Vietoris-Rips thickenings: Problems for birds and frogs

Henry Adams
Colorado State University

Birds and Frogs

Freeman Dyson

Some mathematicians are birds, others are frogs. Birds fly high in the air and survey broad vistas of mathematics out to the far horizon. They delight in concepts that unify our thinking and bring together diverse problems from different parts of the landscape. Frogs live in the mud below and see only the flowers that grow nearby. They delight in the details of particular objects, and they solve problems one at a time. I happen to be a frog, but many of my best friends are birds. The main theme of my talk tonight is this. Mathematics needs both birds and frogs. Mathematics is rich and beautiful because birds give it broad visions and frogs give it intricate details. Mathematics is both great art and important science, because it combines generality of concepts with depth of structures. It is stupid to claim that birds are better than frogs because they see farther, or that frogs are better than birds because they see deeper. The world of mathematics is both broad and deep, and we need birds and frogs working together to explore it.

This talk is called the Einstein lecture, and I am grateful to the American Mathematical Society for inviting me to do honor to Albert Einstein. Einstein was not a mathematician, but a physicist who had mixed feelings about mathematics. On the one hand, he had enormous respect for the power of mathematics to describe the workings of nature, and he had an instinct for mathematical beauty which led him onto the right track to find nature's laws. On the other hand, he had no interest in pure mathematics, and he had no technical skill as a mathematician. In his later years he hired younger colleagues with the title of assistants to do mathematical calculations for him. His way of thinking was physical rather than mathematical. He was supreme among physicists as a bird who saw further than others. I will not talk about Einstein since I have nothing new to say.

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This article is a written version of his AMS Einstein Lecture, which was to have been given in October 2008 but which unfortunately had to be canceled.

Francis Bacon and René Descartes

At the beginning of the seventeenth century, two great philosophers, Francis Bacon in England and René Descartes in France, proclaimed the birth of modern science. Descartes was a bird, and Bacon was a frog. Each of them described his vision of the future. Their visions were very different. Bacon said, "All depends on keeping the eye steadily fixed on the facts of nature." Descartes said, "I think, therefore I am." According to Bacon, scientists should travel over the earth collecting facts, until the accumulated facts reveal how Nature works. The scientists will then induce from the facts the laws that Nature obeys. According to Descartes, scientists should stay at home and deduce the laws of Nature by pure thought. In order to deduce the laws correctly, the scientists will need only the rules of logic and knowledge of the existence of God. For four hundred years since Bacon and Descartes led the way, science has raced ahead by following both paths simultaneously. Neither Baconian empiricism nor Cartesian dogmatism has the power to elucidate Nature's secrets by itself, but both together have been amazingly successful. For four hundred years English scientists have tended to be Baconian and French scientists Cartesian. Faraday and Darwin and Rutherford were Baconians; Pascal and Laplace and Poincaré were Cartesians. Science was greatly enriched by the cross-fertilization of the two contrasting cultures. Both cultures were always at work in both countries. Newton was at heart a Cartesian, using

