\ g_0(2s-1) & 0 \le s < \frac{1}{2} \end{cases} = (f_0 \bullet g_0)(s), \\ \text{and similarly } H(s,1) = (f_1 \bullet g_1)(s) \end{array} \right)$$

Ex $\pi_1(S^1) \underset{\uparrow}{\cong} \mathbb{Z} \cong \langle a \rangle$, the free group on one generator.

group isomorphism



1/22/2017 <u>Identity and inverses in $\pi_1(X,p)$</u>

Let $c_p : I \to X$ be the constant loop defined by $c_p(s) = p \quad \forall s \in I$.

Given $f : I \to X$, let $\bar{f} : I \to X$ be defined by $\bar{f}(s) = f(1-s)$.

<u>Thm 7.11</u> For $f, g, h : I \to X$ loops in X based at p,

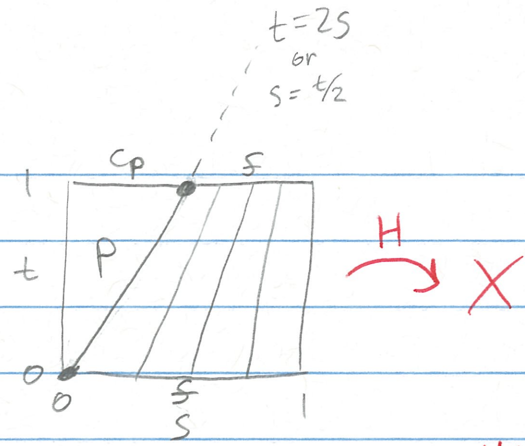
(a) $[c_p] \cdot [f] = [f] = [f] \cdot [c_p]$ and hence $[c_p]$ is the identity in $\pi_1(X,p)$.

(b) $[f] \cdot [\bar{f}] = [c_p] = [\bar{f}] \cdot [f]$, and hence the inverse of $[f]$ in $\pi_1(X,p)$ is $[\bar{f}]$.

(c) $[f] \cdot ([g] \cdot [h]) = ([f] \cdot [g]) \cdot [h]$, and hence the multiplication in $\pi_1(X,p)$ is associative.

<u>Corollary 7.12</u> Hence $\pi_1(X,p)$ is a group.

Pf of Thm 7.11 (a)



Note $H: I \times I \to X$ defined by

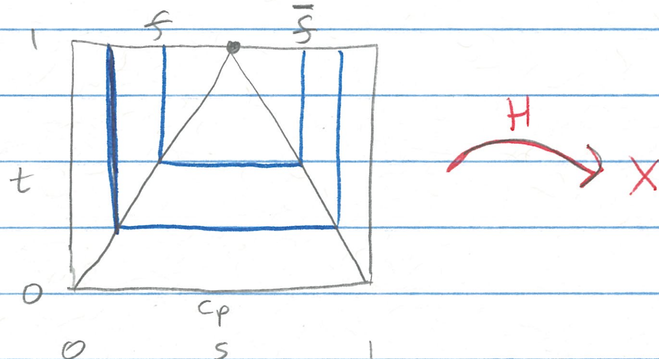
$$H(s,t) = \begin{cases} f\left(\frac{2s-t}{2-t}\right) & t \le 2s \\ p & t \ge 2s \end{cases}$$

is a homotopy from $H(s,0) = f(s)$

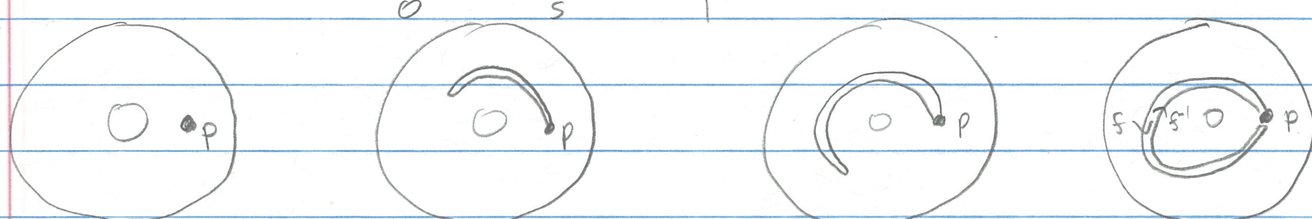
to $H(s,1) = (c_p \cdot f)(s)$

Note as s varies from $t/2$ to 1, $\left(\frac{2s-t}{2-t}\right)$ varies from 0 to 1.

(b)



The blue curves are "iso curves" of H.



$H(s,0) = c_p(s)$ $\quad$ $H(s,\frac{1}{3})$ $\quad$ $H(s,\frac{2}{3})$ $\quad$ $H(s,1) = (f \cdot \bar{f})(s)$

Note $H: I \times I \to X$ defined by

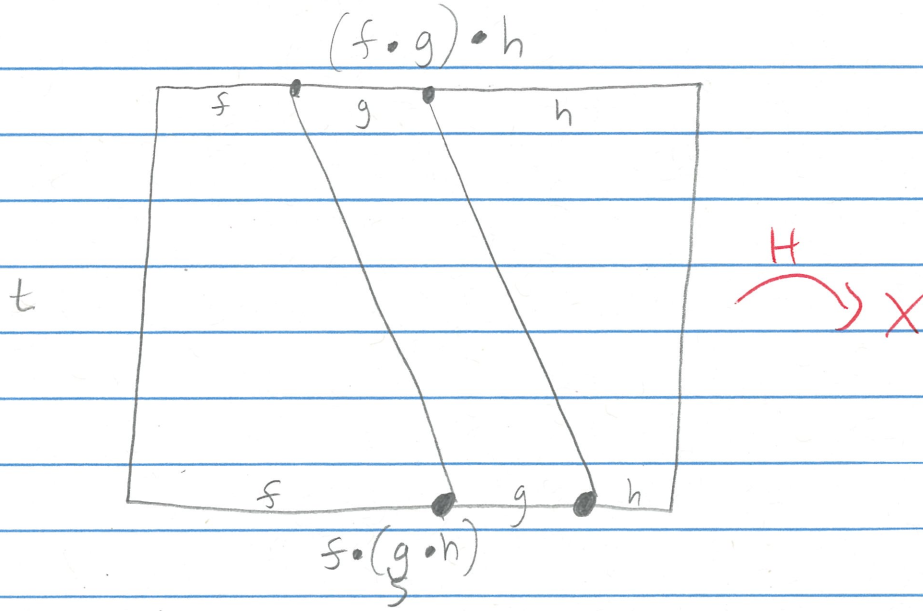
$$H(s,t) = \begin{cases} f(2s) & 0 \le s \le t/2 \\ f(t) & t/2 \le s \le 1 - t/2 \\ f(2-2s) & 1 - t/2 \le s \le 1 \end{cases}$$

Note $f(2-2s) = f(1-(2s-1)) = \bar{f}(2s-1)$.

is a homotopy from $H(s,0) = c_p(s)$

to $H(s,1) = f \cdot \bar{f}(s)$

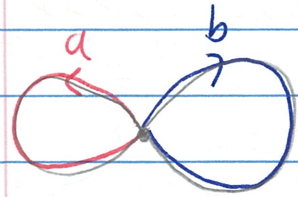
(c)



Writing H down explicitly is possible but not very instructive.

More Examples

- $\pi_1(S^1 \vee S^1) \cong \langle a,b \rangle$



See Chapter 9

Here $\langle a,b \rangle$ is the free group on two generators, with elements the finite strings in a, b, a^{-1}, b^{-1}, modulo "combining like terms":

$$abba^{-1}a^{-1}a^{-1}a = ab^2a^{-3}a = ab^2a^{-2}.$$

Multiplication is defined via

$$(ab^2a^{-2})(a^2b^{-2}ab^5a^{-1}) = ab^2a^{-2}a^2b^{-2}ab^5a^{-1}$$
$$= ab^2b^{-2}ab^5a^{-1}$$
$$= a^2b^5a^{-1}.$$

Note this group is not commutative:

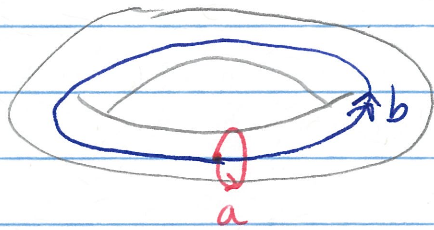
$$(a^2b^{-2}ab^5a^{-1})(ab^2a^{-2}) = ab^{-2}ab^5a^{-1}ab^2a^{-2}$$
$$= ab^{-2}ab^7a^{-2}$$

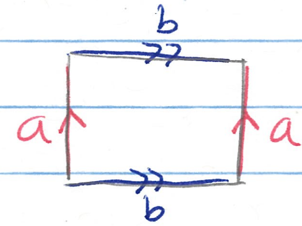
which is different.

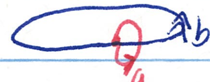
The identity element is the empty word!

2 generators — 1 relation

- $\pi_1(S^1 \times S^1) \cong \langle a, b \mid aba^{-1}b^{-1} = 1 \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$





This space can be formed by starting with $S^1 \vee S^1 =$  and then gluing a disk in along it's boundary via $aba^{-1}b^{-1}$, giving the relation $aba^{-1}b^{-1} = 1$.

This group is commutative:

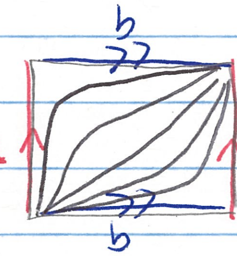
$aba^{-1}b^{-1} = 1$

$\Rightarrow aba^{-1} = b^{-1}$

$\Rightarrow ab = ba$

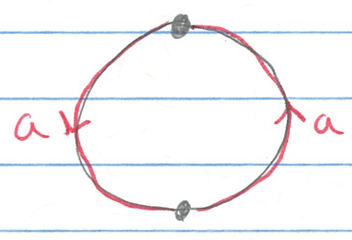
Algebra!

Topology!

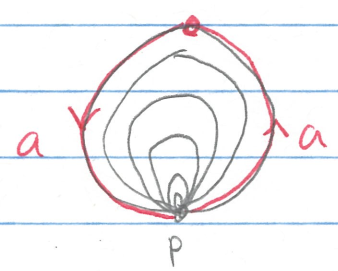


(Homotopy from loop $a \cdot b$ to $b \cdot a$.)

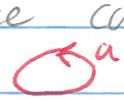
- $\pi_1(\mathbb{R}P^2) \cong \langle a \mid a^2 = 1 \rangle \cong \mathbb{Z}/2\mathbb{Z}$

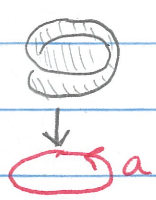


$\mathbb{R}P^2$



(Homotopy from loop a^2 to the constant loop at p.)

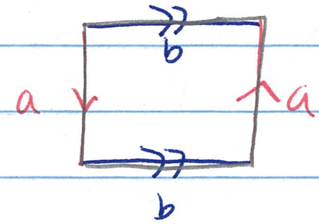
This space can be formed by starting with $S^1 =$  and then gluing a disk in along it's boundary via a^2 (wrapping around twice!)



Disk (that self-intersects in $\mathbb{R}^3$)

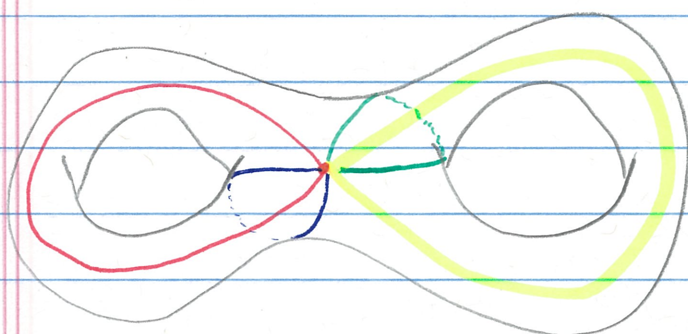
9/25/2017

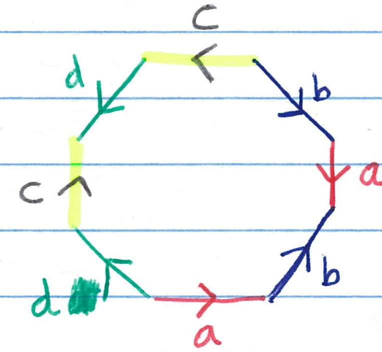
- $\pi_1(\text{Klein bottle}) \cong \langle a,b \mid abab^{-1}=1\rangle$



People stop writing "$=1$".

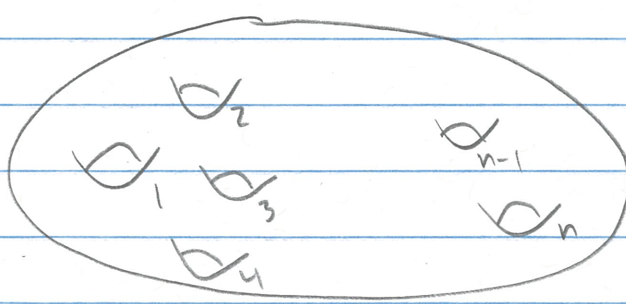
- $\pi_1(\text{Genus 2 torus}) \cong \langle a,b,c,d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1}\rangle$





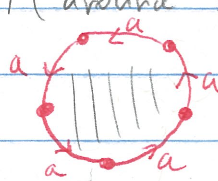
See the YouTube video "From an octagon to a genus 2 surface – Mathlapse":
www.youtube.com/watch?v=G1yyfPShgqw

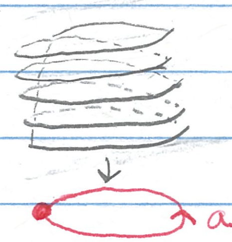
- $\pi_1(\text{Genus } n \text{ torus}) \cong \langle a_1,b_1,a_2,b_2,\dots,a_n,b_n \mid a_1b_1a_1^{-1}b_1^{-1}\cdots a_nb_na_n^{-1}b_n^{-1}\rangle$.



- $\pi_1\left(\begin{matrix}\text{circle with disk wrapped}\\ \text{around } n \text{ times}\end{matrix}\right) \cong \langle a \mid a^n\rangle \cong \mathbb{Z}/n\mathbb{Z}$.

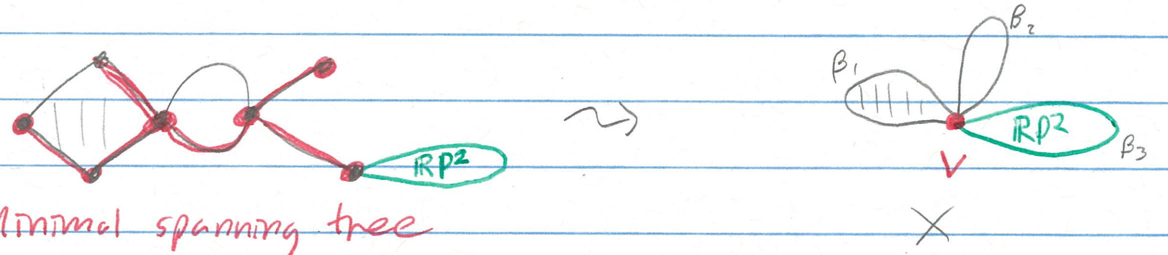
$n=5$





Disk (that self-intersects in $\mathbb{R}^3$)

More generally, let X be a finite connected cell complex with a single vertex v. This is not a big restriction:



Theorem 10.15 will say

$$\pi_1(X,v) \cong \langle \underset{\substack{\uparrow\\ \text{generator for each 1-cell}}}{\beta_1, \dots, \beta_n} \mid \underset{\substack{\uparrow\\ \text{relation for each 2-cell}}}{r_1, \dots, r_k} \rangle$$

For X drawn above we have

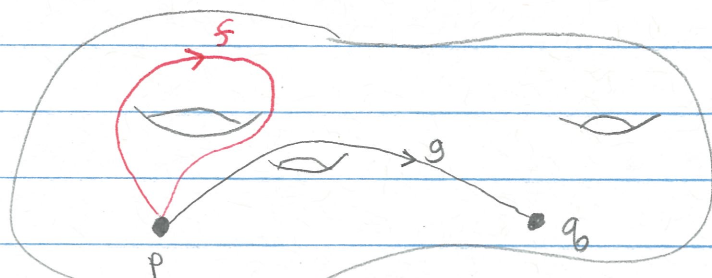
$$\begin{aligned}\pi_1(X,v) &\cong \langle \beta_1, \beta_2, \beta_3 \mid \beta_1, \beta_3^2 \rangle \\ &\cong \langle \beta_2, \beta_3 \mid \beta_3^2 \rangle \\ &\cong \mathbb{Z} * \mathbb{Z}/2\mathbb{Z}\end{aligned}$$

(free product of groups, which we'll see is the coproduct in the category of groups.)

10/2/2017

Our justification for writing $\pi_1(X)$ instead of $\pi_1(X,p)$ is:

Thm 7.13 Suppose X is a path-connected space with $p,q \in X$ and g a path from p to q. Then $\phi_g : \pi_1(X,p) \to \pi_1(X,q)$ defined by $\phi_g[f] = [\bar{g}]\cdot[f]\cdot[g]$ is a group isomorphism, with inverse $\phi_{\bar{g}}$.

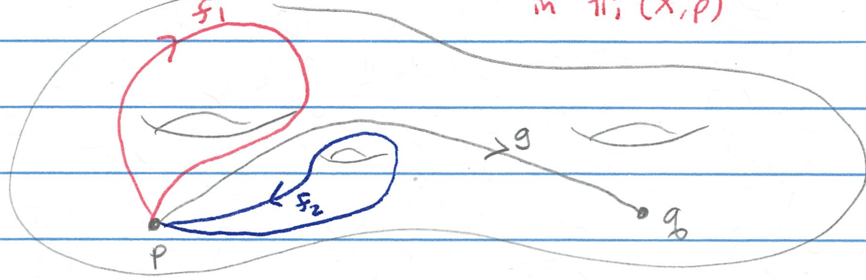


To see that ϕ_g is a group homomorphism, note

$$\phi_g[f_1] \cdot \phi_g[f_2] = [\bar{g}][f_1][g][\bar{g}][f_2][g]$$
$$= [\bar{g}][f_1][c_p][f_2][g]$$
$$= [\bar{g}][f_1][f_2][g]$$
$$= \phi_g([f_1] \cdot [f_2])$$

multiplication in $\pi_1(X,q)$

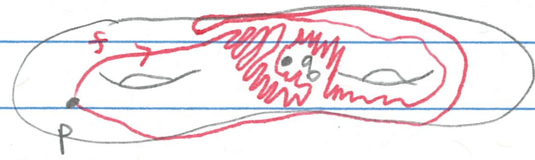
multiplication in $\pi_1(X,p)$



Fundamental groups of spheres

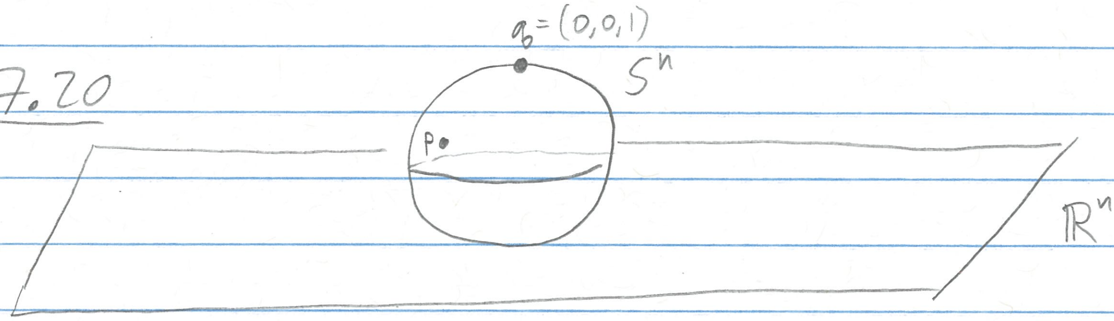
Thm 7.20 For $n \geq 2$, the n-dimensional sphere S^n is "simply connected" (S^n is path-connected and $\pi_1(S^n)$ is the trivial group).

Lemma 7.19 If M is a manifold of dimension $n \geq 2$, f is a loop in M based at p, and $q \neq p$, then f is homotopic to a loop not passing through q.



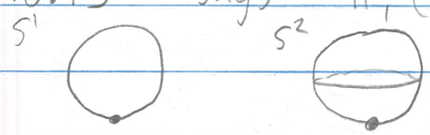
Pf Analysis. Not obvious due to space-filling curves (there exist surjective continuous maps $I \to I^n$ $\forall n \geq 1$).

Pf sketch of Thm 7.20



Let f be a loop in S^n based at $p \neq (0,0,1)$.
By Lemma 7.19, we can assume f doesn't pass through $(0,0,1)$.
Stereographic projection gives a homeomorphism $S^n \setminus \{(0,0,1)\} \longrightarrow \mathbb{R}^n$; see HW2 #3.
In $\mathbb{R}^n$ any loop is homotopic to the constant loop, (via a straight line homotopy)
and hence f is homotopic to the constant loop in $S^n \setminus \{(0,0,1)\}$.
This shows $\pi_1(S^n)$ is the trivial group.

