

# The Vietoris-Rips Complex of the Circle

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## Notation

Let  $S^1$  be the circle of unit circumference with the geodesic metric.

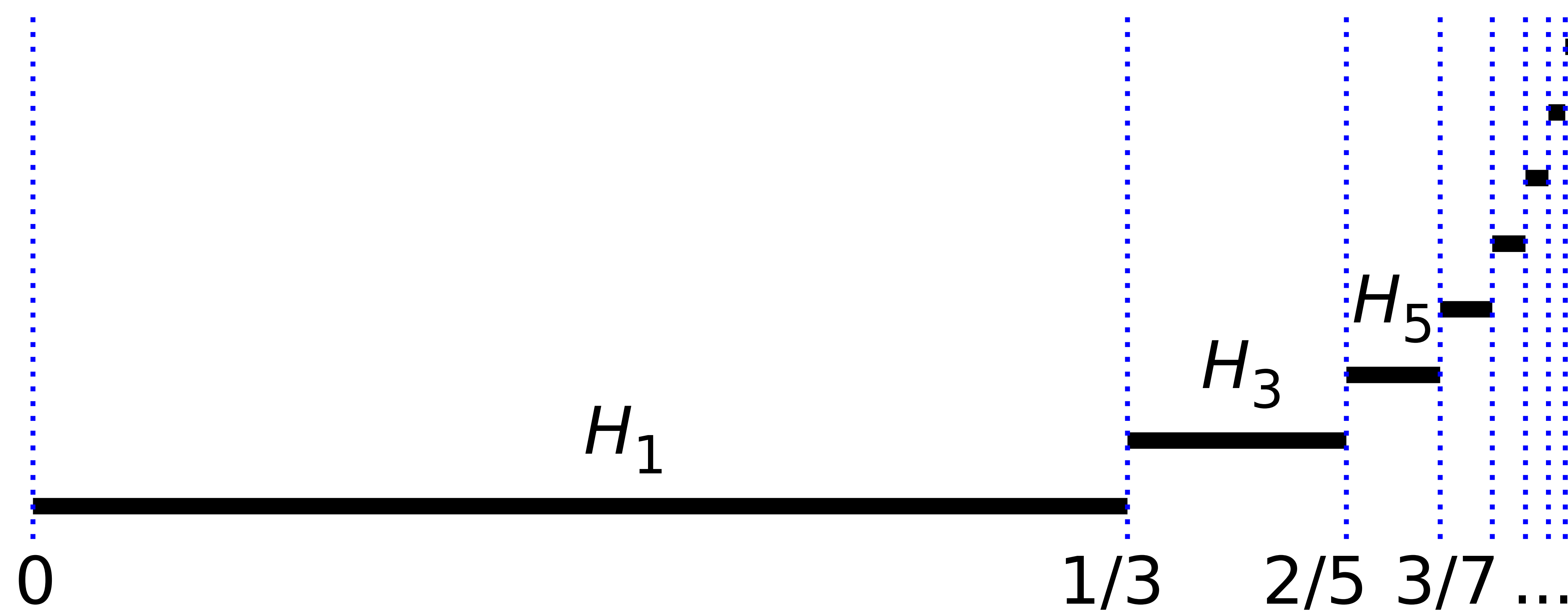
**Definition.** For  $X \subset S^1$  and  $r \geq 0$ , let  $\text{VR}(X, r)$  be the Vietoris-Rips complex with vertex set  $X$  and connectivity parameter  $r$ . That is, simplex  $\sigma \subset X$  is in  $\text{VR}(X, r)$  when  $\text{diam}(\sigma) \leq r$ .

## Persistent Homology of $\text{VR}(S^1, r)$

**Theorem 1.** The odd-dimensional persistent homology of  $\text{VR}(S^1, r)$  is

$$\text{dgm}\left(H_{2l+1}(\text{VR}(S^1, r))\right) = \left\{ \left( \frac{l}{2l+1}, \frac{l+1}{2l+3} \right) \right\},$$

and within this interval  $\text{VR}(S^1, r) \simeq S^{2l+1}$ . The even-dimensional persistent homology has no intervals of positive length.