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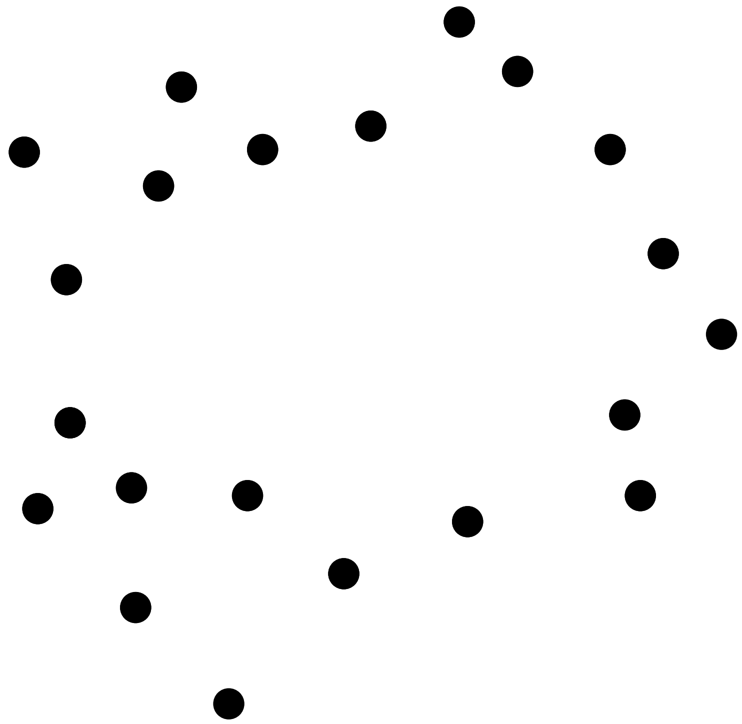
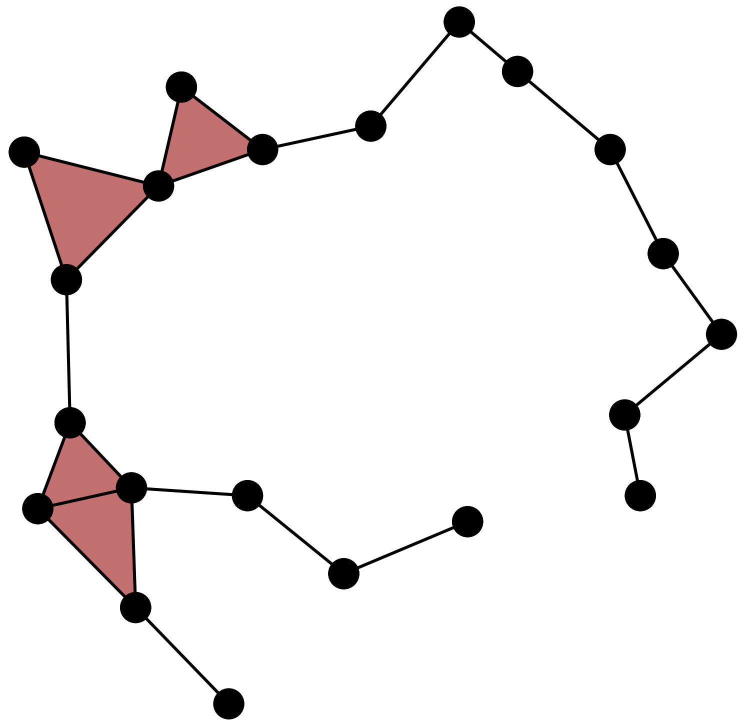


Definition

For X a metric space and scale $r \geq 0$, the *Vietoris–Rips simplicial complex* $\mathrm{VR}(X; r)$ has

- vertex set X
- simplex $\{x_0, \ldots, x_k\}$ when $\mathrm{diam}(\{x_0, \ldots, x_k\}) \leq r$.



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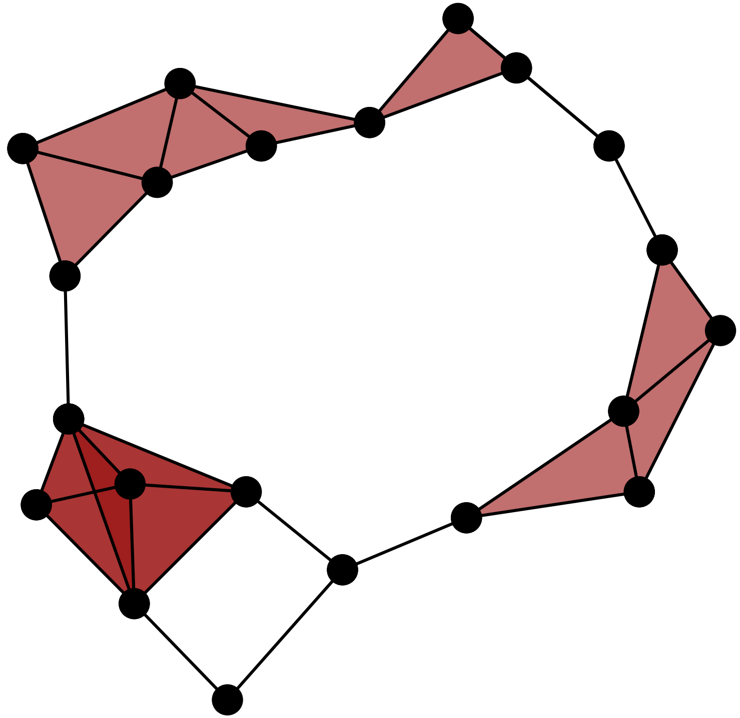