Rmk Since S^n is a cell complex (CW complex) with one 0-cell and one n-cell, Thm 10.15 says $\pi_1(S^n) \cong$ trivial group for $n \geq 2$



Thm 7.21 The fundamental group of a manifold is countable.

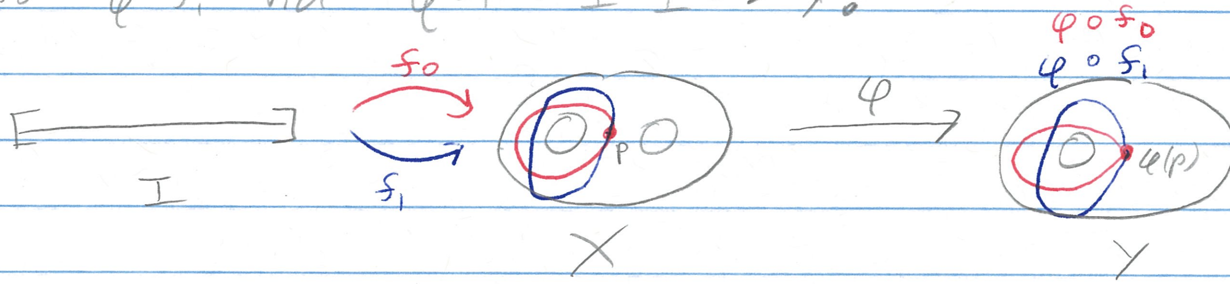
Pf Analysis, de-emphasized.

Homomorphisms induced by continuous maps

Prop 7.22 If $f_0, f_1 : I \to X$ are path-homotopic and $\varphi : X \to Y$ is continuous, then $\varphi \circ f_0, \varphi \circ f_1 : I \to Y$ are path-homotopic.

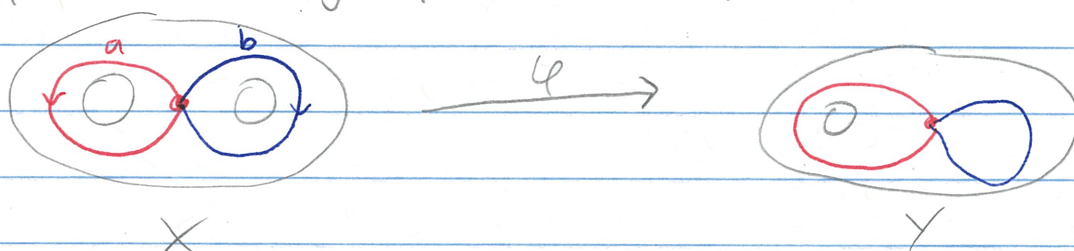
Pf Sketch $f_0 \sim f_1$ via $F : I \times I \to X \Rightarrow$
$\varphi \circ f_0 \sim \varphi \circ f_1$ via $\varphi \circ F : I \times I \to Y$.

Pic



Def Given $\varphi : X \to Y$, let $\varphi_* : \pi_1(X, p) \to \pi_1(Y, \varphi(p))$ be defined by $\varphi_*[f] = [\varphi \circ f]$.
This is well-defined by Prop 7.22.

Prop 7.24 φ_* is a group homomorphism



$\pi_1(X) \cong \langle a, b \rangle$ $\qquad$ $\pi_1(Y) \cong \langle c \rangle \cong \mathbb{Z}$

Here $\varphi_* : \pi_1(X) \to \pi_1(Y)$ via
$\varphi_*(a) = c$ (or 1), $\varphi_*(b) = \{\}$ (or 0)

Pf
$$\begin{aligned}
\varphi_*([f] \cdot [g]) = \varphi_*([f \cdot g]) &= [\varphi \circ (f \cdot g)] \\
&= [(\varphi \circ f) \cdot (\varphi \circ g)] \\
&= [\varphi \circ f] \cdot [\varphi \circ g] \\
&= \varphi_*[f] \cdot \varphi_*[g].
\end{aligned}$$

(Indeed, note we have $\varphi \circ (f \cdot g) = (\varphi \circ f) \cdot (\varphi \circ g)$ on the nose.)

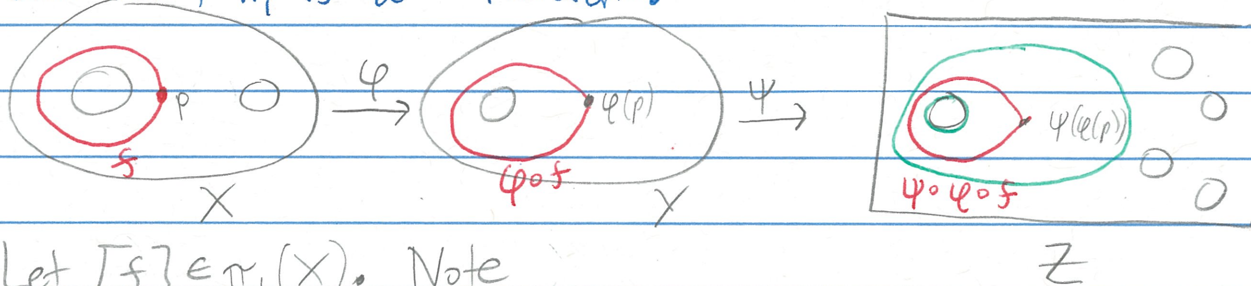
10/4/2017

Prop 7.25 The homomorphism $\varphi_*: \pi_1(X,p) \to \pi_1(Y, \varphi(p))$ induced by $\varphi: X \to Y$ satisfies:

(a) If $\varphi: X \to Y$ and $\psi: Y \to Z$, then $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$.

(b) If $Id_X: X \to X$ denotes the identity map of X, then $(Id_X)_* = Id_{\pi_1(X)}$.

Rmk In other words, π_1 is a "functor".



Pf Let $[f] \in \pi_1(X)$. Note

$\psi_*(\varphi_*[f]) = \psi_*[\varphi \circ f] = [\psi \circ (\varphi \circ f] = (\psi \circ \varphi)_*[f]$.

$(Id_X)_*[f] = [Id_X \circ f] = [f] = Id_{\pi_1(X)}[f]$.

Corollary 7.26 If $\varphi: X \to Y$ is a homeomorphism, then $\varphi_*: \pi_1(X) \to \pi_1(Y)$ is an isomorphism.

Pf

$$\begin{tikzcd} X \arrow[r, bend left, "\varphi"] & Y \arrow[l, bend left, "\varphi^{-1}"] \end{tikzcd} \qquad \begin{tikzcd} \pi_1(X) \arrow[r, bend left, "\varphi_*"] & \pi_1(Y) \arrow[l, bend left, "(\varphi^{-1})_* = (\varphi_*)^{-1}"] \end{tikzcd}$$

Note $(\varphi^{-1})_* \circ \varphi_* = (\varphi^{-1} \circ \varphi)_* = (Id_X)_* = Id_{\pi_1(X)}$

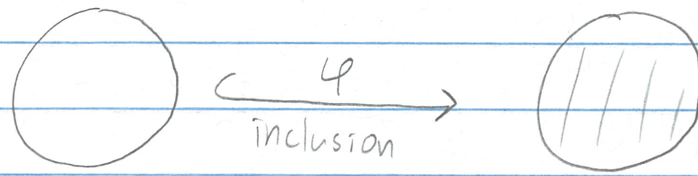
and $\varphi_* \circ (\varphi^{-1})_* = \ldots = Id_{\pi_1(Y)}$,

so φ_* has an inverse homomorphism $(\varphi^{-1})_* = (\varphi_*)^{-1}$.

10/6/2017

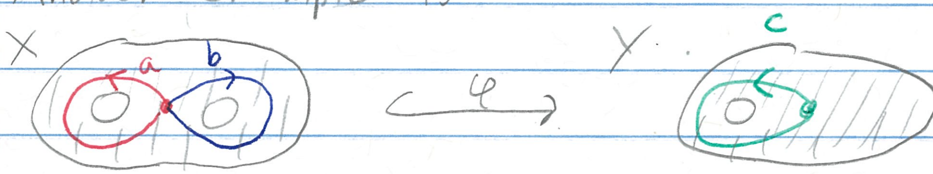
Warning $\varphi: X \to Y$ injective need not imply $\varphi_*: \pi_1(X) \to \pi_1(Y)$ injective.

$X = S^1$ $\qquad\qquad$ $Y = \overline{B^2} = \{ x \in \mathbb{R}^2 \mid \|x\| \leq 1 \}$



Note $\underset{\cong\ \mathbb{Z}}{\varphi_*: \pi_1(X)} \longrightarrow \underset{\cong\ \{id\}}{\pi_1(Y)}$ is not injective.

Another example is

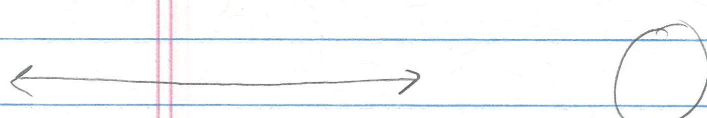


Note $\varphi_*: \underset{\cong\ \langle a, b \rangle}{\pi_1(X)} \longrightarrow \underset{\cong\ \langle c \rangle}{\pi_1(Y)}$

via $a \mapsto c$ and $b \mapsto id$ is not injective.

Warning $\varphi: X \to Y$ surjective need not imply $\varphi_*: \pi_1(X) \to \pi_1(Y)$ surjective.

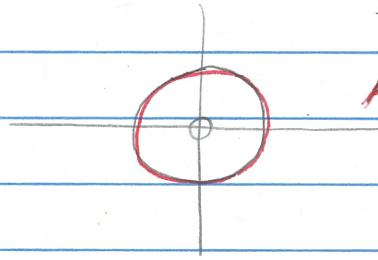
$X = \mathbb{R}$ $\qquad$ $Y = S^1$ $\qquad$ $\varphi: \mathbb{R} \to S^1$ via
$\varphi(t) = e^{2\pi i t}$



Note $\varphi_*: \underset{\cong\ \{id\}}{\pi_1(\mathbb{R})} \longrightarrow \underset{\cong\ \mathbb{Z}}{\pi_1(S^1)}$ is not surjective.

Def For X a space and $A \subseteq X$, a <u>retraction</u> is a continuous map $r: X \to A$ such that $r|_A = Id_A$. Equivalently, $r \circ \iota_A = Id_A$ where $\iota_A : A \hookrightarrow X$ is the inclusion map.

Ex $r: \mathbb{R}^{n+1} \setminus \{0\} \to S^n$ via $r(x) = x/\|x\|$

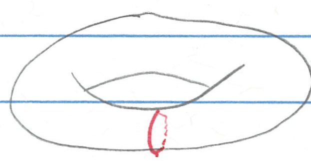


$X = \mathbb{R}^{n+1} \setminus \{0\}$

$A = S^n$

$\boxed{r_* : \pi_1(\mathbb{R}^{n+1} \setminus \{0\}) \to \pi_1(S^n) \text{ is the identity on } \{id\} \text{ or } \mathbb{Z}.}$

Ex $r: S^1 \times S^1 \to S^1$ via $r(z,w) = (z,1)$

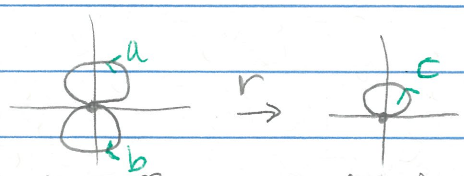


$X = S^1 \times S^1$

$A = S^1$

$\boxed{r_* : \underbrace{\pi_1(S^1 \times S^1)}_{\cong\, \mathbb{Z}\times\mathbb{Z}} \to \underbrace{\pi_1(S^1)}_{\cong\, \mathbb{Z}} \text{ via } (a,b) \mapsto a}$

Ex



X = figure 8 $\qquad A$ = top circle

via the "fold" map

$\boxed{r_* : \underbrace{\pi_1(8)}_{\cong \langle a,b\rangle} \to \underbrace{\pi_1(\bigcirc)}_{\cong \langle c\rangle} \text{ via } \begin{array}{l} a \mapsto c \\ b \mapsto id \end{array}}$

Non-Ex There is no retract from B^{n+1} onto its boundary sphere S^n. A proof (for $n=1$) is...

Prop 7.28 If $r: X \to A$ is a retraction, then $r_* : \pi_1(X) \to \pi_1(A)$ is surjective and $(\iota_A)_* : \pi_1(A) \to \pi_1(X)$ is injective.

Pf Since $r \circ \iota_A = Id_A$, by Prop 7.25 we have

$$r_* \circ (\iota_A)_* \underset{\text{by (a)}}{=} (Id_A)_* \underset{\text{by (b)}}{=} Id_{\pi_1(A)}$$

and hence $(\iota_A)_*$ is injective and r_* is surjective.

10/9/2017 Functors

Def A (covariant) functor from category C to D is a collection of mappings $F: Ob(C) \to Ob(D)$ such that for each $X, Y \in Ob(C)$, there is a mapping $F: Hom_C(X,Y) \to Hom_D(F(X), F(Y))$ such that

- $F(g \circ h) = F(g) \circ F(h)$
- $F(Id_X) = Id_{F(X)}$.

$$\boxed{\begin{array}{l} \text{In } C: \; X \xrightarrow{h} Y \xrightarrow{g} Z \\ \text{In } D: \; F(X) \xrightarrow[F(h)]{} F(Y) \xrightarrow[F(g)]{} F(Z) \end{array}}$$

(We denote the entire functor by $F: C \to D$)

Ex The fundamental group π_1 is a functor from the category of pointed topological spaces to the category of groups.

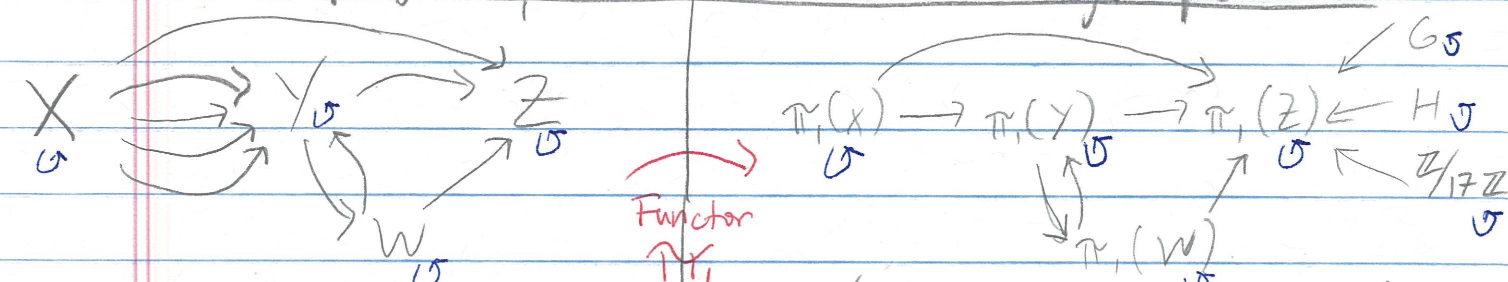
For $\varphi: X \to Y$ a pointed map of spaces, we denote $\pi_1(\varphi): \pi_1(X) \to \pi_1(Y)$ by φ_*.

Note

- $\pi_1(g \circ h) = \pi_1(g) \circ \pi_1(h)$ $\quad [(g \circ h)_* = g_* \circ h_*]$
- $\pi_1(Id_X) = Id_{\pi_1(X)}$ $\quad [(Id_X)_* = Id_{\pi_1(X)}]$

follow from Prop 7.25.

"Land" of topological spaces | "Land" of groups



Only a subcollection of objects/morphisms are drawn!

Blue loops are identity morphisms.

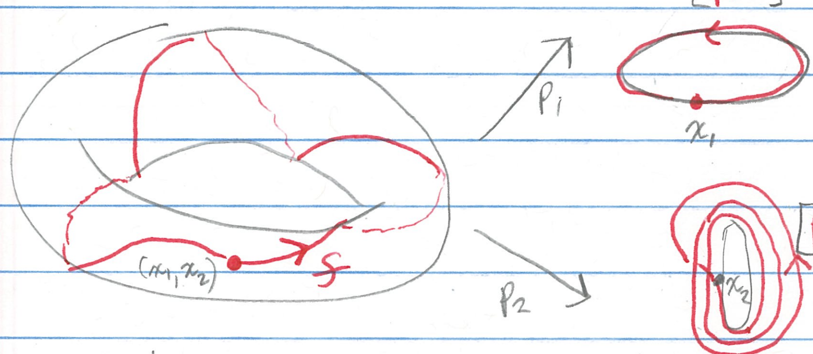
Ex The forgetful functor $F: \mathrm{Top} \to \mathrm{Set}$ assigns to each topological space its underlying set, and to each continuous map its underlying function.

Ex There is a similar forgetful functor $F: \mathrm{Group} \to \mathrm{Set}$.

Fundamental groups of product spaces

Prop 7.34 For topological spaces $X_1, X_2, \ldots, X_n$, the map
$P: \pi_1(X_1 \times \cdots \times X_n, (x_1, \ldots, x_n)) \to \pi_1(X_1, x_1) \times \cdots \times \pi_1(X_n, x_n)$
defined by $P[f] = (p_{1*}[f], \ldots, p_{n*}[f])$
where $p_i : X_1 \times \cdots \times X_n \to X_i$ is the canonical projection map, is a group isomorphism.

Pic



$[p_1 f] \longleftrightarrow 1$ $\quad S^1 \quad \pi_1(S^1) \cong \mathbb{Z}$

$[p_2 f] \longleftrightarrow 3$ $\quad S^1 \quad \pi_1(S^1) \cong \mathbb{Z}$

$S^1 \times S^1$

$\pi_1(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z}$

$[f] \longleftrightarrow (1, 3)$

Pf P is a group homomorphism since each p_{i*} is:

$$\begin{aligned}
P([f]\cdot[g]) = P([f\cdot g]) &= (p_{1*}[f\cdot g], \ldots, p_{n*}[f\cdot g]) \\
&= (p_{1*}[f]\cdot p_{1*}[g], \ldots, p_{n*}[f]\cdot p_{n*}[g]) \\
&= (p_{1*}[f], \ldots, p_{n*}[g])\,(p_{n*}[f], \ldots, p_{n*}[g]) \\
&= P([f])\,P([g]).
\end{aligned}$$

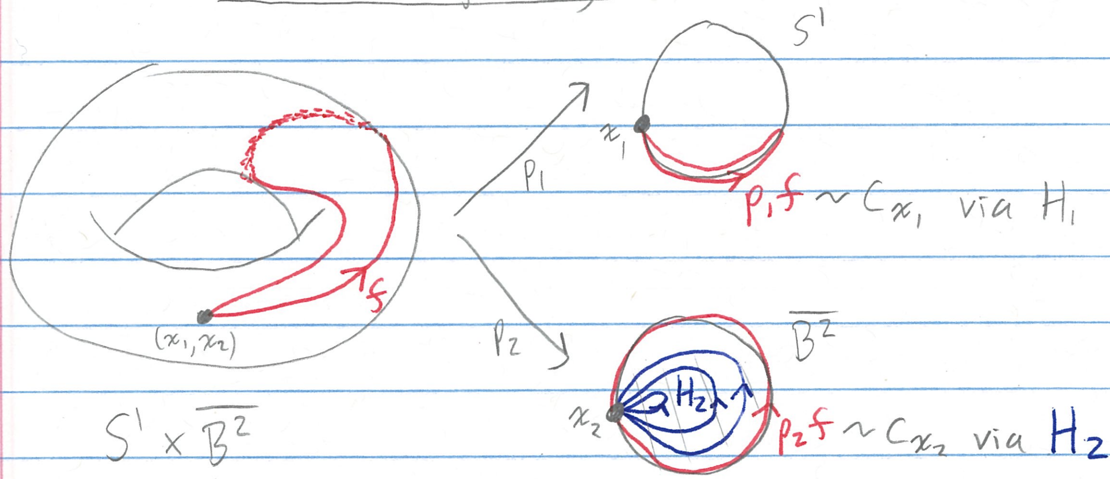
P is surjective since if $[f_i] \in \pi_1(X_i, x_i)$, then we can define a continuous loop $f: I \to X_1 \times \dots \times X_n$ by $f(s) = (f_1(s), \dots, f_n(s))$.

Note $P[f] = (p_{1*}[f], \dots, p_{n*}[f])$

$= ([p_1 \circ f], \dots, [p_n \circ f])$

$= ([f_1], \dots, [f_n])$.

<u>Picture of injectivity</u>



P is injective since if $P[f] = $ identity in $\pi_1(X_1, x_1) \times \dots \times \pi_1(X_n, x_n)$, where $f(s) = (f_1(s), \dots, f_n(s))$, then

$[c_{x_i}] = p_{i*}[f] = [p_i \circ f] = [f_i]$ for all i, so

$f_i \sim c_{x_i}$ via $H_i: I \times I \to X_i$ for all i.

Define $H: I \times I \to X_1 \times \dots \times X_n$ via

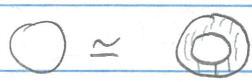
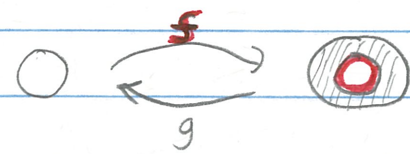
$H(s,t) = (H_1(s,t), \dots, H_n(s,t))$.

