Vietoris–Rips thickenings of spheres

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Thanks to my graduate students: Johnathan Bush, Brittany Carr, Mark Heim, Lara Kassab, Joshua Mirth, Alex Williams
Idea 1: Vietoris–Rips complexes of the circle

Idea 2: Metric thickenings

Idea 3: Vietoris–Rips thickenings of spheres

Idea 4: Borsuk–Ulam theorems
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   With Michał Adamaszek

Idea 2: Metric thickenings
   With Michał Adamaszek and Florian Frick
   SIAM Journal on Applied Algebra and Geometry 2, 597–619, 2018

Idea 3: Vietoris–Rips thickenings of spheres

Idea 4: Borsuk–Ulam theorems
   With Johnathan Bush and Florian Frick
   Mathematika 66, 79–102, 2020
Definition

For $X$ a metric space and scale $r \geq 0$, the \textit{Vietoris–Rips simplicial complex} $VR(X; r)$ has

- vertex set $X$
- simplex \( \{x_0, \ldots, x_k\} \) when $\text{diam}(\{x_0, \ldots, x_k\}) \leq r$. 

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\[ \text{Out}[72]= \text{Cech simplicial complex} \]
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Theorem (Hausmann, 1995)

For $M$ a compact Riemannian manifold and $r$ small, 
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Idea 1: Let $S^1$ be the circle of unit circumference.

**Theorem (Adamaszek, A)**

$$\text{VR}(S^1; r) \cong \begin{cases} S^{2k+1} & \text{if } \frac{k}{2k+1} < r < \frac{k+1}{2k+3} \\ S^1 & \text{for some } k \in \mathbb{N}. \end{cases}$$
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$r = \frac{1}{3}$, $r = \frac{2}{5}$, $r = \frac{3}{7}$
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**Theorem (Adamaszek, A)**

$$VR(S^1; r) \simeq \begin{cases} S^{2k+1} & \text{if } \frac{k}{2k+1} < r < \frac{k+1}{2k+3} \\ \text{for some } k \in \mathbb{N}. \end{cases}$$
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$$\text{VR}(S^1; r) = \begin{cases} S^{2k+1} & \text{if } \frac{k}{2k+1} < r < \frac{k+1}{2k+3} \\ & \text{for some } k \in \mathbb{N}. \end{cases}$$

![Diagram showing the relationship between $r$ and $S^k$ for different values of $k$.](Torus.nb)
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$$VR(S^1; r) \begin{cases} S^{2k+1} & \text{if } \frac{k}{2k+1} < r < \frac{k+1}{2k+3} \\ \sqrt{\infty} S^{2k} & \text{if } r = \frac{k}{2k+1} \\ S^{2k+1} & \text{if } r = \frac{k}{2k+1} \\ S^{2k+1} & \text{for some } k \in \mathbb{N}. \end{cases}$$
Idea 2: Metric thickenings

The Vietoris–Rips simplicial complex may not be metrizable!

Metric space $M$ $\sim$ $X \subseteq M$ $\sim$ $\text{VR}(X; r)$
Let $X$ be a metric space and $r > 0$.

**Definition**

The *Vietoris–Rips metric thickening* is $\text{VR}^m(X; r)$

$$
\left\{ \sum_{i=0}^{k} \lambda_i x_i \middle| \lambda_i \geq 0, \sum_{i} \lambda_i = 1, x_i \in X, \text{diam} \{x_0, \ldots, x_k\} \leq r \right\},
$$
equipped with the 1-Wasserstein metric.
Let $X$ be a metric space and $r > 0$.

**Definition**

The *Vietoris–Rips metric thickening* is $\text{VR}^m(X; r)$

$$\left\{ \sum_{i=0}^{k} \lambda_i \delta_{x_i} \middle| \lambda_i \geq 0, \sum_i \lambda_i = 1, x_i \in X, \text{diam}\{x_0, \ldots, x_k\} \leq r \right\},$$

equipped with the 1-Wasserstein metric.
Theorem (Adamaszek, A, Frick)

If $M$ is a Riemannian manifold and $r$ is sufficiently small, then

$\text{VR}^m(M; r) \simeq M$.

Proof.

$\text{VR}^m(M; r) \xrightarrow{\Sigma_i \lambda_i \delta_{x_i}} \text{Fréchet mean} \xrightarrow{M}$

$\delta_x \leftrightarrow x$
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With Joshua Mirth: version for Euclidean sets of positive reach.
Idea 3: We can now say something about the $n$-sphere $S^n$.

**Theorem (Adamaszek, A, Frick)**

$$VR^m(S^n; r) \simeq \begin{cases} S^n & r < r_c \\ S^n \times \frac{SO(n+1)}{A_{n+2}} & r = r_c \end{cases}$$

$r_c$ = diameter of inscribed regular $\Delta^{n+1}$.

$SO(n+1) =$ group of rotations in $\mathbb{R}^{n+1}$.

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$$S^n \times \left( \frac{SO(n+1)}{A_{n+2}} \right)$$
Lovász’ strongly self-dual polytopes

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Theorem (Borsuk–Ulam)

For \( f: S^n \to \mathbb{R}^n \), there exists a point \( x \in S^n \) with \( f(x) = f(-x) \).

Figure credit: Jiří Matoušek, Using the Borsuk–Ulam theorem
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Gromov’s “waist of sphere” theorem is for \( f: S^n \to \mathbb{R}^k \) with \( k \leq n \).

Figure credit: Benjamin Matschke, Journal of Topology & Analysis
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Gromov’s “waist of sphere” theorem is for \( f: S^n \to \mathbb{R}^k \) with \( k \leq n \).

What about \( f: S^n \to \mathbb{R}^k \) with \( k \geq n \)?
Theorem (Borsuk–Ulam)

For $f: S^n \to \mathbb{R}^n$, there exists a point $x \in S^n$ with $f(x) = f(-x)$.

Theorem (A, Bush, Frick)

For $f: S^1 \to \mathbb{R}^{2k+1}$, there exists a set $\{x_0, \ldots, x_{2k+1}\}$ of diameter at most $\frac{k}{2k+1}$ such that $\sum \lambda_i f(x_i) = \sum \lambda_i f(-x_i)$. 
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Proof: $S^{2k+1} \cong VR^m(S^1; \frac{k}{2k+1}) \xrightarrow{f} \mathbb{R}^{2k+1}$

Sharpness: $f = (\cos \theta, \sin \theta, \cos 3\theta, \sin 3\theta, \cos 5\theta, \sin 5\theta, \ldots)$
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**Theorem (A, Bush, Frick)**

For $f: S^n \to \mathbb{R}^{n+2}$, there exists a set $\{x_0, \ldots, x_{n+2}\}$ of diameter at most $r_c$ such that $\sum \lambda_i f(x_i) = \sum \lambda_i f(-x_i)$.

Proof: $S^{n+2} \subseteq \text{VR}^m(S^n; r_c) \xrightarrow{f} \mathbb{R}^{n+2}$
Open questions & future work

1. Čech and Nerve complexes (Borsuk)
2. Other manifolds $M$?
3. $\text{VR}_<(X; r) \simeq \text{VR}_<^m(X; r)$?
4. Morse, Morse–Bott, and Bestvina–Brady Morse theories.
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References


Thank you!