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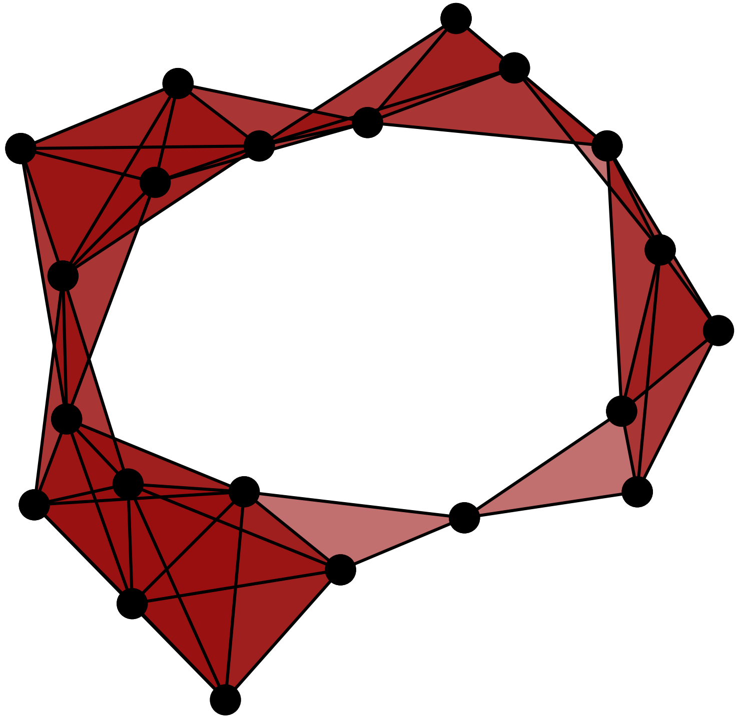

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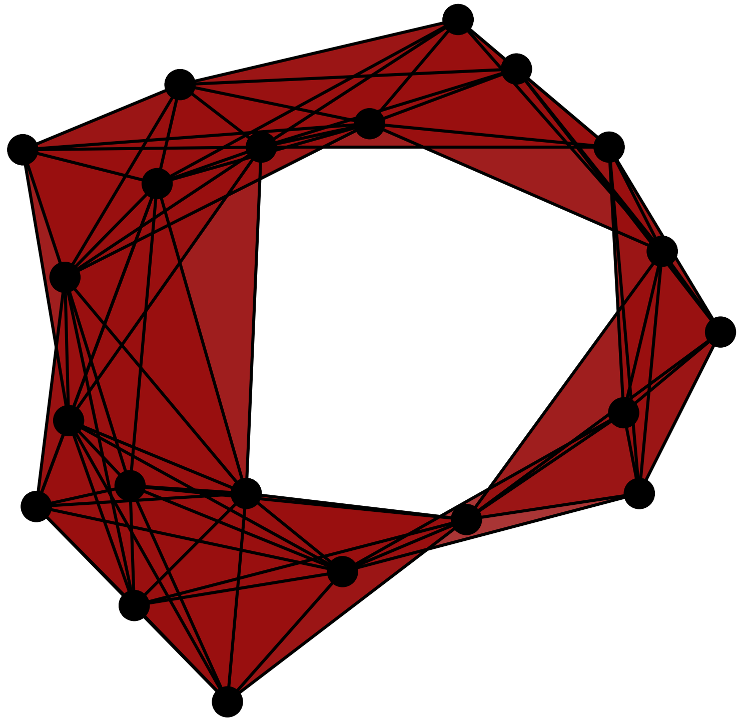