This is a continuous homotopy from f to $c_{(x_1, \dots x_n)}$.

Hence $[f] = $ identity in $\pi_1(X_1 \times \dots \times X_n)$.

Homotopy Equivalence

Recall Spaces X, Y are <u>homotopy equivalent</u> ($X \simeq Y$) if $\exists$ continuous maps $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ s.t. $g \circ f \simeq Id_X$ and $f \circ g \simeq Id_Y$.

Ex  via 

(Here $g \circ f = Id_{circle}$ whereas $f \circ g \simeq Id_{annulus}$.)

Thm 7.40 If $\varphi : X \to Y$ is a homotopy equivalence, then $\varphi_* : \pi_1(X) \to \pi_1(Y)$ is an isomorphism.

Rmk Indeed, $\varphi_* : \pi_1(X, p) \to \pi_1(Y, \varphi(p))$ is an isomorphism for all $p \in X$.

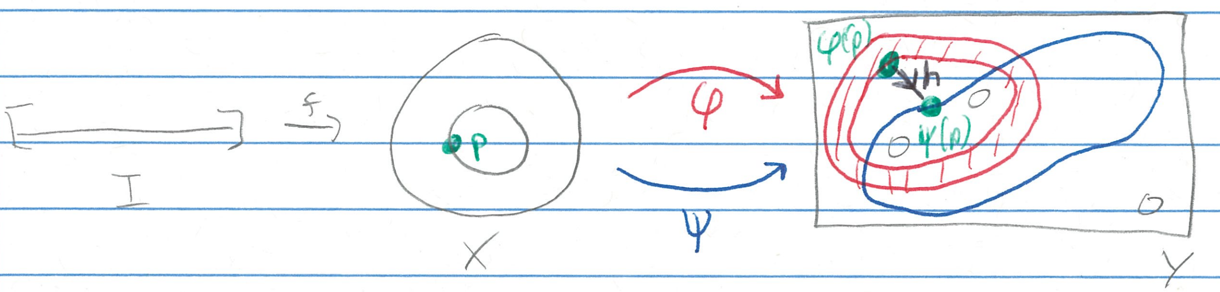
Rmk If we could ignore the subtleties of basepoints, then our proof would be analogous to Corollary 7.26 for homeomorphisms.

Simplified Lemma 7.45 Suppose $\varphi, \psi : X \to Y$ with $\varphi(p) = \psi(p)$ for some $p \in X$, and furthermore $\varphi \simeq \psi$ rel $\{p\}$. Then $\varphi_* = \psi_* : \pi_1(X, p) \to \pi_1(Y, \underset{\psi(p)}{\underset{\shortparallel}{\varphi(p)}})$.

Pf Let f be any loop in X based at p.
We have $\varphi_*[f] = [\varphi \circ f]$
$= [\psi \circ f]$
$= \psi_*[f]$.

The middle step is since $\varphi \circ f \sim \psi \circ f$ (i.e. $\varphi \circ f \simeq \psi \circ f$ rel $\{0,1\}$), which we only have since we made the extra assumption $\varphi \simeq \psi$ rel $\{p\}$.

Rmk In the actual Lemma 7.45 we don't get $\varphi_* = \psi_*$ but only $\Phi_h \circ \varphi_* = \psi_*$ for some path h in X from $\varphi(p)$ to $\psi(p)$.



10/13/17

Pf of Thm 7.40 under the simplified assumption that homotopy equivalence $\varphi: X \to Y$ has a homotopy inverse $\psi: Y \to X$ satisfying

- $\psi(\varphi(p)) = p$ for some $p \in X$
- $\psi \circ \varphi \simeq Id_X$ rel $\{p\}$
- $\varphi \circ \psi \simeq Id_Y$ rel $\{\varphi(p)\}$.

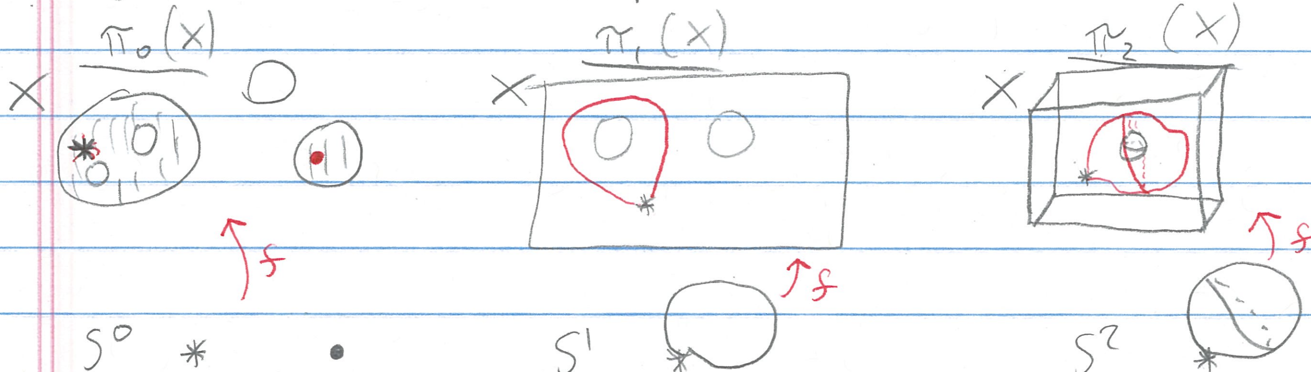
Pf

$$\begin{tikzcd} X \arrow[r, bend left, "\varphi"] & Y \arrow[l, bend left, "\psi"] \end{tikzcd} \qquad \begin{tikzcd} \pi_1(X,p) \arrow[r, bend left, "\varphi_*"] & \pi_1(Y,\varphi(p)) \arrow[l, bend left, "\psi_* = (\varphi_*)^{-1}"] \end{tikzcd}$$

Note $\psi_* \circ \varphi_* = (Id_X)_*$ by Simplified Lemma 7.45
$= Id_{\pi_1(X,p)}$
and $\varphi_* \circ \psi_* = (Id_Y)_*$ by Simplified Lemma 7.45
$= Id_{\pi_1(Y,\varphi(p))}$

Higher homotopy groups

For X a space and $p \in X$, fundamental group $\pi_1(X,p)$ can be recast as the collection of based maps $S^1 \to X$, up to homotopy equivalence rel basepoint.



More generally, homotopy group $\pi_n(X,p)$ is the collection of based maps $S^n \to X$, up to homotopy equivalence rel basepoint.

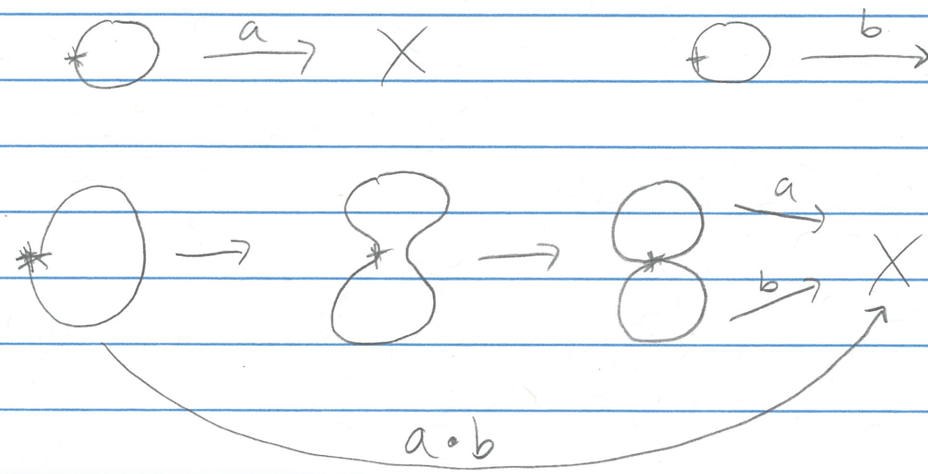
$\pi_0(X,p)$ is a <u>set</u> that "measures the path-connected components of X".

$\pi_1(X,p)$ is a <u>group</u> that "measures the 1-dimensional holes of X".

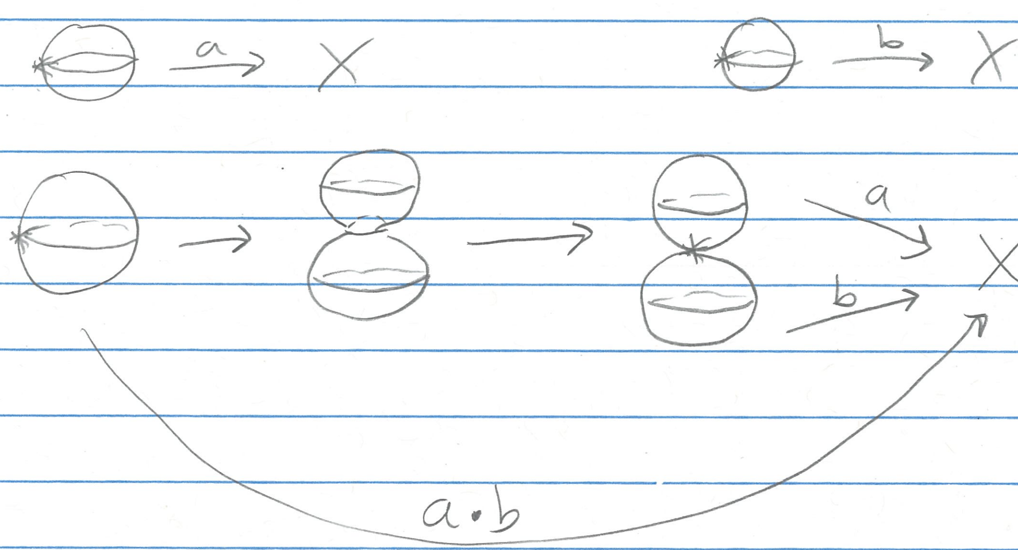
$\pi_n(X,p)$ for $n \geq 2$ is an <u>abelian group</u> that "measures the n-dimensional holes of X".

What's the group operation in $\pi_n(X)$ for $n \geq 1$?

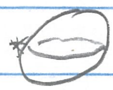
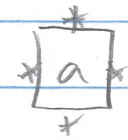
<u>π_1</u>

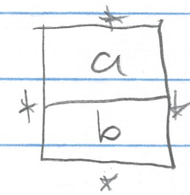


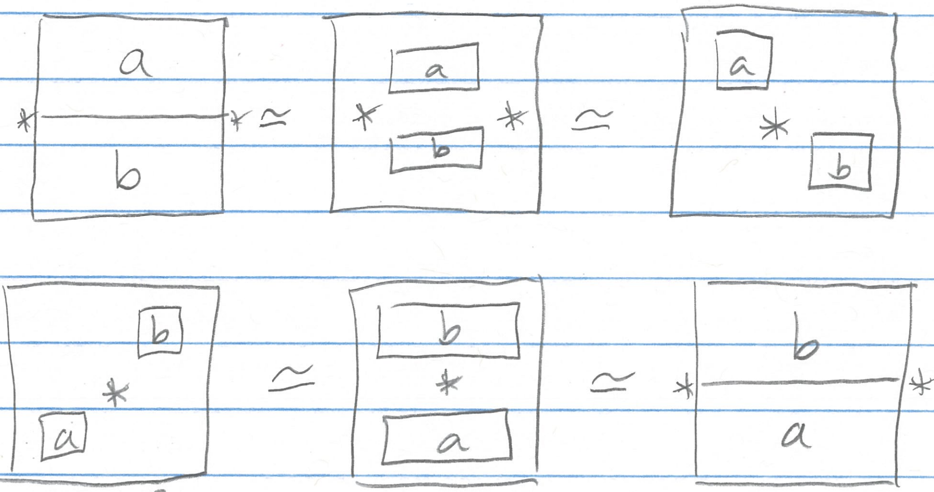
π_2



Why is $\pi_n(X)$ abelian for $n \geq 2$?

Data  $\xrightarrow{a} X$ is the same as  $\to X$

Data  $\xrightarrow{a \cdot b} X$ is the same as  $\to X$.

Note 

Remarks Let $n \geq 1$. Then

- π_n is a functor from pointed topological spaces to groups (or abelian groups for $n \geq 2$).
- $\pi_n(X_1 \times \dots \times X_k) \cong \pi_n(X_1) \times \dots \times \pi_n(X_k)$ by an analogous proof.
- $\pi_n(S^n) \cong \mathbb{Z}$
- $\pi_n(S^k) \cong \{id\}$ for $1 \leq n \leq k$
- $\pi_n(S^k)$ is notorriously hard to compute for $n > k$.

For example $\pi_3(S^2) \cong \mathbb{Z}$ via the Hopf map. See the table at the wikipedia page "Homotopy groups of spheres".

10/16/2017

The Hopf fibration (not tested on)

- The Hopf fibration is a map $p: S^3 \to S^2$ that is not homotopy equivalent to the constant map.
- This shows $\pi_3(S^2) \not\cong \{id\}$.
  Indeed, $\pi_3(S^2) \cong \mathbb{Z} = \langle p \rangle$ is generated by the Hopf fibration.
- See the Youtube video "Hopf fibration — fibers and base"
  www.youtube.com/watch?v=AKotMPGFJYk
  The preimage $p^{-1}(\{x\})$ of each point $x \in S^2$ is a circle S^1 in S^3
- Fiber Total space Base

Fiber		Total space		Base	
S^0	$\longrightarrow$	S^1	$\longrightarrow$	S^1	Real version
S^1	$\longrightarrow$	S^3	$\longrightarrow$	S^2	Defined using complex #'s $\mathbb{C}$
S^3	$\longrightarrow$	S^7	$\longrightarrow$	S^4	Quaternion version
S^7	$\longrightarrow$	S^{15}	$\longrightarrow$	S^8	Octonian version

Hopf fibration

JF Adams proved such fibrations b/w spheres occur only in these dimensions.
This is related to his proof that $\mathbb{R}^n$ is an algebra in which division (except by zero) is allowed only for $n=1$ (reals), $n=2$ (complex), $n=4$ (quaternions), and $n=8$ (octonions).

- Note $\pi_3(S^2) \cong \mathbb{Z}$
  $\pi_7(S^4) \cong \mathbb{Z} \times \mathbb{Z}/12\mathbb{Z}$
  $\pi_{15}(S^8) \cong \mathbb{Z} \times \mathbb{Z}/120\mathbb{Z}$
  are not finite groups.

Chapter 5 Cell complexes

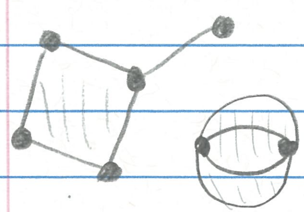
We'll study two types of cell complexes: CW complexes and simplicial complexes.

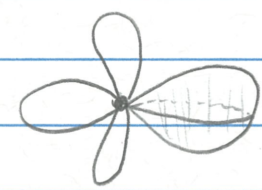
— CW complexes remove many wild phenomena in point-set topology.

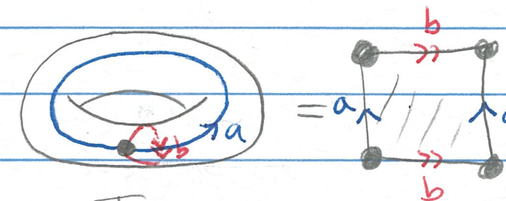
— Simplicial complexes have even more restrictions (every simplicial complex is a CW complex) that allow them to be defined purely combinatorially.

CW complexes

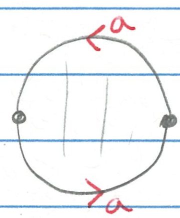
Ex



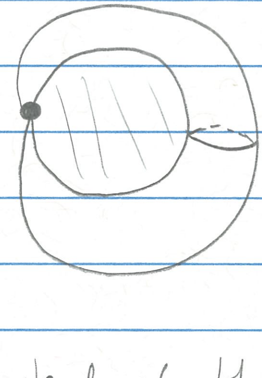




Torus



$\mathbb{R}P^2$



Pinched torus with a disk glued in.

Def (See Prop 5.18, Thm 5.20 instead of the definition on pg 132)

A <u>CW complex</u> is a topological space constructed as follows:

(i) Start with a nonempty discrete set X_0.

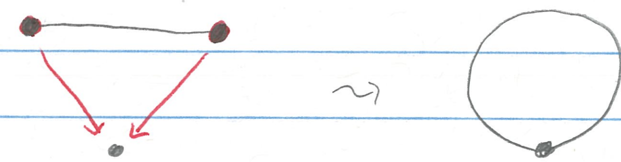
(ii) Inductively, form <u>n-skeleton</u> X_n from X_{n-1} by attaching n-cells along their boundary spheres.

(iii) Equip $X = \bigcup_n X_n$ with the topology such that $A \subseteq X$ is declared to be open (resp. closed) iff $A \cap X_n$ is open (resp. closed) in X_n for all n.

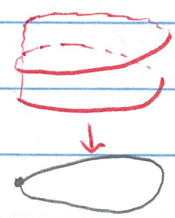
Ex $\mathbb{RP}^2$

0-skeleton $\cdot$

1-skeleton



2-skeleton



$\rightsquigarrow \mathbb{RP}^2$

Rmks

- Each X_n has the quotient space (or "adjunction space") topology $X_n = (X_{n-1} \coprod_{\alpha \in A} D^n_\alpha)/\sim$ where each <u>n-cell</u> D^n_α is a closed n-ball $(\overline{B^n})$, where $\varphi_\alpha : \partial D^n_\alpha \to X_{n-1}$ are the <u>attaching maps</u>, and where $\varphi_\alpha(x) \sim x$ for all $x \in \partial D^n_\alpha$.
- When X is finite dimensional ($X = X_n$ for some n), then (iii) is superfluous.

Ex S^∞ $\quad$ $X_0 = S^0$ $\quad$ $X_1 = S^1$ $\quad$ $X_2 = S^2$ $\quad$ $X_3 = S^3$

This CW complex has two n-cells in each dimension n, and $X_n = S^n$ for all n. Surprisingly, S^∞ is contractible.

10/18/2017 CW complexes have nice point-set topological properties.

- They're Hausdorff
- They're connected $\iff$ they're path-connected (Thm 5.11).
- Etc

Simplicial Complexes

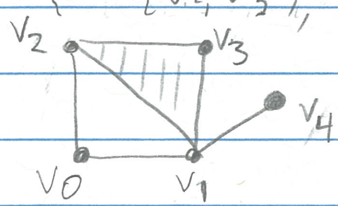
A simplicial complex is a cell complex that can be defined purely combinatorially.

Def An abstract simplicial complex on a vertex set V is a collection K of finite subsets of V such that if $\sigma \in K$ and $\tau \subseteq \sigma$, then $\tau \in K$.

Ex $V = \{v_0, v_1, v_2, v_3, v_4\}$

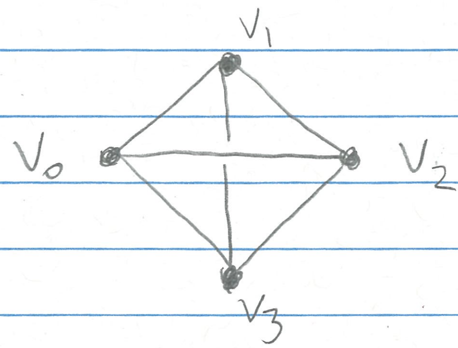
$K = \{ \{v_1, v_2, v_3\}, \{v_0, v_1\}, \{v_0, v_2\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_3\}, \{v_0\}, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\} \}.$

$\emptyset$ often excluded



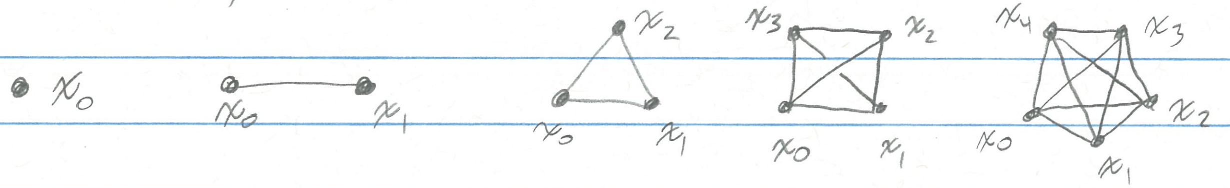
Ex $V = \{v_0, v_1, v_2, v_3\}$

$K =$ the set of all subsets of V



This is a (solid) tetrahedron, or 3-simplex.

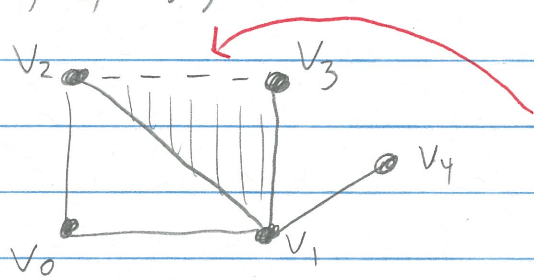
The elements of K are called <u>simplices</u> or <u>faces</u>. A <u>k-dimensional simplex</u> or <u>k-simplex</u> has $k+1$ vertices, often indexed from 0 to k.



0-simplex	1-simplex	2-simplex	3-simplex	4-simplex
vertex	edge	triangle	tetrahedron	

<u>Non-Ex</u> $V = \{v_0, v_1, v_2, v_3, v_4\}$

$\{v_1, v_2, v_3\} \in K$ but $\{v_2, v_3\} \notin K$



Can't have a triangle with a missing edge!

