CSU Math 301 **Practice Final A**

Remarks: The final exam will be comprehensive. The questions on this practice final are roughly ordered according to when we learned about them; this will not be the case for the actual final. Certainly not all topics from our class are represented on this practice final.

Name: ________________________________

- Unless stated otherwise, explain your logic and write out complete sentences.
- No notes, books, calculators, or other electronic devices are permitted.
- Please sign below to indicate you accept the following statement:
  "I will not give, receive, or use any unauthorized assistance."

Signature: ____

Problem	Total Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total	100	

CSU Math 301 **Practice Final A**

$\boxed{1}$ Short answer questions. No English words required (except for partial credit).

(a) How many different ways are there to distribute 10 identical pieces of paper to Amy, Bob, and Carl? The only requirement is that all 10 pieces of paper need to be handed out.

(b) Suppose Diane has 9 pencils (all distinct) and Eve has 6 erasers (all distinct). In how many different ways could Diane trade 4 of her pencils for 2 of Eve's erasers?

(c) How many flush poker hands are there? A flush poker hand, such as $\{2\heartsuit, 4\heartsuit, 9\heartsuit, Q\heartsuit, K\heartsuit\}$, consists of any five cards of the same suit.
Remark: For the purposes of this problem, I have defined flush poker hands so that we are including the straight flush poker hands, such as $\{5\heartsuit, 6\heartsuit, 7\heartsuit, 8\heartsuit, 9\heartsuit\}$.

$\boxed{2}$ Prove the identity $1^2 + 2^2 + 3^2 + \ldots + (n-1)^2 + n^2 = \frac{n(n+1)(2n+1)}{6}$ holds for all integers $n \geq 1$.

$\boxed{3}$ Suppose you are given a set of 112 positive integers. Prove that at least 12 of these integers have the same ones digit.

$\boxed{4}$ (a) You have n dollars to spend. Each day you buy exactly one item: either a banana for \$1 or a bagel for \$4, and you continue until you have exactly \$0. Let S_n be the number of ways you can spend your n dollars (buying a banana the first day and a bagel the second is different from a bagel the first day and a banana the second). Find and explain a recurrence relation for S_n.

(b) Find S_9.

$\boxed{5}$ All variables in this problem are integers. Show that

(a) If $a \mid b$ and $b \mid c$ then $a \mid c$.

(b) If $a \mid b$ and $a \mid c$ then $a \mid (b - c)$.

(c) If $a \mid b$ and $a \nmid c$ then $a \nmid (b - c)$.

$\boxed{6}$ (a) Use the Euclidean algorithm to compute $\gcd(89, 55)$.

(b) Let's define a *division step* in the Euclidean algorithm to be a step where we have $a < b$ and we replace $\gcd(a, b)$ with $\gcd(a, r)$, where r is the remainder of b divided by a. Find inputs a and b so that the calculation of $\gcd(a, b)$ requires at least 100 division steps.

Hint: You may leave your answer in terms of the notation F_k, where F_k is the k-th Fibonacci number. Note $89 = F_{11}$ and $55 = F_{10}$.

$\boxed{7}$ (a) Is the code a 827341 a valid Prüfer code? If so, draw the corresponding tree. If not, explain why not (don't just say "it doesn't make a tree" — say how you could have known in advance before even finding the extended Prüfer code).

(b) Same as (a), except with the code 124625.

(c) Explain why there are n^{n-2} valid Prüfer codes of length $n-2$ (i.e., for trees with n vertices).

$\boxed{8}$ Show that a planar graph G with 8 vertices and 13 edges cannot be 2-colored.

Hint: Show that in a planar map corresponding to G, one of the faces must be a triangle.

$\boxed{9}$ *Remark: Part (a) isn't a great final exam question, because if you draw the graph in (a) wrong then you're in bad shape for (b). However, part (b) is a great final exam question. On an actual final, I would probably give you the graph in (a). But it's nice to see how graph coloring problems show up in many scheduling tasks.*

(a) A set of solar experiments is to be made at observatories. Each experiment begins on a given day of the year and ends on a given day (each experiment is repeated for several years). An observatory can perform only one experiment at a time.
Experiment A runs Sept 2 to Jan 3,
experiment B from Oct 15 to March 10,
experiment C from Nov 20 to Feb 17,
experiment D from Jan 23 to May 30,
experiment E from April 4 to July 28,
experiment F from April 30 to July 28, and
experiment G from June 24 to Sept 30.
The problem we'll solve in (b) is: "What is the minimum number of observatories required to perform a given set of experiments annually?" Model this scheduling problem as a graph-coloring problem, and draw the corresponding graph.

(b) Find a minimal coloring of the graph in (a) (i.e. the smallest k such that this graph is k-colorable), and show that fewer colors will not suffice.

$\boxed{10}$ No justification needed: just say “True” or “False”. No partial credit.

(a) True or False: If a graph has an odd number of vertices, then it has a vertex with even degree.

(b) True or False: There exists a graph G containing both closed and non-closed Eulerian walks.

(c) True or False: There exists a graph with 6 vertices of degrees 2, 2, 2, 2, 2, 4.

(d) True or False: There exists a *tree* with 6 vertices of degrees 1, 1, 2, 2, 3, 3.

(e) True or False: A 2-colorable graph with n connected components can be 2-colored in exactly 2^n different ways.

This page intentionally left blank.