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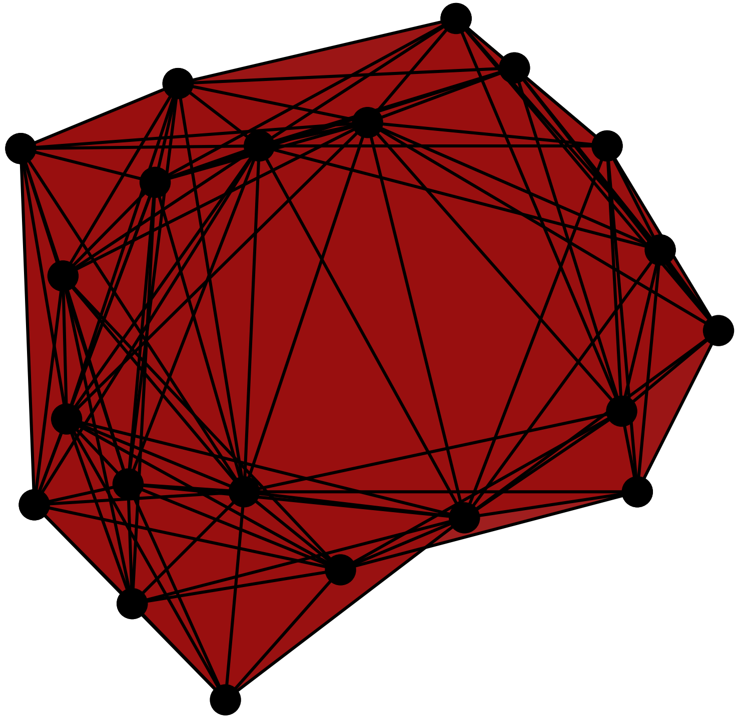

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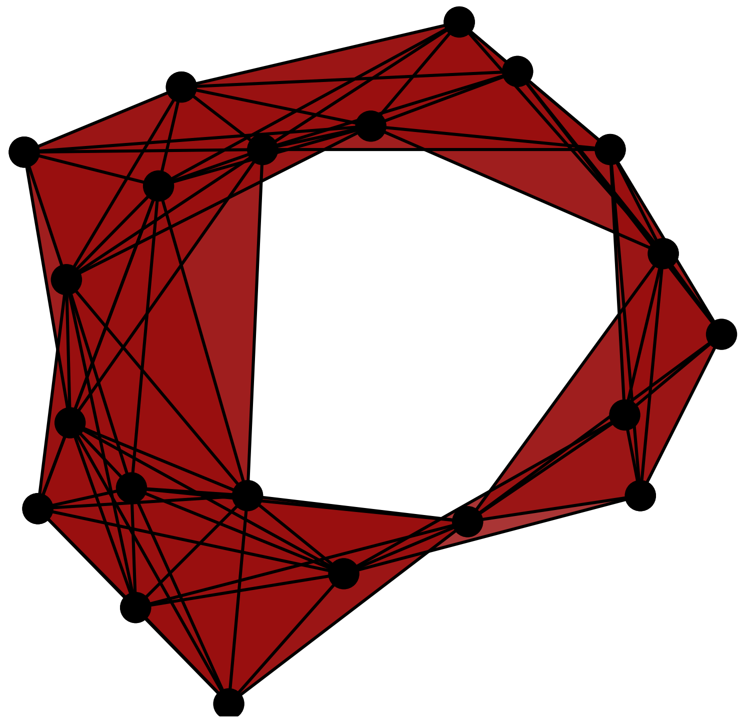


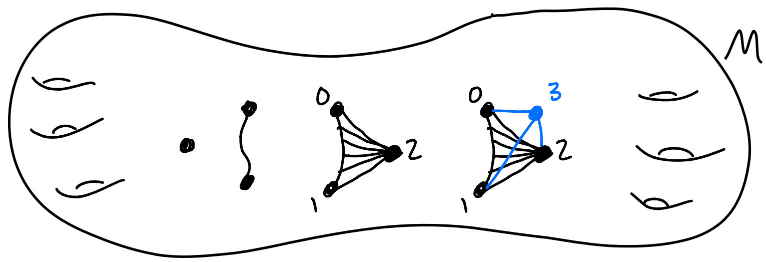
History

- Cohomology theory for metric spaces (Vietoris 1927)
- Geometric group theory (Rips)
- Applied topology