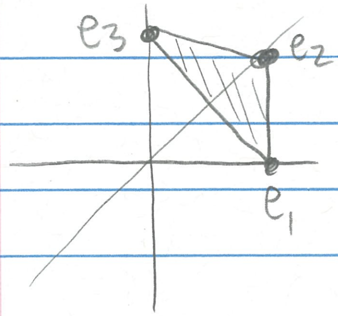
But how is this a topological space?
What are the open sets?

Every abstract simplicial complex K has an associated <u>geometric realization</u> K which is a topological space.

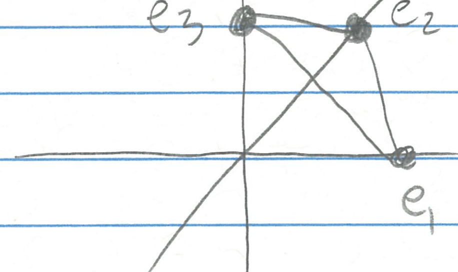
(By an abuse of notation we denote the abstract simplicial complex and its geometric realization by the same symbol K.)

For vertex set $V = \{v_1, \dots, v_n\}$ finite, assign each vertex v_i to the standard basis vector $e_i = (0, \dots, 0, \underset{\substack{\uparrow \\ i\text{-th coordinate}}}{1}, 0, \dots, 0) \in \mathbb{R}^n$.

Assign each simplex $\{v_{i_0}, \dots, v_{i_k}\}$ to the convex hull of $\{e_{i_0}, \dots, e_{i_k}\}$.



$K = \{\{v_1, v_2, v_3\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_1\}, \{v_2\}, \{v_3\}\}$.

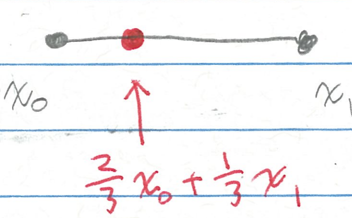


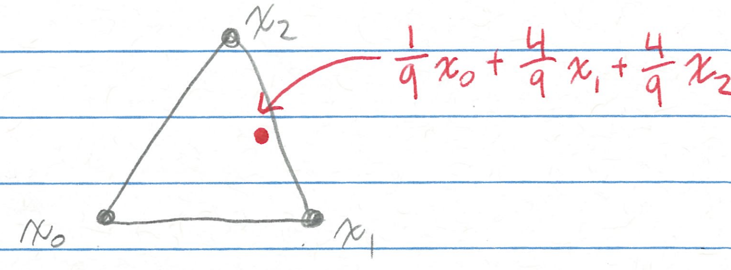
$K = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_1\}, \{v_2\}, \{v_3\}\}$.

The <u>geometric realization</u> of K is then the union of these convex hulls in $\mathbb{R}^n$, equipped with the (Euclidean) subspace topology.

More generally (V is possibly infinite), the geometric realization of K as a set is

$$\Big\{ \underbrace{\sum_{i=0}^{k} \lambda_i x_i}_{\text{barycentric coordinates}} \;\Big|\; k \in \mathbb{N}, \underbrace{\lambda_i \geq 0, \sum_{i=0}^{k} \lambda_i = 1}_{\text{convex combination}}, \{x_0, \dots, x_k\} \in K \Big\}.$$





A subset of the geometric realization of K
is declared to be open (resp. closed)
if its intersection with (the geometric realization of)
each (finite) simplex is open (resp. closed).

10/23/2017 We are momentarily skipping some very important aspects of the fundamental group (the Seifert-Van Kampen Theorem in Chapter 10, covering spaces in Chapters 11-12) to cover...

Chapter 13 Homology

- Much like the homotopy group $\pi_p(X)$, the homology group $H_p(X)$ roughly speaking "measures the # of p-dimensional holes in X".
- Unlike $\pi_p(X)$, homology group $H_p(X)$ is hard to define but easy to compute (linear algebra).
- There are many homology theories.

We'll start with simplicial homology, which is only defined for simplicial complexes.

We'll then move to singular homology, defined for all topological spaces and covered in our book.

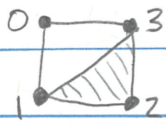
These two homology theories (simplicial and singular) agree on simplicial complexes.

Introduction to simplicial homology

Let X be a simplicial complex.

Def Let $C_p(X)$ be the free abelian group on the set of all oriented p-simplices in X.
I.e., $C_p(X)$ is the set of all formal sums of p-simplices in X with coefficients in $\mathbb{Z}$. Its elements are called <u>p-chains</u>.

Ex

$X =$ 

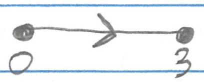
$C_0(X) = \{a[0] + b[1] + c[2] + d[3] \mid a,b,c,d \in \mathbb{Z}\} \cong \mathbb{Z}^4$

Group operation: $(a[0] + \dots + d[3]) + (a'[0] + \dots + d'[3])$
$= (a+a')[0] + \dots + (d+d')[3]$.

So $([0] + [2] + [3]) + ([0] - [2] + 4[3]) = 2[0] + 5[3]$.

$C_1(X) = \{a[0,1] + b[0,3] + c[1,2] + d[1,3] + e[2,3] \mid a,b,c,d,e \in \mathbb{Z}\} \cong \mathbb{Z}^5$

We write a simplex as $[0,3]$ instead of $\{0,3\}$ to denote that it is <u>oriented</u>: $[0,3] = -[3,0]$

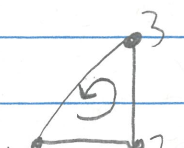


$2[0,3] = -2[3,0]$.

The group operation is analogous:

$([1,2] + [2,3]) + ([1,3] + [2,3]) = [1,2] + [1,3] + 2[2,3]$.

$([1,2] + [2,3]) + ([1,3] + [3,2]) = ([1,2] + [2,3]) + ([1,3] - [2,3])$
$= [1,2] + [1,3]$.



$C_2 = \{a[1,2,3] \mid a \in \mathbb{Z}\} \cong \mathbb{Z}$

$[1,2,3] = [2,3,1] = [3,1,2]$ (differ from $[1,2,3]$ by an even permutation)

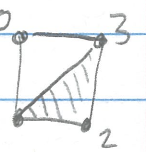
$(-[1,2,3] =)$ $[1,3,2] = [3,2,1] = [2,1,3]$ (differ from $[1,2,3]$ by an odd permutation)

Def Define the <u>boundary operator</u> $\partial_p : C_p(X) \to C_{p-1}(X)$ (or $\partial : C_p(X) \to C_{p-1}(X)$) by setting

$$\partial_p([x_0, \dots, x_p]) = \sum_{i=0}^{p} (-1)^i [x_0, \dots, \hat{x}_i, \dots, x_p]$$

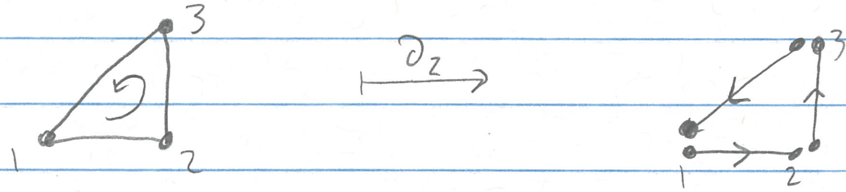
and extending linearly (where the hat $\hat{x}_i$ means x_i is omitted).

Ex $\partial_2 : C_2(X) \to C_1(X)$ via

$X =$ 

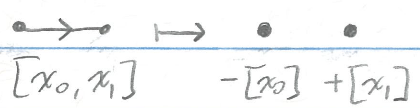
$$\partial_2([1,2,3]) = [2,3] - [1,3] + [1,2] = [1,2] + [2,3] + [3,1]$$

$C_2(X)$ $\quad\quad\quad\quad$ $C_1(X)$



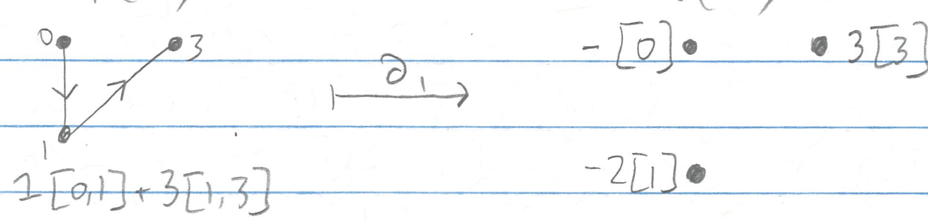
10/25/2017 So $\partial_2(5[1,2,3]) = 5[1,2] + 5[2,3] + 5[3,1]$.

$\partial_1 : C_1(X) \to C_0(X)$ via $\partial_1([x_0, x_1]) = [x_1] - [x_0]$



So $\partial_1([0,1] + 3[1,3]) = \partial_1[0,1] + 3\,\partial_1[1,3]$
$= [1] - [0] + 3([3] - [1])$
$= -[0] - 2[1] + 3[3]$

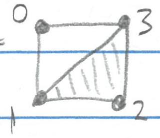
$C_1(X)$ $\quad\quad\quad\quad$ $C_0(X)$



$\partial_0 : C_0(X) \to 0$ sends any 0-chan to zero.

Def Let $Z_p(X) = \ker(\partial_p)$ be the set of <u>p-cycles</u>.
Let $B_p(X) = \operatorname{im}(\partial_{p+1})$ be the set of <u>p-boundaries</u>
Let $H_p(X) = Z_p(X)/B_p(X)$ be the <u>p-dimensional simplicial homology group</u>.

Ex

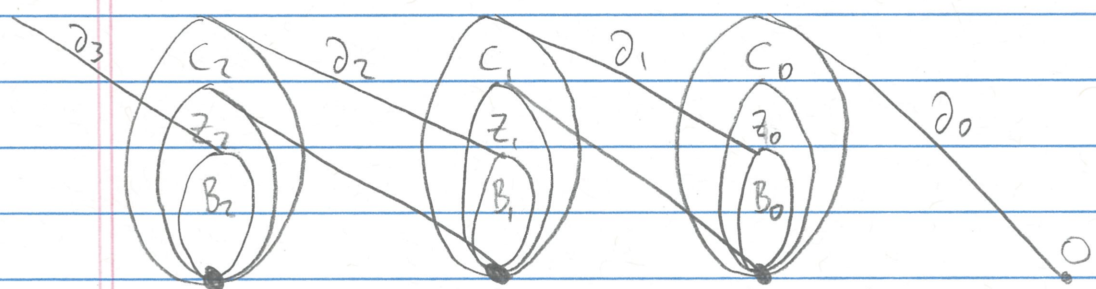
$X =$ 

$Z_1(X) = \ker(\partial_1) = \{a([0,1]+[1,3]+[3,0]) + b([1,2]+[2,3]+[3,1]) \mid a,b \in \mathbb{Z}\} \cong \mathbb{Z}^2$

Note $[0,1]+[1,2]+[2,3]+[3,0] \in Z_1(X)$ by letting $a=b=1$.

$B_1(X) = \operatorname{im}(\partial_2) = \{a([1,2]+[2,3]+[3,1]) \mid a \in \mathbb{Z}\} \cong \mathbb{Z}$

$H_1(X) = Z_1(X)/B_1(X) \cong \mathbb{Z}$ "X has a single 1-dimensional hole"

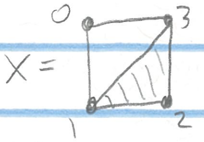


Here $Z_0 = C_0$

$C_2(X) \cong \mathbb{Z}$	$C_1(X) \cong \mathbb{Z}^5$	$C_0(X) \cong \mathbb{Z}^4$
$Z_2(X) = 0$	$Z_1(X) \cong \mathbb{Z}^2$	$Z_0(X) \cong \mathbb{Z}^4$
$B_2(X) = 0$	$B_1(X) \cong \mathbb{Z}$	$B_0(X) \cong \mathbb{Z}^3$
$H_2(X) = 0$	$H_1(X) \cong \mathbb{Z}$	$H_0(X) \cong \mathbb{Z}$
	"X has a single 1-dimensional hole"	"X has a single connected component"

10/27/2017

Ex



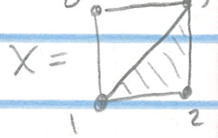
$Z_2(X) = \ker(\partial_2) = 0$ since $\partial_2([1,2,3]) \neq 0$.

$B_2(X) = \mathrm{im}(\partial_3) = 0$ since $C_3(X) = 0$.

$H_2(X) = Z_2(X)/B_2(X) \cong 0$.

"X has no 2-dimensional holes"

Ex



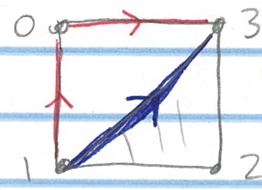
$Z_0(X) = \ker(\partial_0) = C_0(X) = \{a[0]+b[1]+c[2]+d[3] \mid a,b,c,d \in \mathbb{Z}\} \cong \mathbb{Z}^4$.

$B_0(X) = \mathrm{im}(\partial_1)$

$= \{a\partial_1[0,1] + b\partial_1[0,3] + c\partial_1[1,2] + d\partial_1[1,3] + e\partial_1[2,3] \mid a,b,c,d,e \in \mathbb{Z}\}$

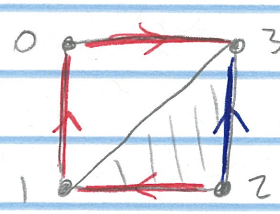
$= \{a([1]-[0]) + b([3]-[0]) + c([2]-[1]) \mid a,b,c \in \mathbb{Z}\} \cong \mathbb{Z}^3$

since $\partial_1[1,3] = [3]-[1] = -([1]-[0]) + ([3]-[0]) = -\partial_1[0,1] + \partial_1[0,3]$



and since $\partial_1[2,3] = [3]-[2] = -([1]-[0]) + ([3]-[0]) - ([2]-[1])$

$= -\partial_1[0,1] + \partial_1[0,3] - \partial_1[1,2]$



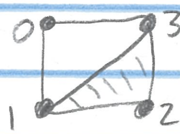
(Note using the 3 edges $[0,1]$, $[0,3]$, $[1,2]$ we can walk from any vertex to any other vertex.)

$H_0(X) = Z_0(X)/B_0(X) \cong \mathbb{Z}$

To see this, note we can "change basis" on Z_0 so that 3 of the 4 generators are $[1]-[0]$, $[3]-[0]$, and $[2]-[1]$.

"X has a single connected component"

In the prior example, how might we algorithmically compute $Z_1(X) \cong \mathbb{Z}^2$ and $B_0(X) \cong \mathbb{Z}^3$, while also finding generators?

$X =$ 

4×5 Matrix representing $\partial_1 : C_1(X) \to C_0(X)$

$$
\begin{array}{c|ccccc}
 & [0,1] & [0,3] & [1,2] & [1,3] & [2,3] \\
\hline
[0] & -1 & -1 & 0 & 0 & 0 \\
[1] & 1 & 0 & -1 & -1 & 0 \\
[2] & 0 & 0 & 1 & 0 & -1 \\
[3] & 0 & 1 & 0 & 1 & 1
\end{array}
\xrightarrow{\text{swap rows}}
\begin{array}{c|ccccc}
 & [0,1] & [0,3] & [1,2] & [1,3] & [2,3] \\
\hline
[1] & 1 & 0 & -1 & -1 & 0 \\
[3] & 0 & 1 & 0 & 1 & 1 \\
[2] & 0 & 0 & 1 & 0 & -1 \\
[0] & -1 & -1 & 0 & 0 & 0
\end{array}
$$

Add third row to first

$$
\begin{array}{c|ccccc}
 & [0,1] & [0,3] & [1,2] & [1,3] & [2,3] \\
\hline
[1] & 1 & 0 & 0 & -1 & -1 \\
[3] & 0 & 1 & 0 & 1 & 1 \\
[2]-[1] & 0 & 0 & 1 & 0 & -1 \\
[0] & -1 & -1 & 0 & 0 & 0
\end{array}
\xrightarrow{\text{Add first and third rows to fourth}}
\begin{array}{c|ccccc}
 & [0,1] & [0,3] & [1,2] & [1,3] & [2,3] \\
\hline
[1]-[0] & 1 & 0 & 0 & -1 & -1 \\
[3]-[0] & 0 & 1 & 0 & 1 & 1 \\
[2]-[1] & 0 & 0 & 1 & 0 & -1 \\
[0] & 0 & 0 & 0 & 0 & 0
\end{array}
$$

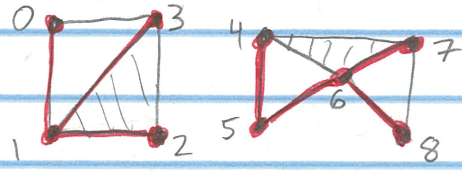
Add col 1 − col 2 to col 4
Add col 1 − col 2 + col 3 to col 5

$$
\begin{array}{c|ccccc}
 & [0,1] & [0,3] & [1,2] & \begin{array}{c}[1,3]+[0,1]-[0,3] \\ \| \\ [0,1]+[1,3]+[3,0]\end{array} & \begin{array}{c}[2,3]+[0,1]-[0,3]+[1,2] \\ \| \\ [0,1]+[1,2]+[2,3]+[3,0]\end{array} \\
\hline
[1]-[0] & 1 & 0 & 0 & 0 & 0 \\
[3]-[0] & 0 & 1 & 0 & 0 & 0 \\
[2]-[1] & 0 & 0 & 1 & 0 & 0 \\
[0] & 0 & 0 & 0 & 0 & 0
\end{array}
$$

This matrix has rank 3 and nullity 2.
The first three rows give a generating set for $B_0(X) = \mathrm{im}(\partial_1)$.
The last two columns give a generating set for $Z_1(X) = \ker(\partial_1)$.

11/8/2017

Simplicial homology: spanning trees and $\partial_1 : C_1(X) \to C_0(X)$

$X =$ 

Pick a spanning tree for each connected component.
This can help you identify generating sets for
$B_0(X) = \text{im}(\partial_1)$ and $Z_1(X) = \ker(\partial_1)$.

12 letters for 12 edges

$$B_0(X) = \{a\partial_1[0,1] + b\partial_1[0,3] + \dots + l\partial_1[7,8] \mid a, b, \dots, l \in \mathbb{Z}\}$$
$$= \{a\partial_1[0,1] + b\partial_1[1,2] + \dots g\partial_1[6,8] \mid a, b, \dots, g \in \mathbb{Z}\} \cong \mathbb{Z}^7$$

7 letters for the 7 edges in the spanning trees

$$Z_1(X) = \left\{ \begin{array}{l} a([0,3]+[3,1]+[1,0]) + b([2,3]+[3,1]+[1,2]) + c([4,6]+[6,5]+[5,4]) \\ + d([4,7]+[7,6]+[6,5]+[5,4]) + e([7,8]+[8,6]+[6,7]) \\ \text{such that } a,b,c,d,e \in \mathbb{Z} \end{array} \right\} \cong \mathbb{Z}^5.$$

Note $\dim(Z_1(X)) + \dim(B_0(X)) = \dim(\mathbb{Z}^5) + \dim(\mathbb{Z}^7) = 5 + 7 = 12$
$= \#\text{ edges} = \dim(C_1(X))$.

More generally,
$\dim(Z_p(X)) + \dim(B_{p-1}(X)) = \dim(\ker \partial_p) + \dim(\text{im}\, \partial_p) = \dim(C_p(X))$
$(= \#\text{ p-simplices})$

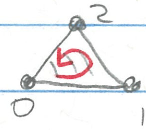
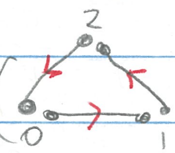
$$C_p(X) \xrightarrow{\partial_p} C_{p-1}(X)$$

In the definition $H_p(X) = Z_p(X)/B_p(X)$ why do we know $B_p(X) = \mathrm{im}(\partial_{p+1})$ is a subgroup of $Z_p(X) = \ker(\partial_p)$?

The reason is that $\partial_p \circ \partial_{p+1} = 0$.
(This is often written $\partial \circ \partial = 0$ or $\partial^2 = 0$.)

