Math 51
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These notes should help you with exercises 12.1-12.11 in Levandosky.

What do rank(A) and nullity(A) tell you about the existence and uniqueness of solutions x to $Ax = b$?

Let A be an $m \times n$ matrix. Consider the function which takes an input vector $x \in \mathbb{R}^n$ and multiplies it by matrix A to get output vector $Ax \in \mathbb{R}^m$. This is a function that goes from $\mathbb{R}^n$, the set of vectors of with n components, to $\mathbb{R}^m$, the set of vectors with m components.

Value rank(A) tells you about the existence of solutions x to $Ax = b$.

Here's how:

- If $\text{rank}(A) = m$, then for any vector $b \in \mathbb{R}^m$ there exists at least one solution x to $Ax = b$.
- If $\text{rank}(A) < m$, then there are some vectors $b \in \mathbb{R}^m$ (namely, those $b \notin C(A)$) for which there exist no solutions x to $Ax = b$.

Why is this true? We will use Proposition 9.1, which says the system $Ax = b$ has a solution x if and only if $b \in C(A)$.

Suppose $\text{rank}(A) = m$. By definition, $\dim(C(A)) = m$. Since $C(A)$ is a subspace of $\mathbb{R}^m$, it follows that $C(A) = \mathbb{R}^m$. Therefore any vector $b \in \mathbb{R}^m$ satisfies $b \in C(A)$. By Proposition 9.1, for any vector $b \in \mathbb{R}^m$ there exists at least one solution x to $Ax = b$.

Conversely, suppose $\text{rank}(A) < m$. By definition, $\dim(C(A)) < m$. Since $C(A)$ is a subspace of $\mathbb{R}^m$, it follows that $C(A)$ is not all of $\mathbb{R}^m$. Therefore there are some vectors $b \in \mathbb{R}^m$ with $b \notin C(A)$. By Proposition 9.1, there are some vectors $b \in \mathbb{R}^m$ for which there exists no solutions x to $Ax = b$.

Value nullity(A) tells you about the uniqueness of solutions x to $Ax = b$.

Here's how:

- If $\text{nullity}(A) = 0$, then any solution x to $Ax = b$ is unique.
- If $\text{nullity}(A) > 0$, then no solution x to $Ax = b$ can be unique.

Why is this true? We will use Proposition 8.2, which says that if there exists a solution x to $Ax = b$, then the set of all solutions is a translation of $N(A)$.

Suppose $\text{nullity}(A) = 0$. By definition, $\dim(N(A)) = 0$. Hence $N(A) = \{\vec{0}\}$ consists of a single vector. Hence a translation of $N(A)$ is a single vector. By Proposition 8.2, if there exists a solution x to $Ax = b$, then the set of all solutions is a single vector, so that solution x is unique.

Conversely, suppose $\text{nullity}(A) > 0$. By definition, $\dim(N(A)) > 0$. Hence $N(A)$ consists of many vectors. Hence a translation of $N(A)$ consists of many vectors. By Proposition 8.2, if there exists a solution x to $Ax = b$, then the set of all solutions consists of many vectors, so that solution x is not unique.