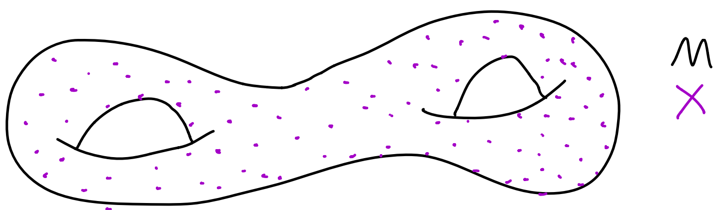
Thm (Hausmann 1995, Virk 2021)

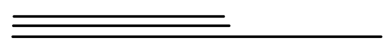
M compact Riemannian manifold.
Then for r sufficiently small, $VR(M;r) \simeq M$.

$$\begin{tikzcd} VR(M;r) \arrow[d] \\ M \arrow[u, dashed, bend right] \end{tikzcd}$$



Thm (Latschev 2001) M, r as above.
$\exists \delta > 0$ such that if $d_{GH}(X, M) < \delta$, then $VR(X; r) \simeq M$.



$PH_1(VR(M;r))$ 

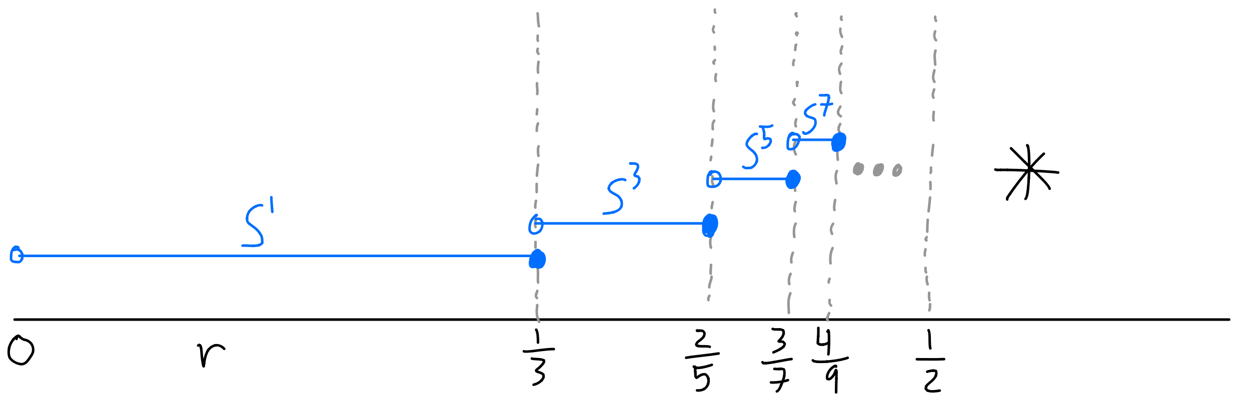
$PH_1(VR(X;r))$ 

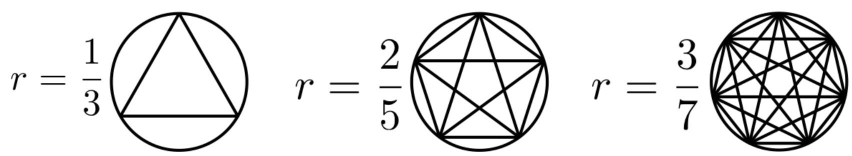
Stability (Chazal, de Silva, Oudot 2014)
(Chazal, Cohen-Steiner, Guibas, Mémoli, Oudot 2009)
If two totally bounded metric spaces are close, then their persistent homology is close.
Metric geometry: Gromov-Hausdorff distance

S^1 is circle with geodesic metric, unit circumference.