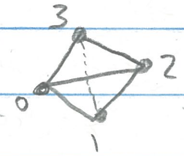
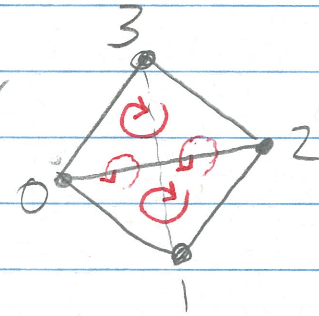
10/30/2017

Pic of $\partial_1 \circ \partial_2 = 0$

$\partial_1 (\partial_2 ($  $)) = \partial_1 ($  $) = 0$

$$\partial_1(\partial_2([0,1,2])) = \partial_1([1,2] - [0,2] + [0,1])$$
$$= ([2]-[1]) - ([2]-[0]) + ([1]-[0])$$
$$= 0.$$

Pic of $\partial_2 \circ \partial_3 = 0$

$\partial_2 (\partial_3 ($  $)) = \partial_2 ($  $) = 0$

$$\partial_2(\partial_3([0,1,2,3])) = \partial_2([1,2,3] - [0,2,3] + [0,1,3] - [0,1,2])$$
$$= ([2,3]-[1,3]+[1,2]) - ([2,3]-[0,3]+[0,2]) + ([1,3]-[0,3]+[0,1]) - ([1,2]-[0,2]+[0,1])$$
$$= 0.$$

Proof of $\partial_p \circ \partial_{p+1} = 0$

Since the boundary operators are linear, it suffices to show that applying $\partial_p \circ \partial_{p+1}$ to a single $(p+1)$-simplex gives zero. Note

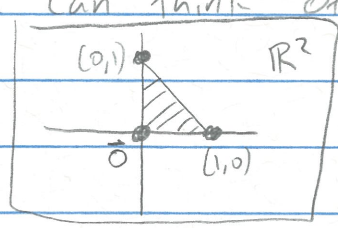
$$\begin{aligned}
&\partial_p(\partial_{p+1}([x_0, \ldots, x_{p+1}])) \\
&= \partial_p\left(\textstyle\sum_{i=0}^{p+1} (-1)^i [x_0, \ldots, \hat{x}_i, \ldots, x_{p+1}]\right) \\
&= \textstyle\sum_{j<i} (-1)^j (-1)^i [x_0, \ldots, \hat{x}_j, \ldots, \hat{x}_i, \ldots, x_{p+1}] \\
&\quad + \textstyle\sum_{i<j} (-1)^{j-1} (-1)^i [x_0, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{p+1}] \\
&= 0 \qquad \text{(by symmetry — terms cancel in pairs!)}
\end{aligned}$$

Singular homology

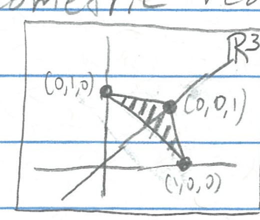
- This is in our book
- Singular homology is more complicated than simplicial homology, but the upshot is that given a homotopy equivalence $X \simeq Y$, it's easier to prove with singular homology that $H_p(X) \cong H_p(Y)$.

Let Δ_p denote both an abstract p-simplex $[e_0, \ldots, e_p]$ and its geometric realization.

Rmk You can think of the geometric realization of Δ_2 as



or as

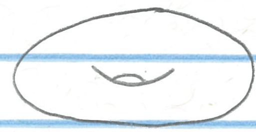


or as 2-dimensional convex hull of 3 points in Euclidean space.

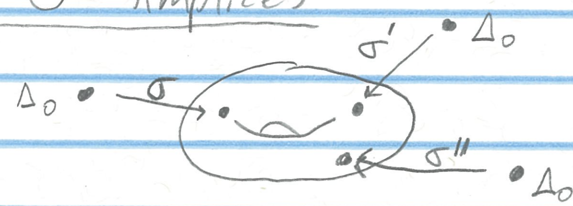
Let X be a topological space.

<u>Def</u> A <u>singular p-simplex in X</u> is a continuous map $\sigma : \Delta_p \to X$

Here we're using the geometric realization

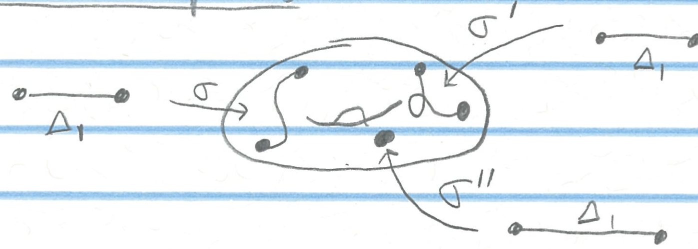
<u>Ex</u> $X =$ 

<u>Singular 0-simplices</u>



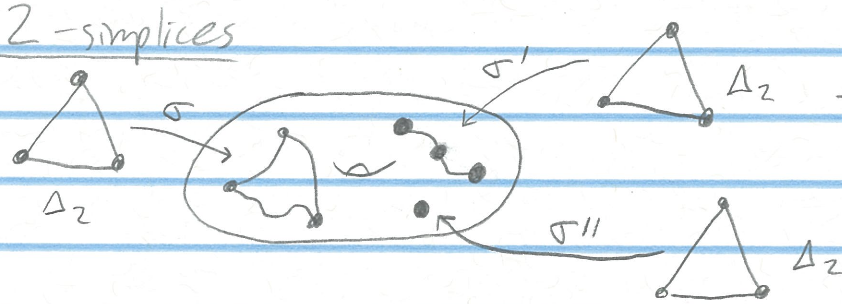
$17\sigma + 5\sigma' - 3\sigma''$ is a singular 0-chain.

<u>Singular 1-simplices</u>



$-3\sigma + 5\sigma' - 9\sigma''$ is a singular 1-chain.

<u>Singular 2-simplices</u>



$-\sigma - 8\sigma' + 3\sigma''$ is a singular 2-chain.

Non-injectivity is allowed; hence the name singular!

using same symbol as in simplicial homology

<u>Def</u> Let $C_p(X)$ be the free abelian group on the set of all singular p-simplices in X.
An element of $C_p(X)$, which is called a <u>singular p-chain</u>, is a formal sum of singular p-simplices with coefficients in $\mathbb{Z}$.

What's the group operation on $C_p(X)$?
If $\sigma, \sigma', \sigma'', \sigma'''$ are singular p-simplices, then
$$(5\sigma + 7\sigma' - 3\sigma'') + (7\sigma' - 2\sigma'' + \sigma''') = 5\sigma + 14\sigma' - 5\sigma'' + \sigma'''$$

11/6/2017

The boundary operator

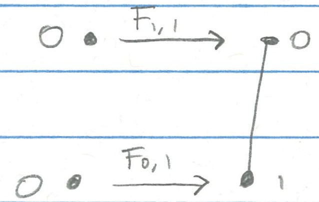
Def For $i = 0, 1, \ldots, p$, define $F_{i,p} : \Delta_{p-1} \to \Delta_p$ to be the affine map taking Δ_{p-1} homeomorphically onto the face of Δ_p opposite the i-th vertex:

$$\begin{aligned} e_0 &\mapsto e_0 \\ &\vdots \\ e_{i-1} &\mapsto e_{i-1} \\ e_i &\mapsto e_{i+1} \\ &\vdots \\ e_{p-1} &\mapsto e_p \end{aligned}$$

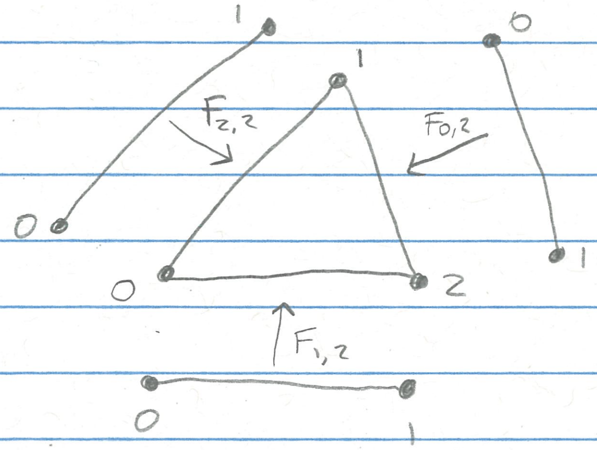
$$\Delta_{p-1} = [e_0, \ldots, e_{p-1}]$$
$$\Delta_p = [e_0, \ldots, e_p]$$

This is the *i-th face map in dimension p*.

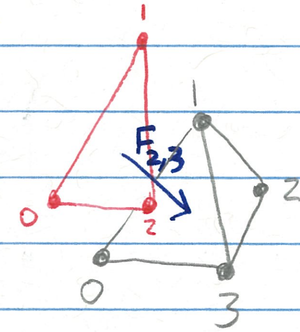
Ex $p=1$



Ex $p=2$



Ex $p=3$

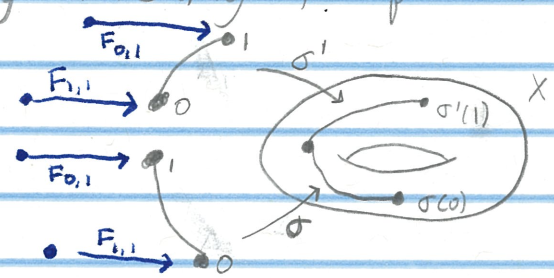


Def Define the (singular) boundary operator $\partial_p : C_p(X) \to C_{p-1}(X)$ (or $\partial : C_p(X) \to C_{p-1}(X)$) by setting

$$\partial\sigma = \sum_{i=0}^{p} (-1)^i \sigma \circ F_{i,p}$$

for any singular p-simplex $\sigma : \Delta_p \to X$, and then extending linearly to p-chains.

Ex $p=1$



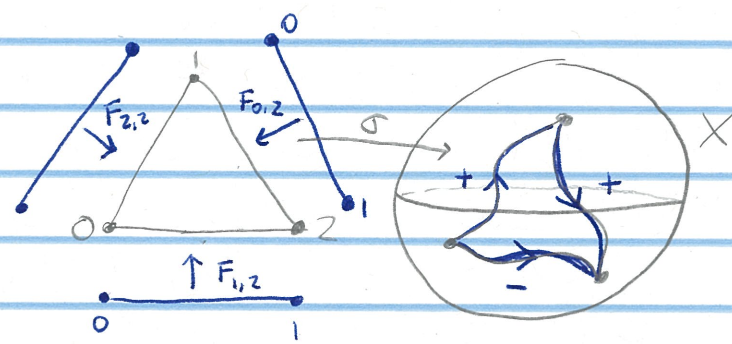
$$\partial(\sigma+\sigma') = \partial\sigma + \partial\sigma'$$

$$= \cancel{\sigma \circ F_{0,1}} - \sigma \circ F_{1,1} + \sigma' \circ F_{0,1} - \cancel{\sigma' \circ F_{1,1}}$$

Note $\sigma \circ F_{0,1} = \sigma' \circ F_{1,1}$.

$$= -\sigma \circ F_{1,1} + \sigma' \circ F_{0,1}.$$

Ex $p=2$



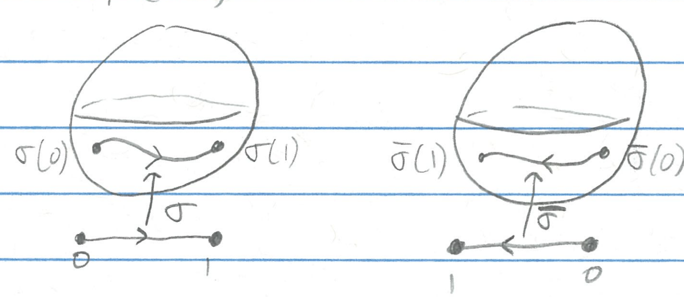
$$\partial\sigma = \sigma \circ F_{0,2} - \sigma \circ F_{1,2} + \sigma \circ F_{2,2}$$

Minor remark In simplicial homology, $C_p(X)$ was generated by all <u>oriented</u> p-simplices.
For example we had $[x_0, x_1] = -[x_1, x_0]$ on the nose (i.e. in $C_1(X)$).

$$\partial[x_0, x_1, x_2] = [x_1, x_2] - [x_0, x_2] + [x_0, x_1]$$
$$= [x_0, x_1] + [x_1, x_2] + [x_2, x_0]$$

In singular homology, $C_p(X)$ is generated by all singular p-simplices (no orientation data).

For example, in $C_1(X)$ we have
$\sigma \neq -\bar{\sigma}$

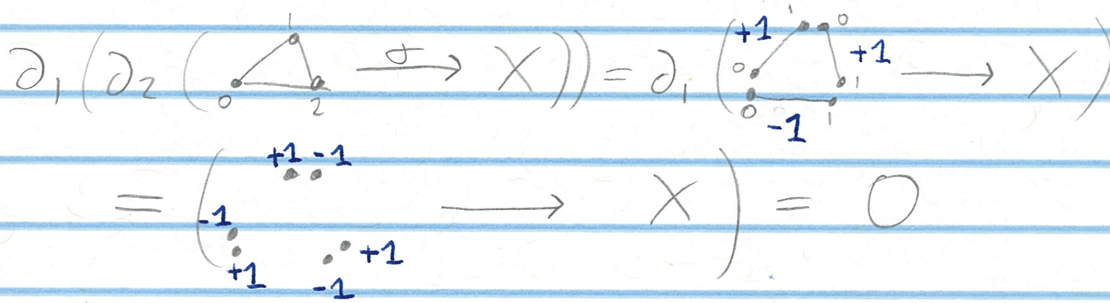


However, we will have that σ and $-\bar{\sigma}$ differ by an element of $B_1(X)$.

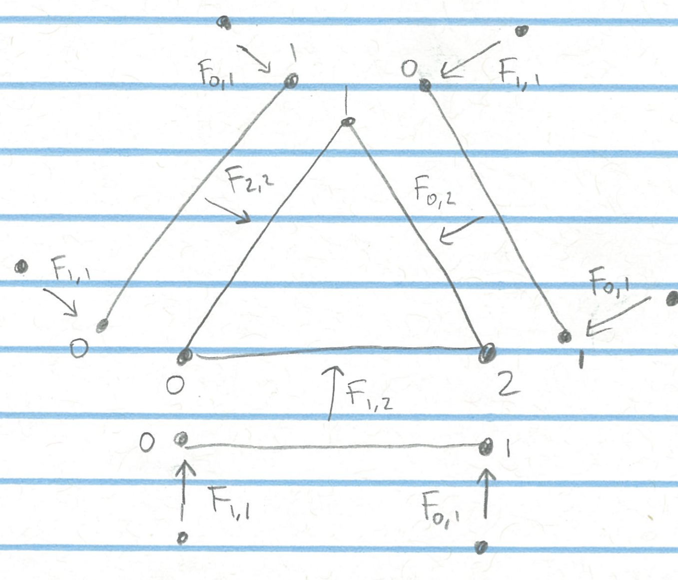
11/10/2017

Lemma 13.1 If $c \in C_p(X)$ is a singular p-chain, then $\partial_{p-1}(\partial_p(c)) = 0$.
In other words, $\partial_{p-1} \circ \partial_p = 0$, or $\partial \circ \partial = 0$.

Pic of $\partial_1 \circ \partial_2 = 0$



$$\partial_1(\partial_2(\sigma)) = \partial_1(\sigma \circ F_{0,2} - \sigma \circ F_{1,2} + \sigma \circ F_{2,2})$$
$$= \partial_1(\sigma \circ F_{0,2}) - \partial_1(\sigma \circ F_{1,2}) + \partial_1(\sigma \circ F_{2,2})$$
$$= (\sigma \circ \cancel{F_{0,2} \circ F_{0,1}} - \sigma \circ \cancel{F_{0,2} \circ F_{1,1}}) - (\sigma \circ \cancel{F_{1,2} \circ F_{0,1}} - \sigma \circ \cancel{F_{1,2} \circ F_{1,1}}) + (\sigma \circ \cancel{F_{2,2} \circ F_{0,1}} - \sigma \circ \cancel{F_{2,2} \circ F_{1,1}})$$
$= 0$ since
- $F_{1,2} \circ F_{0,1} = F_{0,2} \circ F_{0,1}$
- $F_{2,2} \circ F_{0,1} = F_{0,2} \circ F_{1,1}$
- $F_{2,2} \circ F_{1,1} = F_{1,2} \circ F_{1,1}$



These are examples of the commutation relation

$$F_{i,p} \circ F_{j,p-1} = F_{j,p} \circ F_{i-1,p-1} \quad \text{when } i > j.$$

This is equation (13.1) in our book.

$$
\begin{array}{ccccc}
 & F_{j,p-1} & & F_{i,p} & \\
e_0 & \longmapsto & e_0 & \longmapsto & e_0 \\
 & \vdots & & & \\
e_{j-1} & \longmapsto & e_{j-1} & \longmapsto & e_{j-1} \\
e_j & \longmapsto & e_{j+1} & \longmapsto & e_{j+1} \\
 & \vdots & & & \\
e_{i-2} & \longmapsto & e_{i-1} & \longmapsto & e_{i-1} \\
e_{i-1} & \longmapsto & e_i & \longmapsto & e_{i+1} \\
 & \vdots & & & \\
e_{p-2} & \longmapsto & e_{p-1} & \longmapsto & e_p
\end{array}
\qquad
\begin{array}{ccccc}
 & F_{i-1,p-1} & & F_{j,p} & \\
e_0 & \longmapsto & e_0 & \longmapsto & e_0 \\
 & & & & \\
e_{j-1} & \longmapsto & e_{j-1} & \longmapsto & e_{j-1} \\
e_j & \longmapsto & e_j & \longmapsto & e_{j+1} \\
 & & & & \\
e_{i-2} & \longmapsto & e_{i-2} & \longmapsto & e_{i-1} \\
e_{i-1} & \longmapsto & e_i & \longmapsto & e_{i+1} \\
 & & & & \\
e_{p-2} & \longmapsto & e_{p-1} & \longmapsto & e_p
\end{array}
$$

<u>Proof of Lemma 13.1</u> Since the boundary operators are linear and $C_p(X)$ is generated by singular p-simplices, it suffices to show $\partial_{p-1}(\partial_p(\sigma)) = 0$ when σ is a singular p-simplex.

We compute

$$\partial_{p-1}(\partial_p(\sigma)) = \partial_{p-1}\left(\sum_{i=0}^{p} (-1)^i \sigma \circ F_{i,p}\right)$$

$$= \sum_{j=0}^{p-1} \sum_{i=0}^{p} (-1)^{i+j} \sigma \circ F_{i,p} \circ F_{j,p-1}$$

$$= \sum_{0 \le j < i \le p} (-1)^{i+j} \sigma \circ F_{i,p} \circ F_{j,p-1} + \sum_{0 \le i \le j \le p-1} (-1)^{i+j} \sigma \circ F_{i,p} \circ F_{j,p-1}$$

In this sum replace i with j and j with $i-1$

$$= \sum_{0 \le j < i \le p} (-1)^{i+j} \sigma \circ F_{i,p} \circ F_{j,p-1} + \sum_{0 \le j \le i-1 \le p-1} (-1)^{i+j-1} \sigma \circ F_{j,p} \circ F_{i-1,p-1}$$

$$= \sum_{0 \le j < i \le p} (-1)^{i+j} \sigma \circ F_{i,p} \circ F_{j,p-1} + \sum_{0 \le j < i \le p} (-1)^{i+j-1} \sigma \circ F_{j,p} \circ F_{i-1,p-1}$$

$= 0$ since $F_{i,p} \circ F_{j,p-1} = F_{j,p} \circ F_{i-1,p-1}$. $\square$

We have $\cdots \to C_{p+1}(X) \xrightarrow{\partial_{p+1}} C_p(X) \xrightarrow{\partial_p} C_{p-1}(X) \to \cdots$

Define the <u>p-cycles</u> by $Z_p(X) = \ker(\partial_p) \subseteq C_p(X)$.

Define the <u>p-boundaries</u> by $B_p(X) = \operatorname{im}(\partial_{p+1}) \subseteq C_p(X)$.

By Lemma 13.1 we have

$$B_p(X) = \operatorname{im}(\partial_{p+1}) \subseteq \ker(\partial_p) \subseteq Z_p(X).$$

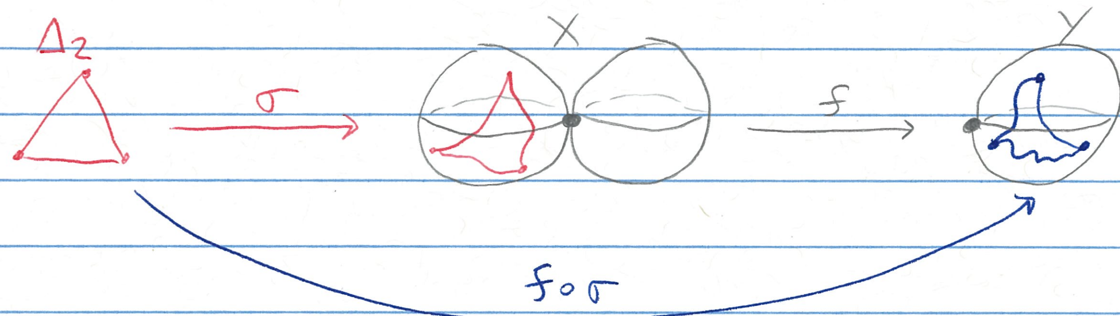
Hence we can define the p-th singular homology group by

$$H_p(X) = Z_p(X)/B_p(X) = \ker(\partial_p)/\operatorname{im}(\partial_{p+1}).$$

11/13/17

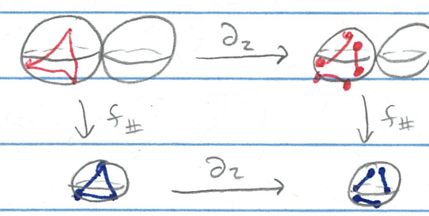
Singular homology is a functor

Let $f: X \to Y$ be a continuous map of topological spaces. This induces a homomorphism $f_\#: C_p(X) \to C_p(Y)$ of chain groups via $f_\#(\sigma) = f \circ \sigma$
(and hence $f_\#(17\sigma - 3\sigma' + 4\sigma'') = 17 f\circ\sigma - 3 f\circ\sigma' + 4 f\circ\sigma''$).