Thm (Adamaszek, Adams 2017)

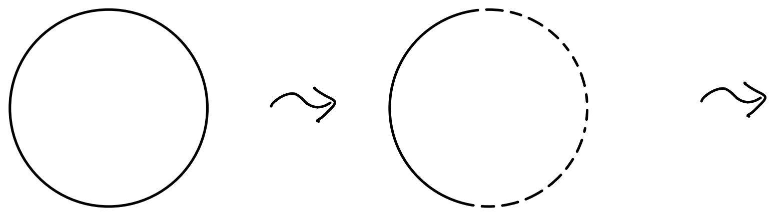
$$VR_<(S^1;r) \simeq S^{2k+1} \quad \text{if } \frac{k}{2k+1} < r \leq \frac{k+1}{2k+3}, \quad k \in \mathbb{N}.$$





Why $VR_<(S^1; \frac{1}{3}+\varepsilon) \simeq S^3$?

Metric Reconstruction



metric space M $\qquad$ $X \subseteq M$

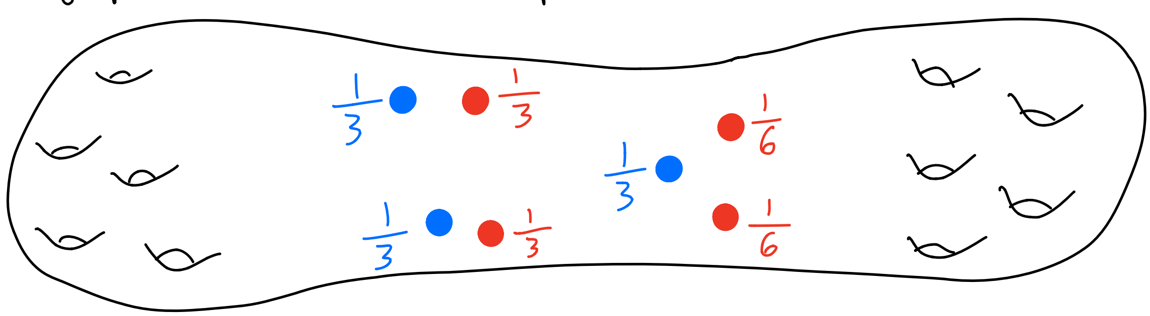
$VR(X;r) \simeq M$

often not metrizable

Def X metric space, $r \geq 0$.
The Vietoris–Rips metric thickening is

$$VR^m(X;r) = \left\{ \sum_{i=0}^{k} \lambda_i \delta_{x_i} \;\middle|\; \begin{array}{ll} x_i \in X, & \mathrm{diam}(\{x_0, \ldots, x_k\}) \leq r \\ \lambda_i \geq 0 & \sum \lambda_i = 1 \end{array} \right\},$$

equipped with the p-Wasserstein metric.



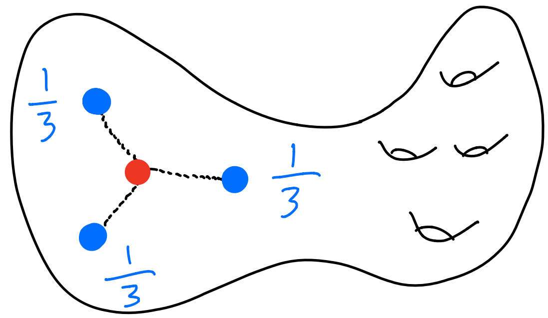
measure theory:
optimal transport

Thm (Adamaszek, Adams, Frick 2018)

M complete Riemannian manifold.
Then for r sufficiently small, $VR^m(M;r) \simeq M$.

$$\begin{tikzcd} VR^m(M;r) \arrow[d, bend right] \\ M \arrow[u, bend right] \end{tikzcd}$$

$$\sum \lambda_i \delta_{x_i} \mapsto \text{Karcher or Frechét mean}$$



Thm (Moy, 2021)

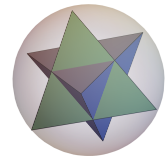
For X totally bounded, $VR^m(X;r)$ and $VR(X;r)$ have the same persistent homology diagrams.

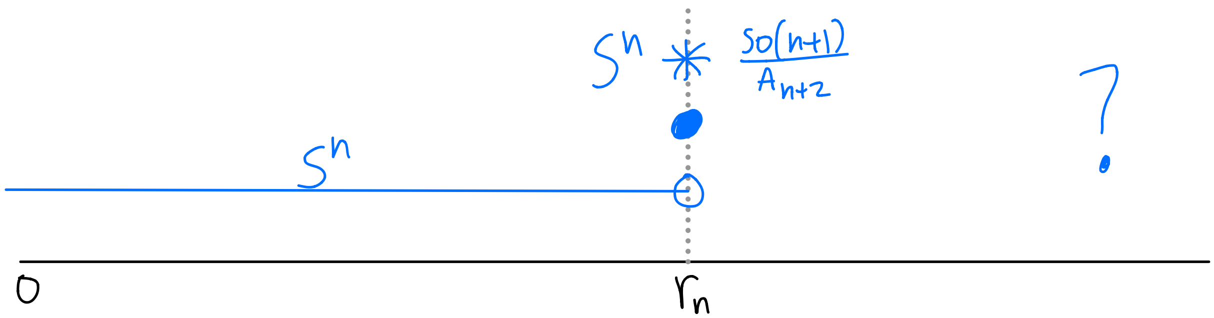
Consequence VR^m is stable.

Conjecture $VR^m_<(X;r) \simeq VR_<(X;r)$.

Thm (Adamaszek, Adams, Frick 2018)

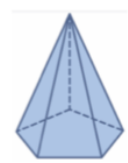
$$VR^m_{\leq}(S^n;r) \simeq \begin{cases} S^n & r < r_n \\ S^n * \frac{SO(n+1)}{A_{n+2}} & r = r_n. \end{cases}$$





Thm (Lim, Mémoli, Okutan 2020, Katz 1991)

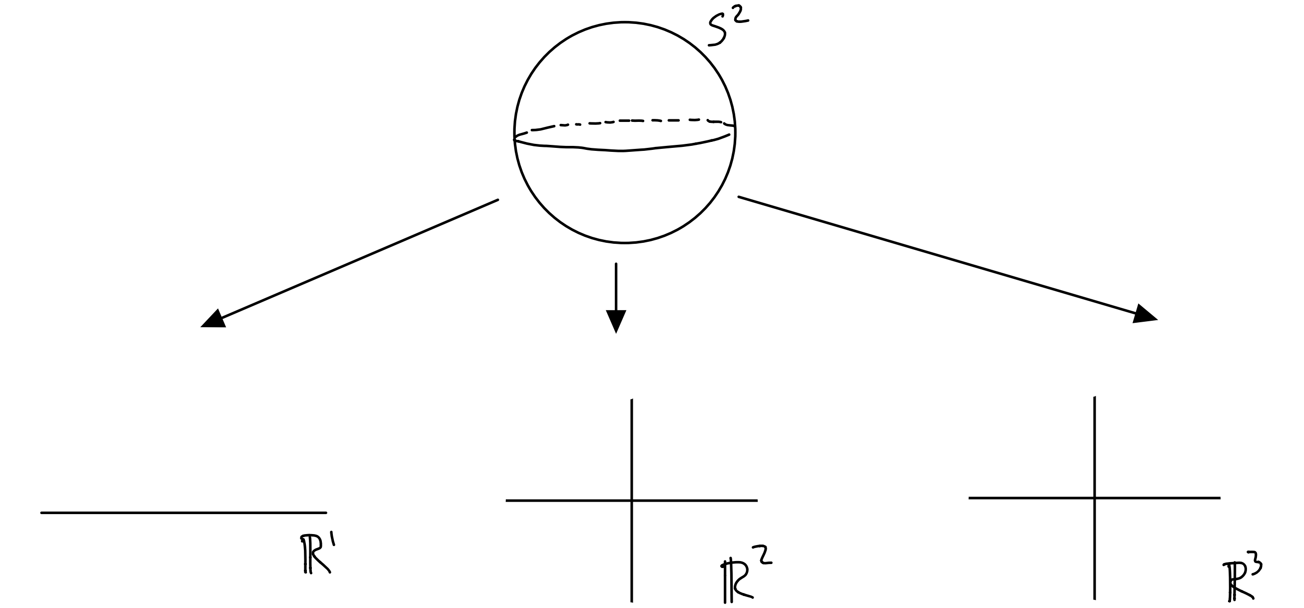
$$VR(S^2;r) \simeq S^2 * \frac{SO(3)}{A_4} \qquad r_2 < r < \alpha_2$$



Quantitative topology:
Filling radius

Topological combinatorics:
Polytopes

Consequences of homotopy connectivity of $VR(S^n;r)$ or $VR^m(S^n;r)$:
Borsuk-Ulam theorems for $S^n \to \mathbb{R}^k$, $k \geq n$.