Key fact $f_\# \circ \partial = \partial \circ f_\#$ as maps from $C_p(X)$ to $C_{p-1}(Y)$.

$$\begin{tikzcd}
C_p(X) \arrow[r, "\partial_p"] \arrow[d, "f_\#"] & C_{p-1}(X) \arrow[d, "f_\#"] \\
C_p(Y) \arrow[r, "\partial_p"] & C_{p-1}(Y)
\end{tikzcd}$$



Proof
$$\begin{aligned}
f_\#(\partial\sigma) &= f_\#\left(\textstyle\sum_{i=0}^{p} \sigma \circ F_{i,p}\right) \\
&= \textstyle\sum_{i=0}^{p} f\circ(\sigma\circ F_{i,p}) \\
&= \textstyle\sum_{i=0}^{p} (f\circ\sigma)\circ F_{i,p} \\
&= \partial(f\circ\sigma) \\
&= \partial(f_\#\sigma)
\end{aligned}$$

Key consequences

$f_\#: Z_p(X) \to Z_p(Y)$ since $c \in Z_p(X) \Rightarrow \partial c = 0$
$\Rightarrow \partial(f_\# c) = f_\#(\partial c) = f_\#(0) = 0 \Rightarrow f_\# c \in Z_p(Y)$.

$f_\#: B_p(X) \to B_p(Y)$ since $c \in B_p(X) \Rightarrow c = \partial d$ for some $d \in C_{p+1}(X)$
$\Rightarrow f_\# c = f_\#(\partial d) = \partial(f_\# d) \Rightarrow f_\# c \in B_p(Y)$.

$$\begin{tikzcd} c \arrow[rrr, mapsto] \arrow[dddd, mapsto] & & & 0 \arrow[dddd, mapsto] \\ & C_p(X) \arrow[r, "\partial"] \arrow[d, "f_\#"] & C_{p-1}(X) \arrow[d, "f_\#"] & \\ & C_p(Y) \arrow[r, "\partial"] & C_{p-1}(Y) & \\ & & & \\ f_\# c \arrow[rrr, mapsto] & & & \partial(f_\# c) = 0 \end{tikzcd} \qquad \begin{tikzcd} d \arrow[rrr, mapsto] \arrow[dddd, mapsto] & & & c \arrow[dddd, mapsto] \\ & C_{p+1}(X) \arrow[r, "\partial"] \arrow[d, "f_\#"] & C_p(X) \arrow[d, "f_\#"] & \\ & C_{p+1}(Y) \arrow[r, "\partial"] & C_p(Y) & \\ & & & \\ f_\# d \arrow[rrr, mapsto] & & & \partial(f_\# d) = f_\# c \end{tikzcd}$$

We therefore get an induced homomorphism
$f_*: H_p(X) \to H_p(Y)$ on the quotient, defined by
$f_*(\underbrace{c + B_p(X)}) = f_\# c + B_p(Y)$ for $c \in Z_p(X)$.

often written as $[c]$

(Recall $H_p(X) = Z_p(X)/B_p(X)$ and $H_p(Y) = Z_p(Y)/B_p(Y)$)

Prop 13.2 Singular p-dimensional homology is a functor from the category of topological spaces to the category of (abelian) groups.
A space X is assigned to $H_p(X)$.
A map $f: X \to Y$ is assigned to $f_*: H_p(X) \to H_p(Y)$.

Our book's proof It is easy to check that the necessary properties hold already for $f_\#: C_p(X) \to C_p(Y)$.

More detailed proof Let X be a space.
Given $c + B_p(X) \in H_p(X)$, note
$Id_*(c + B_p(X)) = Id_\# c + B_p(X) = c + B_p(X)$.
Hence $Id_* = Id_{H_p(X)}$, as required.

Let $f: X \to Y$ and $g: Y \to Z$.
Given $c + B_p(X) \in H_p(X)$, note

$$\begin{aligned} g_*(f_*(c + B_p(X))) &= g_*(f_\# c + B_p(Y)) \\ &= g_\#(f_\# c) + B_p(Z) \\ &= (g \circ f)_\#(c) + B_p(Z) \\ &= (g \circ f)_*(c + B_p(X)) \end{aligned}$$

Hence $(g \circ f)_* = g_* \circ f_* : H_p(X) \to H_p(Z)$.

<u>Corollary 13.3</u> If $f: X \to Y$ is a homeomorphism then $f_*: H_p(X) \to H_p(Y)$ is an isomorphism.

<u>Pf</u> Functors take isomorphisms to isomorphisms.

<u>Corollary 13.4</u> If $A \subseteq X$ is a retract of X (meaning there exists $r: X \to A$ with $r \circ \iota_A = id_A$), then $r_*: H_p(X) \to H_p(A)$ is surjective and $(\iota_A)_*: H_p(A) \to H_p(X)$ is injective.

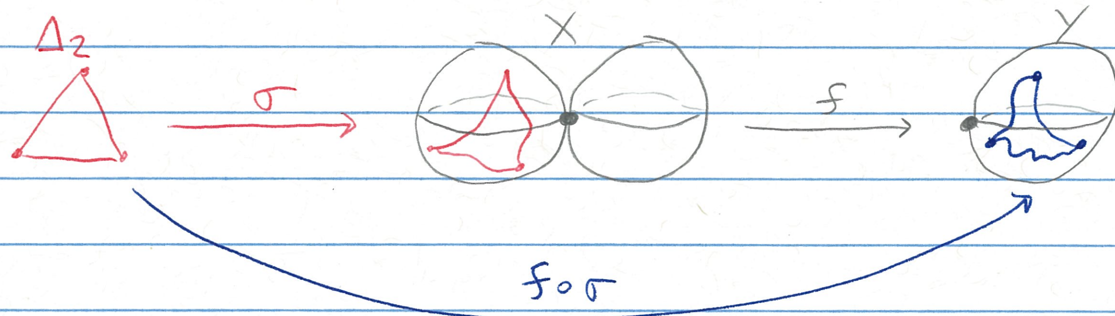
<u>Picture</u>

$$\begin{tikzcd} A \arrow[r, "\iota_A"] \arrow[rr, bend right, "r \circ \iota_A = id_X"'] & X \arrow[r, "r"] & A \end{tikzcd}
\qquad
\begin{tikzcd} H_p(A) \arrow[r, "(\iota_A)_*"] \arrow[rr, bend right, "r_* \circ (\iota_A)_* = id_{H_p(A)}"'] & H_p(X) \arrow[r, "r_*"] & H_p(A) \end{tikzcd}$$

11/13/17

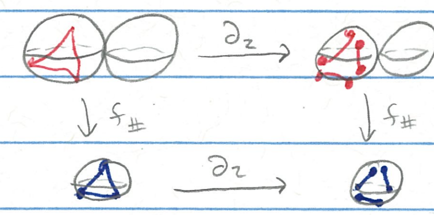
Singular homology is a functor

Let $f: X \to Y$ be a continuous map of topological spaces. This induces a homomorphism $f_\#: C_p(X) \to C_p(Y)$ of chain groups via $f_\#(\sigma) = f \circ \sigma$ (and hence $f_\#(17\sigma - 3\sigma' + 4\sigma'') = 17 f\circ\sigma - 3 f\circ\sigma' + 4 f\circ\sigma''$).



Key fact $f_\# \circ \partial = \partial \circ f_\#$ as maps from $C_p(X)$ to $C_{p-1}(Y)$.

$$\begin{array}{ccc} C_p(X) & \xrightarrow{\partial_p} & C_{p-1}(X) \\ \downarrow f_\# & & \downarrow f_\# \\ C_p(Y) & \xrightarrow{\partial_p} & C_{p-1}(Y) \end{array}$$



Proof
$$\begin{aligned} f_\#(\partial\sigma) &= f_\#\left(\textstyle\sum_{i=0}^{p} (-1)^i \sigma \circ F_{i,p}\right) \\ &= \textstyle\sum_{i=0}^{p} (-1)^i f \circ (\sigma \circ F_{i,p}) \\ &= \textstyle\sum_{i=0}^{p} (-1)^i (f\circ\sigma) \circ F_{i,p} \\ &= \partial(f\circ\sigma) \\ &= \partial(f_\# \sigma) \end{aligned}$$

Key consequences

$f_\#: Z_p(X) \to Z_p(Y)$ since $c \in Z_p(X) \Rightarrow \partial c = 0$
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$$\begin{tikzcd}
c \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & 0 \arrow[ddd, mapsto] \\
& C_p(X) \arrow[r, "\partial"] \arrow[d, "f_\#"] & C_{p-1}(X) \arrow[d, "f_\#"] & \\
& C_p(Y) \arrow[r, "\partial"] & C_{p-1}(Y) & \\
f_\# c \arrow[rrr, mapsto] & & & \partial(f_\# c) = 0
\end{tikzcd}
\qquad
\begin{tikzcd}
d \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & c \arrow[ddd, mapsto] \\
& C_{p+1}(X) \arrow[r, "\partial"] \arrow[d, "f_\#"] & C_p(X) \arrow[d, "f_\#"] & \\
& C_{p+1}(Y) \arrow[r, "\partial"] & C_p(Y) & \\
f_\# d \arrow[rrr, mapsto] & & & \partial(f_\# d) = f_\# c
\end{tikzcd}$$

We therefore get an induced homomorphism $f_*: H_p(X) \to H_p(Y)$ on the quotient, defined by
$$f_*(\underbrace{c + B_p(X)}_{\text{often written as } [c]}) = f_\# c + B_p(Y) \quad \text{for } c \in Z_p(X).$$
(Recall $H_p(X) = Z_p(X)/B_p(X)$ and $H_p(Y) = Z_p(Y)/B_p(Y)$)

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Hence $(g \circ f)_* = g_* \circ f_* : H_p(X) \to H_p(Z)$.

11/15/17

<u>Corollary 13.3</u> If $f: X \to Y$ is a homeomorphism then $f_*: H_p(X) \to H_p(Y)$ is an isomorphism.

<u>Pf</u> Functors take isomorphisms to isomorphisms.

<u>Corollary 13.4</u> If $A \subseteq X$ is a retract of X (meaning there exists $r: X \to A$ with $r \circ \iota_A = id_A$), then $r_*: H_p(X) \to H_p(A)$ is surjective and $(\iota_A)_*: H_p(A) \to H_p(X)$ is injective.

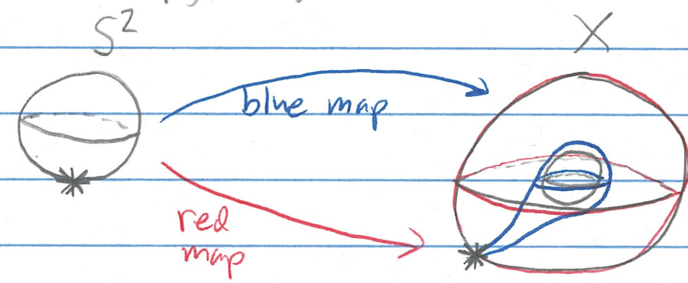
<u>Picture</u>

$$\begin{tikzcd} A \arrow[r, "\iota_A"] \arrow[rr, bend right, "r \circ \iota_A = id_X"'] & X \arrow[r, "r"] & A \end{tikzcd} \qquad \begin{tikzcd} H_p(A) \arrow[r, "(\iota_A)_*"] \arrow[rr, bend right, "r_* \circ (\iota_A)_* = id_{H_p(A)}"'] & H_p(X) \arrow[r, "r_*"] & H_p(A) \end{tikzcd}$$

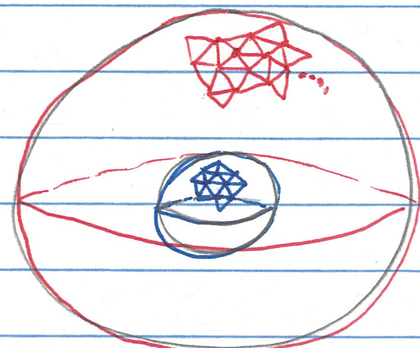
Now that I've played a bit with simplicial and singular homology, how should I think about the homology groups?

This is a 3-manifold with boundary $S^2 \amalg S^2$

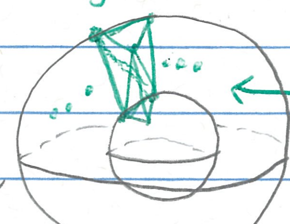
Let $X = \underbrace{S^2 \times I}$ be a hollow coconut.
Recall $\pi_2(X) \cong \pi_2(S^2) \cong \mathbb{Z}$.
The red and blue pointed maps $S^2 \to X$ represent the same element of $\pi_2(X)$ because they are homotopy equivalent (relative basepoints).



It turns out that $H_2(X) \cong H_2(S^2 \times I) \cong \mathbb{Z}$
The red and blue (simplicial or singular) 2-cycles represent the same element of $H_2(X)$ because they are <u>homologous</u>, i.e. their difference is a 2-boundary.



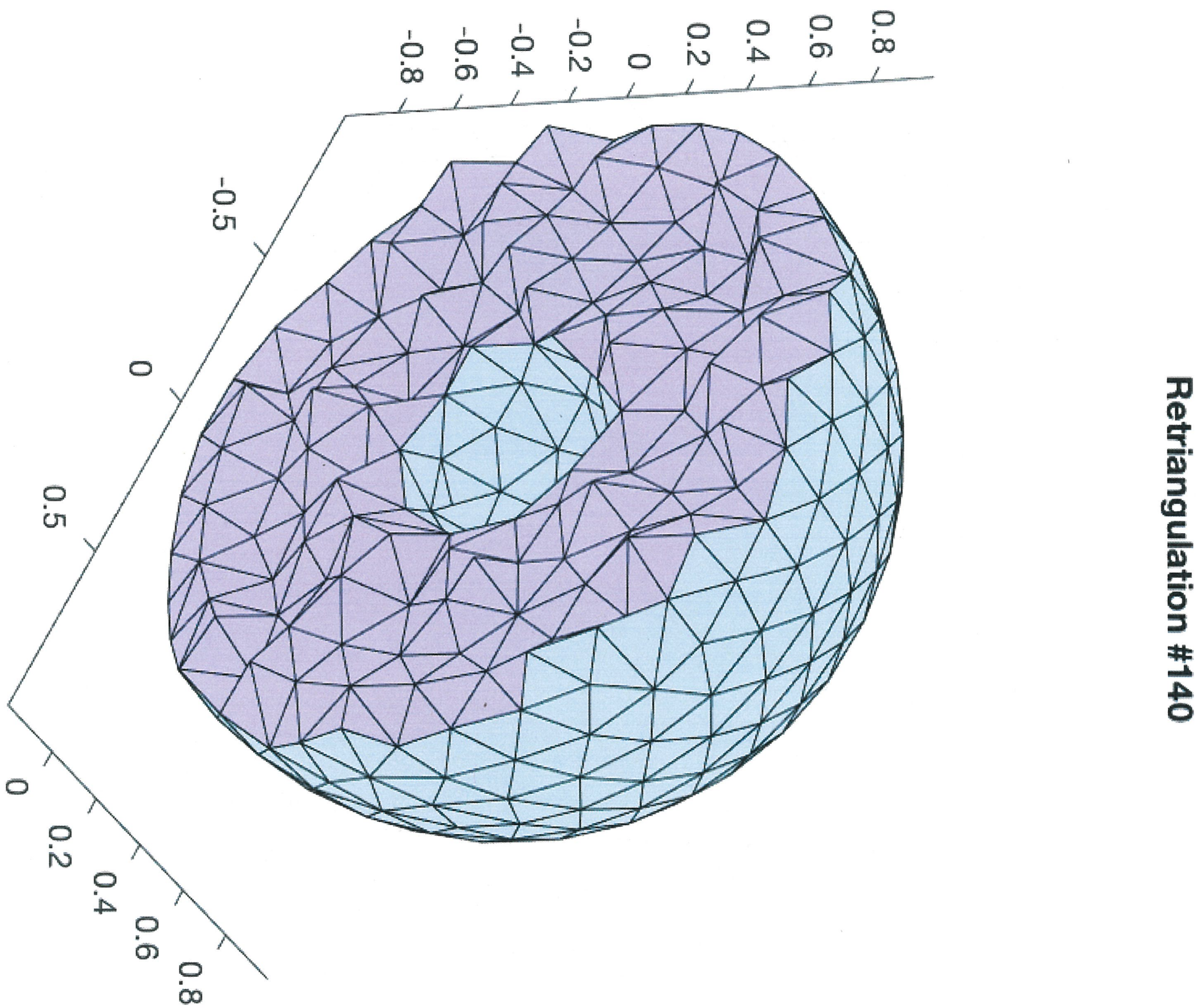
The red 2-cycle minus the blue 2-cycle is the boundary of the green 3-chain



filled with tetrahedra, all the way around.

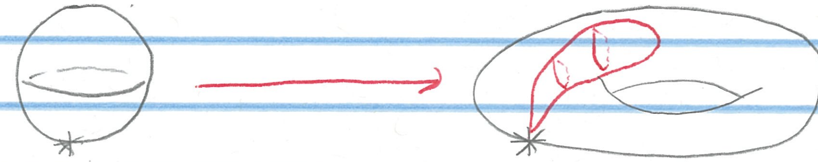
From the perspective of either π_2 or H_2, X has a "single 2-dimensional hole".

Here's a picture of a triangulation of the "hollow coconut" $X = S^2 \times I$ we considered on the previous page of notes.



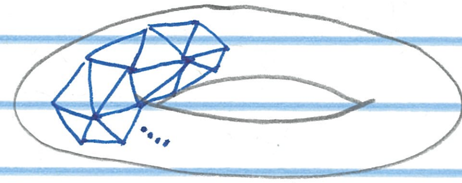
Homotopy and homology groups often can't come to an agreement whether they think there's a hole or not!

$\pi_2(S^1 \times S^1) \cong \pi_2(S^1) \times \pi_2(S^1) \cong$ trivial group



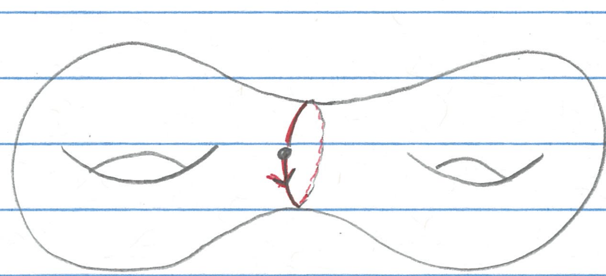
Homotopy doesn't think there's a 2-dimensional hole in the torus since any pointed map $S^2 \to S^1 \times S^1$ is homotopy equivalent to a constant map.

$H_2(S^1 \times S^1) \cong \mathbb{Z}$ since $S^1 \times S^1$ is a compact connected orientable 2-manifold without boundary.



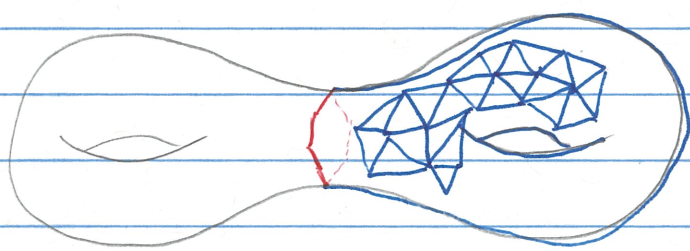
The blue 2-cycle above is not the boundary of any 3-chain, so it represents a nontrivial element of $H_2(S^1 \times S^1)$, and in fact generates $H_2(S^1 \times S^1) \cong \mathbb{Z}$.

Fact More generally, if M is a compact connected orientable n-manifold with boundary, then $H_n(M) \cong \mathbb{Z}$.

$X =$ 

The loop drawn above is a nontrivial element of $\pi_1(X) \cong \langle a, b, c, d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} \rangle$ since it's not homotopy equivalent to the constant loop.

This loop corresponds to the element $aba^{-1}b^{-1}$ or $dcd^{-1}c^{-1}$.

$X =$ 

The red 1-cycle drawn above is a trivial element of $H_1(X) \cong \mathbb{Z}^4$ since it's the boundary of the blue 2-chain.

The projective plane example $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}/2\mathbb{Z}$ and $H_1(\mathbb{R}P^2) \cong \mathbb{Z}/2\mathbb{Z}$ show that the "number of i-dimensional holes" is an imprecise notion that isn't made fully precise by either homotopy groups or homology groups.

11/17/17 Recall from HW 11 #1(b) that a sequence of abelian groups and group homomorphisms

$$\cdots \to G_{p+1} \xrightarrow{\alpha_{p+1}} G_p \xrightarrow{\alpha_p} G_{p-1} \to \cdots$$

is <u>exact</u> if $\operatorname{im}(\alpha_{p+1}) = \ker(\alpha_p)$ $\forall p$.

<u>Thm 13.16</u> (Mayer-Vietoris Theorem) Let X be a topological space, and let U, V be open subsets of X with $X = U \cup V$

$$\begin{tikzcd} & U \arrow[dr, hook, "k"] & \\ U \cap V \arrow[ur, hook, "i"] \arrow[dr, hook, "j"'] & & X \\ & V \arrow[ur, hook, "l"'] & \end{tikzcd}$$

This map ∂_* is related to the boundary maps $\partial: C_p(Y) \to C_{p-1}(Y)$, but the relationship is complicated.