"Waist of sphere" (Almgren 1965, Gromov 2003)

Borsuk-Ulam (1933)

Adams, Bush, Frick (2020)

Quantitative topology

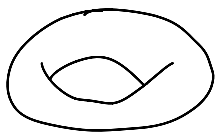
Combinatorial topology:
Barvinok-Novik orbitopes
L^1 norm promotes sparsity

Questions for frogs

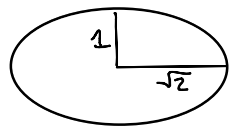
(1) $VR(S^n; r)$ for larger r?

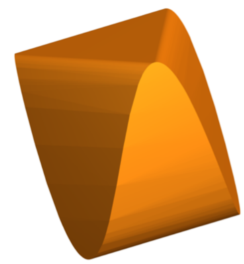
(2) Čech$(S^n; r)$?

(3) Other manifolds: tori, ellipsoids ?



L^∞ metric known
L^2, L^1 metrics open





(4) For $X \subseteq \mathbb{R}^2$, is $VR(X; r)$ a wedge of spheres?
(Chambers, de Silva, Erickson, Ghrist 2010)
(Adamaszek, Frick, Vakili 2017)

(5) Is $VR((\mathbb{Z}^n; L^1); r)$ contractible for $r \geq n$?
(Case $n=2,3$ by Mallery, Zaremsky)
Cayley graphs of groups ?

Geometric group theory: finiteness properties

(6) Hypercube graphs $VR((\{0,1\}^n; L^1); r)$
(Case $r=2$ by Adamaszek, Adams)

(7) Vietoris-Rips complexes of graph vertex sets?

(8) PH_k of Vietoris-Rips complexes of metric graphs and geodesic spaces?

(k=1 by Gasparovic, Gommel, Purvine, Sazdanovic, Wang, Wang, Ziegelmeier 2018, Virk 2020)

(9) Perea for time series: Vietoris-Rips complexes of curve $(z, z^2, z^3, \ldots, z^d) \subseteq (S^1)^d \subseteq \mathbb{C}^d$

(10) Statistics for $PH_k(VR(X;r))$ for $X \subset S^1$
(Case k=0,1 by Bubenik, Kim 2007)

(11) $VR_<^m(X;r) \simeq VR_<(X;r)$?

(12) Homotopy type of all probability measures on (say) S^1 of bounded variance?

Measure theory

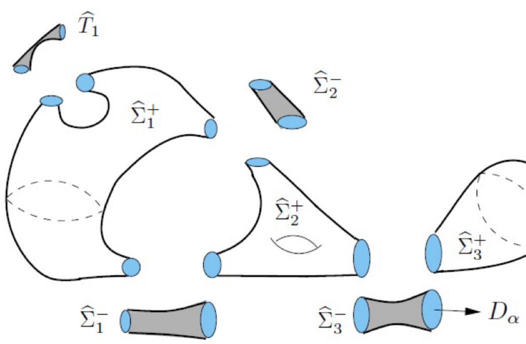
Emerging bridges for birds

(13) Morse and Morse-Bott theories (Mirth PhD thesis)

(14) Borsuk-Ulam theorems $S^n \to \mathbb{R}^k$ for $k \geq n$
Algebraic Topology: Characteristic classes
(Michael Crabb)

(15) Connections to quantitative topology, especially Kuratowski embeddings and filling radii.
(Lim, Mémoli, Okutan 2020, Katz 1983–1991)

(16) Geometric Topology: Thick-thin decompositions



Baris Coskunuzer
2021