Then $\forall p$ there is a homomorphism $\partial_*: H_p(X) \to H_{p-1}(U \cap V)$ such that the following sequence is exact:

$$\xrightarrow{\partial_*} H_p(U \cap V) \xrightarrow{i_* \oplus j_*} H_p(U) \oplus H_p(V) \xrightarrow{k_* - l_*} H_p(X)$$

$$\xrightarrow{\partial_*} H_{p-1}(U \cap V) \xrightarrow{i_* \oplus j_*} H_{p-1}(U) \oplus H_{p-1}(V) \xrightarrow{k_* - l_*} H_{p-1}(X)$$

$$\xrightarrow{\partial_*} H_{p-2}(U \cap V) \xrightarrow{i_* \oplus j_*}$$

$\partial_*: H_p(X) \to H_{p-1}(U \cap V)$ is called the <u>connecting homomorphism</u>.

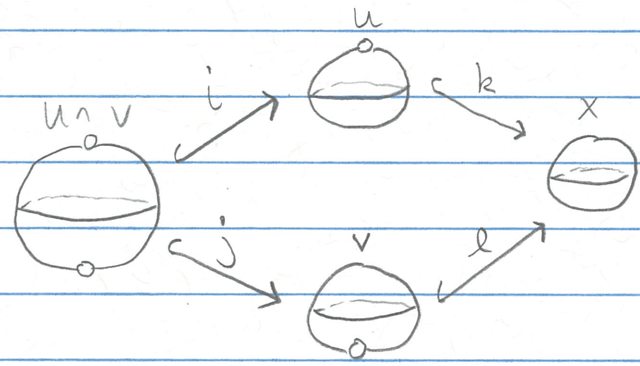
The other maps are defined

$(i_* \oplus j_*)[c] = (i_*[c], j_*[c])$ for $[c] \in H_p(U \cap V)$ and

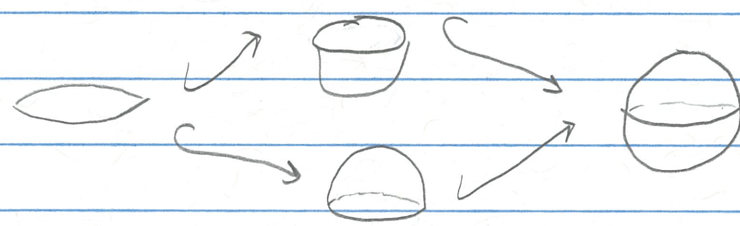
$(k_* - l_*)([c],[c']) = k_*[c] - l_*[c']$ for $[c] \in H_p(U)$ and $[c'] \in H_p(V)$.

Rmk This theorem allows us to understand the homology of $X = U \cup V$ from the homologies of U, of V, and of $U \cap V$.

Ex Let $X = S^2$ with north and south poles $N, S \in S^2$.
Let $U = S^2 \setminus \{N\} \simeq *$ (U is homotopy equivalent to a point)
Let $V = S^2 \setminus \{S\} \simeq *$
So $U \cap V = S^2 \setminus \{N, S\} \simeq S^1$



A homotopy equivalent picture is



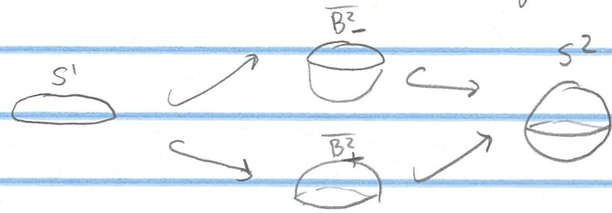
The Mayer-Vietoris Theorem gives an exact sequence

$$\begin{array}{ccccc} 0 & & 0 & & \\ \shortparallel\wr & & \shortparallel\wr & & \\ H_2(U\cap V) & \longrightarrow & H_2(U)\oplus H_2(V) & \longrightarrow & H_2(X) \end{array}$$

$$\begin{array}{ccccc} \xrightarrow{\partial_*} H_1(U\cap V) & \longrightarrow & H_1(U)\oplus H_1(V) & \longrightarrow & \cdots \\ \wr\shortparallel & & \wr\shortparallel & & \\ \mathbb{Z} & & 0 & & \end{array}$$

Let's isolate our attention to $0 \to H_2(X) \xrightarrow{\partial_*} H_1(U\cap V) \to 0$.
Exactness at $H_2(X) \Rightarrow \partial_*$ injective
Exactness at $H_1(U\cap V) \Rightarrow \partial_*$ surjective
Hence ∂_* is an isomorphism showing
$H_2(S^2) = H_2(X) \cong H_1(U\cap V) \cong \mathbb{Z}$.

Rmk We used $H_1(S^1) \cong \mathbb{Z}$ to get $H_2(S^2) \cong \mathbb{Z}$ via



One could use $H_2(S^2) \cong \mathbb{Z}$ to get $H_3(S^3) \cong \mathbb{Z}$ via

$$
\begin{tikzcd}
 & \overline{B^3_-} \arrow[dr, hook] & \\
S^2 \arrow[ur, hook] \arrow[dr, hook] & & S^3 \\
 & \overline{B^3_+} \arrow[ur, hook] &
\end{tikzcd}
$$

Inductively, one could use $H_{n-1}(S^{n-1}) \cong \mathbb{Z}$ to get $H_n(S^n) \cong \mathbb{Z}$ via

$$
\begin{tikzcd}
 & \overline{B^n_-} \arrow[dr, hook] & \\
S^{n-1} \arrow[ur, hook] \arrow[dr, hook] & & S^n \\
 & \overline{B^n_+} \arrow[ur, hook] &
\end{tikzcd}
$$

The Mayer-Vietoris Theorem is proven using...

Lemma 13.17 The Zigzag (or Snake) Lemma

Let $0 \to C_* \xrightarrow{F} D_* \xrightarrow{G} E_* \to 0$

be a short exact sequence of chain complexes:

$$\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C_{p+1} & \xrightarrow{F} & D_{p+1} & \xrightarrow{G} & E_{p+1} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C_p & \xrightarrow{F} & D_p & \xrightarrow{G} & E_p & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C_{p-1} & \xrightarrow{F} & D_{p-1} & \xrightarrow{G} & E_{p-1} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & &
\end{array}$$

"short exact sequence of chain complexes"

(C_*, D_*, and E_* being chain complexes means the composition of any two vertical arrows is zero)

(Short exact here means that each row is short exact, as in HW 11 #1(b).)

Then $\forall p$ there is a connecting homomorphism $\partial_* : H_p(E) \to H_{p-1}(C)$ such that the following sequence is exact:

$$\cdots \xrightarrow{\partial_*} H_p(C_*) \longrightarrow H_p(D_*) \longrightarrow H_p(E_*) \xrightarrow{\partial_*} H_{p-1}(C_*) \longrightarrow H_{p-1}(D_*) \longrightarrow H_{p-1}(E_*) \to \cdots$$

"long exact sequence of homology groups"

In English: A short exact sequence of chain complexes gives a long exact sequence of homology groups.

11/27/2017 **Homotopy Invariance (of Singular Homology)**

Thm 13.8 If $f_0, f_1 : X \to Y$ are homotopic maps, then for each $p \geq 0$ the induced homomorphisms $(f_0)_*, (f_1)_* : H_p(X) \to H_p(Y)$ are equal.

(Recall that the corresponding result for fundamental groups, Lemma 7.45, had a mess of basepoint issues. We don't have that here; we have equality on the nose!)

Corollary 13.9 Suppose $f : X \to Y$ is a homotopy equivalence. Then $f_* : H_p(X) \to H_p(Y)$ is an isomorphism for each $p \geq 0$.

"Homotopy equivalent spaces have isomorphic homology groups"

Pf

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd} \qquad \begin{tikzcd} H_p(X) \arrow[r, bend left, "f_*"] & H_p(Y) \arrow[l, bend left, "g_*"] \end{tikzcd}$$

Note $g \circ f \simeq id_X$ gives

$$id_{H_p(X)} \underset{\text{functoriality}}{=} (id_X)_* \underset{\text{Thm 13.8}}{=} (g \circ f)_* \underset{\text{functoriality}}{=} g_* \circ f_*$$

and $f \circ g \simeq id_Y$ gives

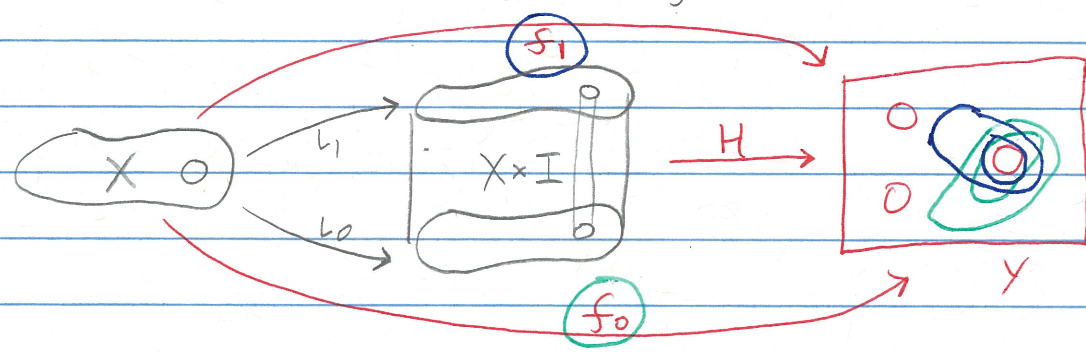
$$id_{H_p(Y)} = (id_Y)_* = (f \circ g)_* = f_* \circ g_*.$$

Hence f_* is an isomorphism (with inverse g_*).

<u>Pf of Thm 13.8</u> By the magic of functoriality, it will suffice to prove the <u>very</u> special case where

- $Y = X \times I$
- $f_0 = \iota_0 : X \to X \times I$ is given by $\iota_0(x) = (x, 0)$, and
- $f_1 = \iota_1 : X \to X \times I$ is given by $\iota_1(x) = (x, 1)$.

(Clearly $\iota_0 \simeq \iota_1$)



Indeed, suppose Y is arbitrary and $f_0, f_1 : X \to Y$ satisfy $f_0 \simeq f_1$. Then there is a homotopy $H : X \times I \to Y$ with $H \circ \iota_0 = f_0$ and $H \circ \iota_1 = f_1$.

Functoriality then gives

$$(f_0)_* = (H \circ \iota_0)_* = H_* \circ (\iota_0)_* \underset{\substack{\uparrow \\ \text{by the very special case}}}{=} H_* \circ (\iota_1)_* = (H \circ \iota_1)_* = (f_1)_*.$$

So it suffices to consider this special case and prove $(\iota_0)_* = (\iota_1)_*$.

<u>Aside</u> To prove $(\iota_0)_* = (\iota_1)_* : H_p(X) \to H_p(X \times I)$ it would be enough to define a map

$h : Z_p(X) \to C_{p+1}(X \times I)$ such that

$\partial h(c) = (\iota_1)_\# c - (\iota_0)_\# c$ for all $c \in Z_p(X)$.

Indeed, we'd then have $(\iota_0)_\# c + B_p(X \times I) = (\iota_1)_\# c + B_p(X \times I)$, meaning $(\iota_0)_*[c] = (\iota_1)_*[c]$ for all $[c] \in H_p(X)$, i.e. meaning $(\iota_0)_* = (\iota_1)_* : H_p(X) \to H_p(X \times I)$.

Instead We'll instead define a homomorphism
$h: C_p(X) \to C_{p+1}(X \times I)$ for all $p \geq 0$ satisfying
$h \circ \partial + \partial \circ h = (\iota_1)_\# - (\iota_0)_\#$.

$$\begin{tikzcd}
\cdots \arrow[r] & C_{p+1}(X) \arrow[r, "\partial"] \arrow[dl] & C_p(X) \arrow[r, "\partial"] \arrow[dl, "h"'] \arrow[d, shift right, "(\iota_0)_\#"'] \arrow[d, shift left, "(\iota_1)_\#"] & C_{p-1}(X) \arrow[r] \arrow[dl, "h"'] & \cdots \arrow[dl] \\
\cdots \arrow[r] & C_{p+1}(X \times I) \arrow[r, "\partial"] & C_p(X \times I) \arrow[r, "\partial"] & C_{p-1}(X \times I) \arrow[r] & \cdots
\end{tikzcd}$$

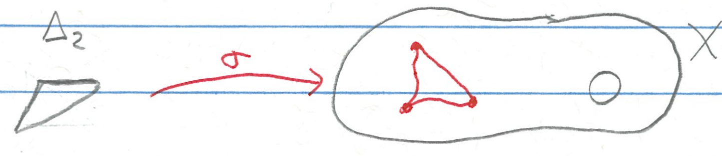
(h is called a chain homotopy from $(\iota_0)_\#$ to $(\iota_1)_\#$)

11/30/17 For $c \in Z_p(X)$ we'll then have

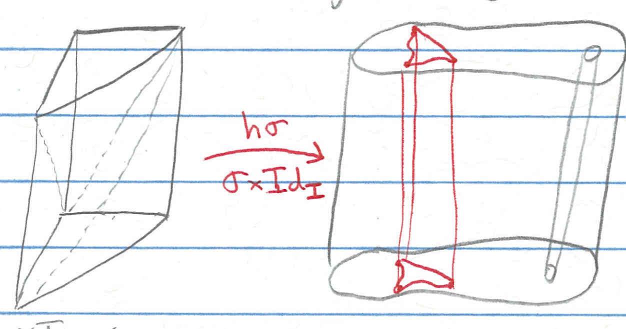
$$\begin{aligned}(\iota_1)_\# c - (\iota_0)_\# c &= h(\partial c) + \partial h(c) \\ &= \partial h(c) \quad \text{since } \partial c = 0.\end{aligned}$$

For the same reason as in the aside, this gives
$(\iota_0)_* = (\iota_1)_* : H_p(X) \to H_p(X \times I)$.

Pic of h Let $\sigma: \Delta_p \to X$ be a singular p-simplex in $C_p(X)$.



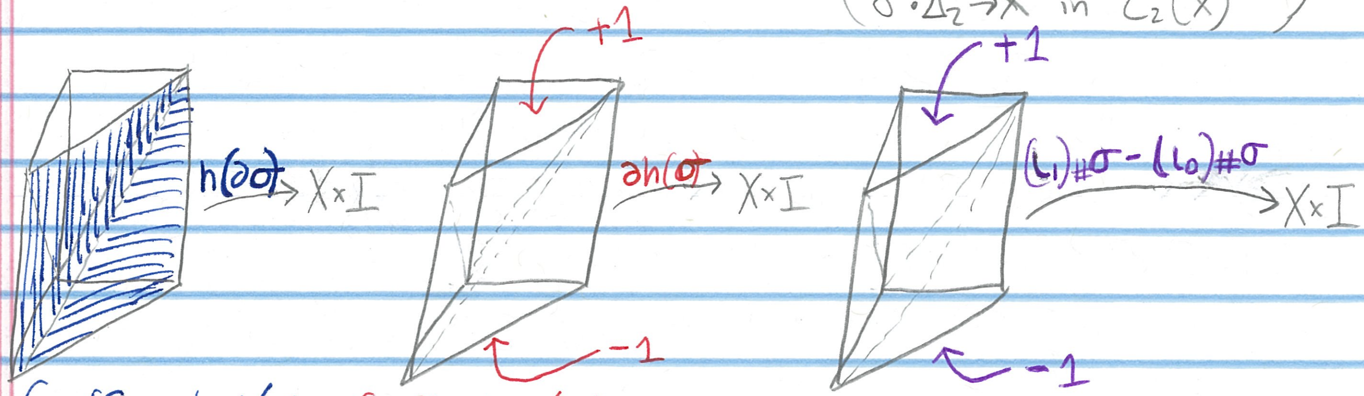
Then $h\sigma$ will be a singular $(p+1)$-chain in $C_{p+1}(X \times I)$.



$\Delta_2 \times I$ is a sum of 3-simplices

$X \times I$

Pic of why $h\circ\partial+\partial\circ h=(\iota_1)_\#-(\iota_0)_\#$ (for a singular 2-simplex $\sigma:\Delta_2\to X$ in $C_2(X)$)



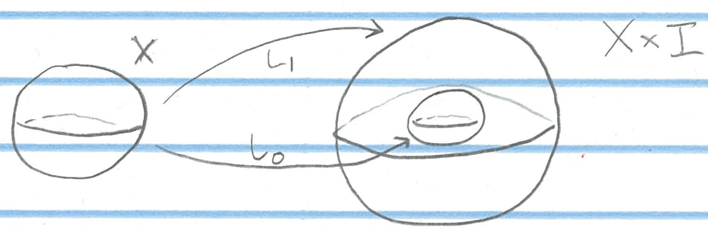
Coefficients +/−1 on all six side triangles.

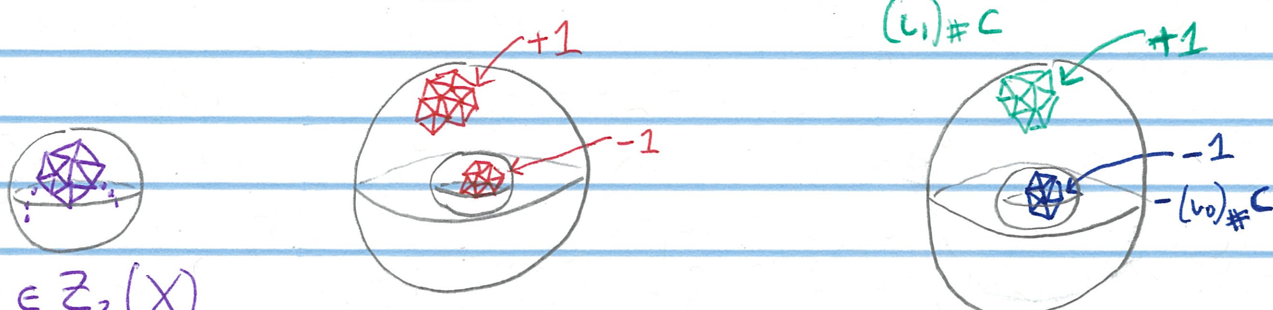
Coefficients −/+1 on all six side triangles, and +1 on top, −1 on bottom.

Coefficient +1 on top and −1 on bottom.

One can check that in $h(\partial\sigma)+\partial h(\sigma)$, the coefficients on the six side triangles cancel, but we won't!

Pic of $\partial h(c)=(\iota_1)_\# c-(\iota_0)_\# c$ for $c\in Z_2(X)$



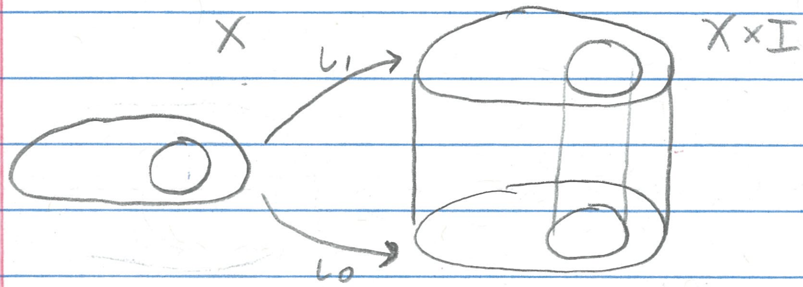


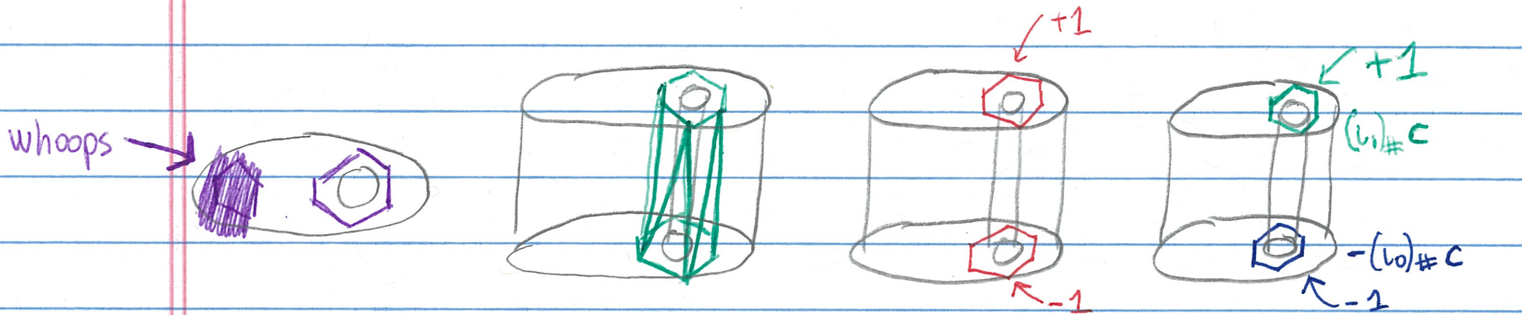
$c\in Z_2(X)$

$\partial h(c)\in C_2(X\times I)$

$(\iota_1)_\# c-(\iota_0)_\# c\in C_2(X\times I)$

Pic of $\partial h(c) = (\iota_1)_\# c - (\iota_0)_\# c$ for $c \in Z_1(X)$





$c \in Z_1(X)$ $\quad h(c) \in C_2(X \times I)$ $\quad \partial h(c) \in C_1(X \times I)$ $\quad (\iota_1)_\# c - (\iota_0)_\# c \in C_1(X \times I)$

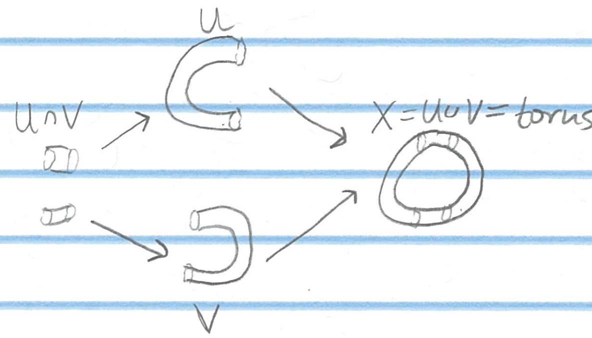
12/1/2017

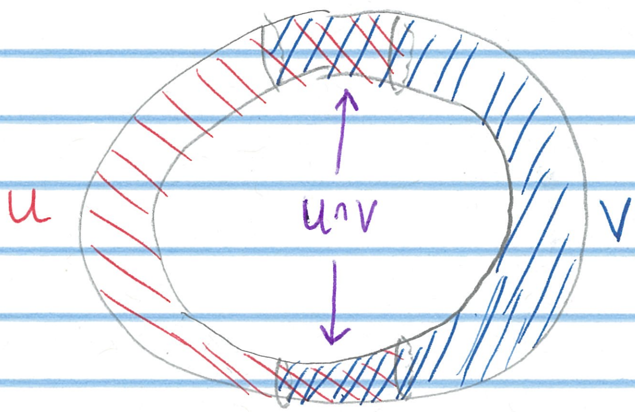
Ex Write the torus as the union of two cylinders and use the Mayer-Vietoris Theorem to deduce H_2(torus)

Rmk Since the torus is an orientable 2-manifold, we know the correct answer will be $H_2(\text{torus}) \cong \mathbb{Z}$.

Solution

$$\begin{tikzcd} & U \arrow[dr, "k"] & \\ U \cap V \arrow[ur, "i"] \arrow[dr, "j"'] & & X = U \cup V \\ & V \arrow[ur, "\ell"'] & \end{tikzcd}$$

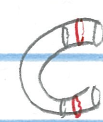




$U \cap V \simeq S^1 \amalg S^1$	$H_1(U \cap V) \cong \mathbb{Z} \oplus \mathbb{Z}$	$H_2(U \cap V) = 0$
$U \simeq S^1$	$H_1(U) \cong \mathbb{Z}$	$H_2(U) = 0$
$V \simeq S^1$	$H_1(V) \cong \mathbb{Z}$	$H_2(V) = 0$

$i_* : H_1(U \cap V) \longrightarrow H_1(U)$ by

$(a,b) \longmapsto a+b.$



$j_* : H_1(U \cap V) \longrightarrow H_1(V)$ by

$(a,b) \longmapsto a+b.$

By Mayer-Vietoris (Thm 13.16) we have a long exact sequence

$$\overset{0}{\overset{\shortparallel}{H_2(U\cap V)}} \xrightarrow{i_*\oplus j_*} \overset{0}{\overset{\shortparallel}{H_2(U)\oplus H_2(V)}} \xrightarrow{k_*-\ell_*} H_2(X)$$

$$\xrightarrow{\partial_*} \underset{\mathbb{Z}\oplus\mathbb{Z}}{\underset{\cong}{H_1(U\cap V)}} \xrightarrow{i_*\oplus j_*} \underset{\mathbb{Z}\oplus\mathbb{Z}}{\underset{\cong}{H_1(U)\oplus H_1(V)}} \longrightarrow \cdots$$

Note ∂_* is injective since $\ker(\partial_*) = \operatorname{im}(k_*-\ell_*) = 0$

So $H_2(\text{torus}) = H_2(X) \cong \operatorname{im} \partial_*$ (by 1^{st} isomorphism theorem)

$= \ker(i_*\oplus j_*)$ (by exactness at $H_1(U\cap V)$)

$\cong \mathbb{Z}$

Since $i_*\oplus j_* : \mathbb{Z}\oplus\mathbb{Z} \to \mathbb{Z}\oplus\mathbb{Z}$ has a 1-dimensional kernel $\ker(i_*\oplus j_*) = \{(a,-a) \mid a\in\mathbb{Z}\} \cong \mathbb{Z}$.

Ex Use this same gluing to deduce $H_1(\text{torus})$.

Rmk We know $H_1(\text{torus}) \cong \mathbb{Z}\oplus\mathbb{Z}$ since H_1 is the abelianization of π_1, and $\pi_1(\text{torus}) \cong \mathbb{Z}\oplus\mathbb{Z}$.

Solution We have a long exact sequence

kernel $\{(a,-a) \mid a \in \mathbb{Z}\}$ kernel $\{(a,a) \mid a \in \mathbb{Z}\}$

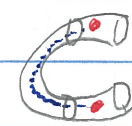
image $\{(a,a) \mid a \in \mathbb{Z}\}$

$$\begin{tikzcd}
\mathbb{Z}\oplus\mathbb{Z} \arrow[d, phantom, "\cong"] & & \mathbb{Z}\oplus\mathbb{Z} \arrow[d, phantom, "\cong"] & & \\
H_1(U\cap V) \arrow[r, "(i_*, j_*)"] & & H_1(U)\oplus H_1(V) \arrow[r, "k_* - l_*"] & H_1(X) \arrow[dll, "\partial_*"] & \\
H_0(U\cap V) \arrow[r, "(i_*, j_*)"] \arrow[d, phantom, "\cong"] & & H_0(U)\oplus H_0(V) \arrow[r] \arrow[d, phantom, "\cong"] & {} & \\
\mathbb{Z}\oplus\mathbb{Z} & & \mathbb{Z}\oplus\mathbb{Z} & &
\end{tikzcd}$$

kernel $\{(a,-a) \mid a \in \mathbb{Z}\}$

$i_* : H_0(U\cap V) \longrightarrow H_0(U)$ by
$(a,b) \longmapsto a+b$

$j_* : H_0(U\cap V) \longrightarrow H_0(V)$ by
$(a,b \longmapsto a+b$



Note
$$\begin{aligned}
\operatorname{im}(\partial_*) &= \ker\left(i_* \oplus j_* : H_0(U\cap V) \to H_0(U)\oplus H_0(V)\right) \\
&= \{(a,-a) \mid a \in \mathbb{Z}\} \\
&\cong \mathbb{Z}.
\end{aligned}$$

Note
$$\begin{aligned}
\ker(\partial_*) &= \operatorname{im}(k_* - l_*) \\
&\cong \left(H_1(U)\oplus H_1(V)\right)/\ker(k_* - l_*) \quad \text{by } 1^{st} \text{ iso thm} \\
&\cong (\mathbb{Z}\oplus\mathbb{Z})/\{(a,a) \mid a \in \mathbb{Z}\} \\
&\cong \mathbb{Z}.
\end{aligned}$$

By the 1^{st} iso thm we have $H_1(X)/\ker(\partial_*) \cong \operatorname{im}(\partial_*)$ with $\ker(\partial_*) \cong \mathbb{Z}$ and $\operatorname{im}(\partial_*) \cong \mathbb{Z}$.
In general these "extension problems" are hard to solve, but in this case we have $H_1(X) \cong \mathbb{Z}\oplus\mathbb{Z}$.

12/4/17 **Degree Theory for Spheres**

Algebra fact Any group homomorphism $h: \mathbb{Z} \to \mathbb{Z}$ is multiplication by some fixed integer d.

Ex

$$\vdots$$
$$h(-1) = -3$$
$$h(0) = 0$$
$$h(1) = 3$$
$$h(2) = 6$$
$$\vdots$$

Ex

$$\vdots$$
$$h(-1) = 8$$
$$h(0) = 0$$
$$h(1) = -8$$
$$h(2) = -16$$
$$\vdots$$

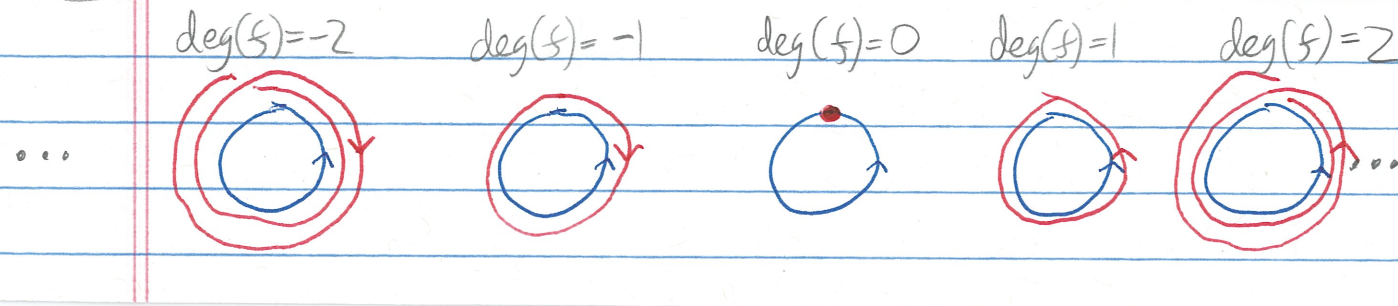
Proof Let $d = h(1)$. Note for any $m \in \mathbb{Z}$ we have
$$h(m) = h(\underbrace{1 + \cdots + 1}_{m \text{ times}}) = \underbrace{h(1) + \cdots + h(1)}_{m \text{ times}} = m \cdot h(1).$$

Let $n \geq 1$. Recall $H_n(S^n) \cong \mathbb{Z}$.

Def We define the degree of a continuous map $f: S^n \to S^n$ to be the unique integer d such that the induced homomorphism $f_*: \underset{\mathbb{Z}}{\underset{\cong}{H_n(S^n)}} \longrightarrow \underset{\mathbb{Z}}{\underset{\cong}{H_n(S^n)}}$ is multiplication by d.

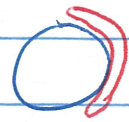
This integer is denoted $\deg(f) = d$.

Ex $f: S^1 \longrightarrow S^1$

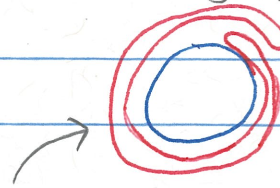


Ex $f: S^1 \to S^1$

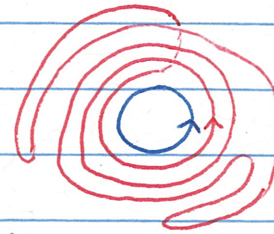
$\deg(f) = 0$



$\deg(f) = 0$



$\deg(f) = 3$



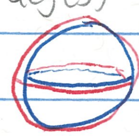
Thm 13.28 will say $\deg(f) \neq 0 \Rightarrow f$ surjective.
This picture shows the converse is false!

Ex $f: S^2 \to S^2$

$\deg(f) = 0$

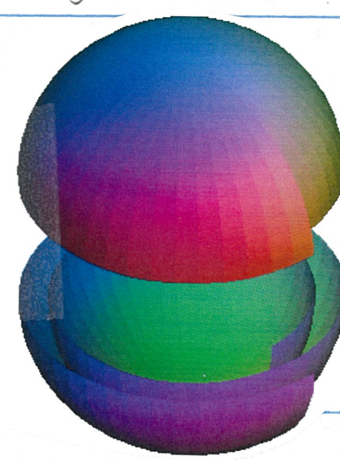
Constant map to a point

$\deg(f) = 1$



Identity map

$\deg(f) = 2$



This map cuts the globe at the prime meridian (say), and then wraps the surface of the earth around twice, fixing the north and south poles.

As Alex looked up for us,
Fort Collins (40.59° latitude, 255.08° longitude) and
At-Bashi, Kyrgyzstan (40.59°, 75.08°) get mapped on top of each other under this map.

Prop 13.25 If $f, g : S^n \to S^n$ are continuous, then

(a) $\deg(g \circ f) = \deg(g)\deg(f)$.

(b) If $f \simeq g$, then $\deg(f) = \deg(g)$

Pf Part (a) follows from $(g\circ f)_* = g_* \circ f_*$.

[Indeed, for any integer $m \in \mathbb{Z}$ we have

$$\begin{aligned}(g\circ f)_*(m) &= g_*(f_*(m)) \\ &= g_*(\deg(f)\, m) \\ &= \deg(g)\deg(f)\, m.\end{aligned}$$
]

Part (b) follows since $f \simeq g$ implies $f_* = g_* : H_n(S^n) \to H_n(S^n)$ (by Thm 13.8).

Picture of (b) $f \simeq g$

$\deg(f) = 1$ $\quad$ $\deg(g) = 1$

Rmk (b) is in fact an $\Leftrightarrow$, but the proof of the reverse direction requires more machinery.

Prop 13.27 Degrees of some common maps.

(a) The identity map $id : S^n \to S^n$ has degree 1.

(b) A constant map $c : S^n \to S^n$ has degree zero.

(c) A reflection map $R : S^n \to S^n$ (about an n-dimensional hyperplane through the origin in $\mathbb{R}^{n+1}$) has degree -1.

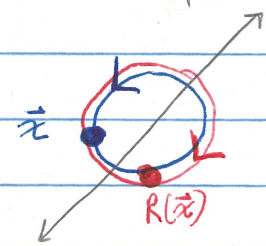
(d) The antipodal map $\alpha : S^n \to S^n$ given by $\alpha(\vec{x}) = -\vec{x}$ has degree $(-1)^{n+1}$.

Pf (a) follows since $id_*: H_n(S^n) \to H_n(S^n)$ is the identity homomorphism.

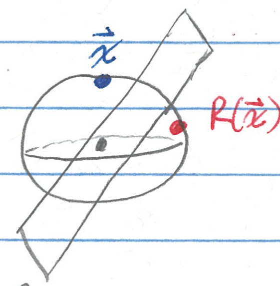
(b) follows since if $c: S^n \to S^n$ is a constant map (meaning $\exists \vec{p} \in S^n$ with $c(\vec{x}) = \vec{p}$ $\forall \vec{x} \in S^n$), then $c_*: H_n(S^n) \to H_n(S^n)$ maps every element to zero.

Instead of proving (c), we'll draw a picture:

n=1



n=2



Reflecting through a hyperplane "turns the sphere inside-out".

(d) follows from (c). Indeed, for $1 \le i \le n+1$ let $R_i: S^n \to S^n$ be the reflection given by

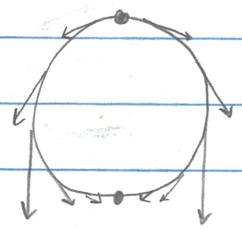
$$R_i(x_1, \ldots, x_i, \ldots, x_{n+1}) = (x_1, \ldots, -x_i, \ldots, x_{n+1}).$$

Since $\alpha(x_1, \ldots, x_{n+1}) = (-x_1, \ldots, -x_{n+1})$, this means $\alpha = R_{n+1} \circ R_n \circ \ldots \circ R_2 \circ R_1$.

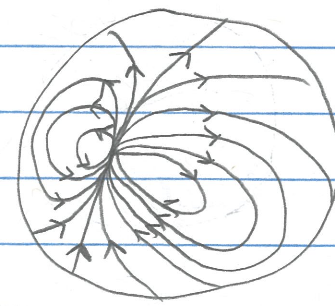
Hence
$$\begin{aligned} \deg(\alpha) &= \deg(R_{n+1}) \deg(R_n) \cdot \ldots \cdot \deg(R_1) && \text{(Prop 13.25(a))} \\ &= (-1) \cdot (-1) \cdot \ldots \cdot (-1) && \text{by (c)} \\ &= (-1)^{n+1}. \end{aligned}$$

12/6/17 **Def** A vector field on S^n is a continuous map $V: S^n \to \mathbb{R}^{n+1}$ such that $V(\vec{x})$ is tangent to S^n at $\vec{x}$ (i.e. $V(\vec{x}) \cdot \vec{x} = 0$) for all $\vec{x} \in S^n$.

Picture $n=1$



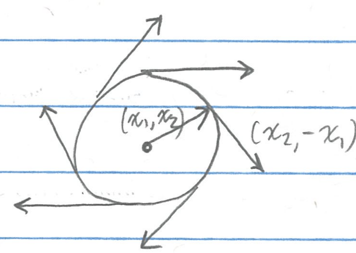
$n=2$



Thm 13.32 (The Hairy Ball Theorem)

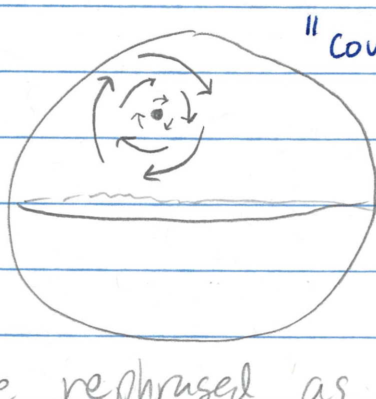
There exists a nowhere vanishing vector field on S^n if and only if n is odd.

Picture $n=1$ You can comb a hairy circle.

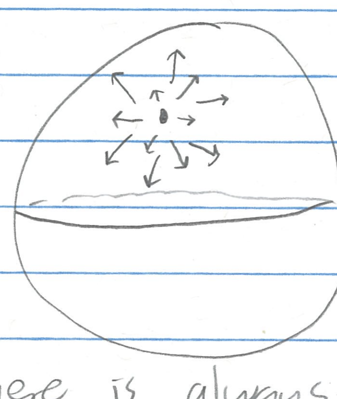


$n=2$ You can't comb a hairy ball.

"cowlick"



"source"



This can be rephrased as saying there is always a spot on the earth where the (tangential) wind speed is zero!

Proof

When $n=2k-1$ is odd, then $S^n \subseteq \mathbb{R}^{2k}$.
The following vector field is tangent to the sphere and nowhere vanishing:

$$V(x_1, x_2, \ldots, x_{2k-1}, x_{2k}) = (x_2, -x_1, x_4, -x_3, \ldots, x_{2k}, -x_{2k-1}).$$

Now let n be even. Suppose for a contradiction that we had a nowhere vanishing vector field V on S^n.
We may assume $\|V(\vec{x})\| = 1$ for all $\vec{x} \in S^n$ (simply replace $\vec{x} \longmapsto V(\vec{x})$ with $\vec{x} \longmapsto V(\vec{x})/\|V(\vec{x})\|$).
We use V to construct a homotopy between the identity map $id: S^n \to S^n$ and the antipodal map $\alpha: S^n \to S^n$ (with $\alpha(\vec{x}) = -\vec{x}$) as follows:

$$H: S^n \times I \longrightarrow S^n \quad \text{via}$$

$$H(\vec{x}, t) = \cos(\pi t)\vec{x} + \sin(\pi t) V(\vec{x}).$$

This map is well-defined (lands in S^n) since

$$\|H(\vec{x},t)\|^2 = \cos^2(\pi t) \underbrace{\|\vec{x}\|^2}_{1} + 2\cos(\pi t)\sin(\pi t) \underbrace{\vec{x} \cdot V(\vec{x})}_{0} + \sin^2(\pi t) \underbrace{\|V(\vec{x})\|^2}_{1}$$

$$= \cos^2(\pi t) + \sin^2(\pi t)$$

$$= 1 \quad \text{for all } \vec{x} \in S^n \text{ and } t \in I.$$

The continuity of this map follows from the continuity of V.
Finally, note $H(\vec{x}, 0) = \vec{x} = id(\vec{x})$
and $H(\vec{x}, 1) = -\vec{x} = \alpha(\vec{x})$.

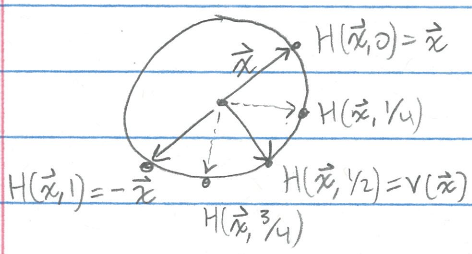
However, this homotopy between the identity and antipodal maps contradicts Prop 13.25 (b) and 13.27 (d), since

$$\deg(\alpha) = (-1)^{n+1} \underset{\substack{\uparrow \\ \text{since } n \text{ is even}}}{=} (-1) \neq 1 = \deg(id).$$

Hence for n even there cannot be a nowhere vanishing vector field on S^n. □

<u>Remark</u> When $n=2k-1$ is odd, the homotopy H can be combined with the vector field
$V(x_1, \ldots, x_{2k}) = (x_2, -x_1, x_4, -x_3, \ldots, x_{2k}, -x_{2k-1})$
to get a homotopy $id \simeq \alpha : S^n \to S^n$

<u>Pic $n=1$</u> $V(x_1, x_2) = (x_2, -x_1)$



$$H(\vec{x},t) = \cos(\pi t)\vec{x} + \sin(\pi t)V(\vec{x})$$
$$H(\vec{x},0) = \vec{x} = id(\vec{x})$$
$$H(\vec{x},\tfrac{1}{2}) = V(\vec{x})$$
$$H(\vec{x},1) = -\vec{x} = \alpha(\vec{x})$$

This essentially gives a proof of...

<u>Prop 13.31</u> The antipodal map $\alpha : S^n \to S^n$ is homotopic to the identity map if and only if n is odd.