## References

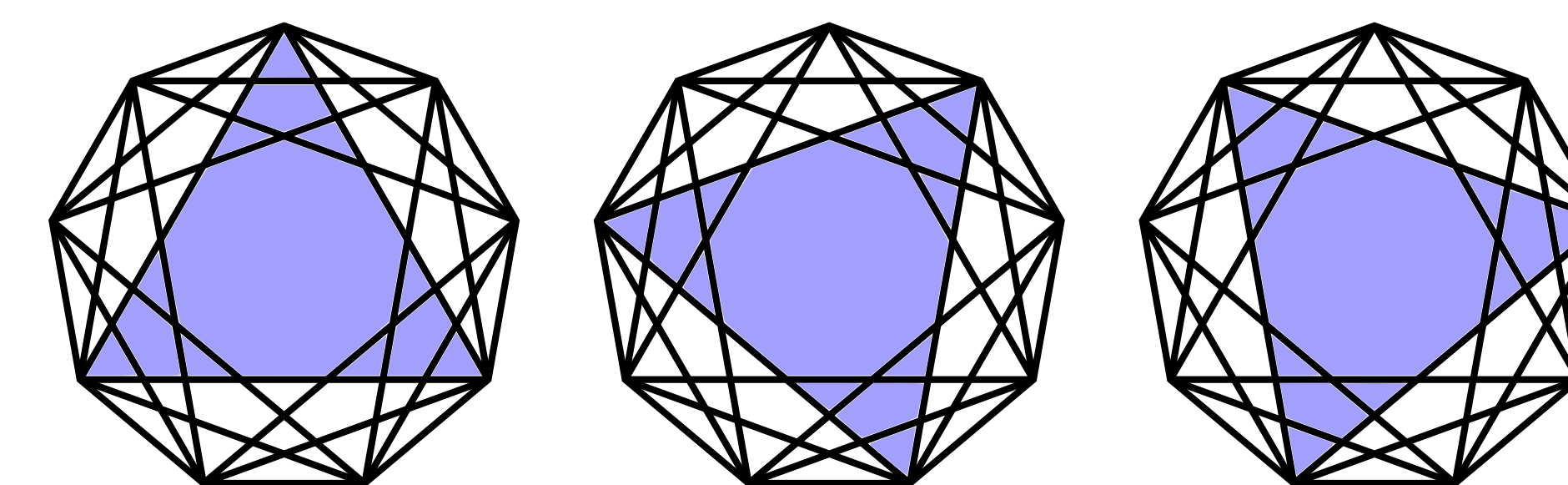
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- [2] Frédéric Chazal, Vin de Silva, Steve Oudot, *Persistence stability for geometric complexes*, Geometriae Dedicata (2013), 1–22.
- [3] Jean-Claude Hausmann, *On the Vietoris-Rips complexes and a cohomology theory for metric spaces*, Annals of Mathematics Studies 138 (1995), 175–188.
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## Homotopy Types of $\text{VR}(X, r)$ for $X \subset S^1$

**Theorem 2.** For  $X \subset S^1$  finite,  $\text{VR}(X, r)$  is homotopy equivalent to either a point, an odd sphere, or a wedge sum of spheres of the same even dimension.

For example, let  $X_n \subset S^1$  consist of  $n$  evenly-spaced points. This case is given in [1].

**Example.** We have  $\text{VR}(X_9, 1/3) \simeq \vee_2 S^2$ .



**Corollary 6.7 from [1].** Let  $k < n/2$ . Then

$$\text{VR}(X_n, k/n) \simeq \begin{cases} \vee_{n-2k-1} S^{2l} & \text{if } \frac{k}{n} = \frac{l}{2l+1} \\ S^{2l+1} & \text{if } \frac{l}{2l+1} < \frac{k}{n} < \frac{l+1}{2l+3} \text{ for some } l \geq 0. \end{cases}$$

	$k=1$	2	3	4	5	6	7	8	9	10	11	12	13
$n=4$	$S^1$	*	*	*	*	*	*	*	*	*	*	*	*
5	$S^1$	*	*	*	*	*	*	*	*	*	*	*	*
6	$S^1$	$S^2$	*	*	*	*	*	*	*	*	*	*	*
7	$S^1$	$S^1$	*	*	*	*	*	*	*	*	*	*	*
8	$S^1$	$S^1$	$S^3$	*	*	*	*	*	*	*	*	*	*
9	$S^1$	$S^1$	$\vee_2 S^2$	*	*	*	*	*	*	*	*	*	*
10	$S^1$	$S^1$	$S^1$	$S^4$	*	*	*	*	*	*	*	*	*
11	$S^1$	$S^1$	$S^1$	$S^3$	*	*	*	*	*	*	*	*	*
12	$S^1$	$S^1$	$S^1$	$\vee_3 S^2$	$S^5$	*	*	*	*	*	*	*	*
13	$S^1$	$S^1$	$S^1$	$S^1$	$S^3$	*	*	*	*	*	*	*	*
14	$S^1$	$S^1$	$S^1$	$S^1$	$S^3$	$S^6$	*	*	*	*	*	*	*
15	$S^1$	$S^1$	$S^1$	$S^1$	$\vee_4 S^2$	$\vee_2 S^4$	*	*	*	*	*	*	*
16	$S^1$	$S^1$	$S^1$	$S^1$	$S^1$	$S^3$	$S^7$	*	*	*	*	*	*
17	$S^1$	$S^1$	$S^1$	$S^1$	$S^1$	$S^3$	$S^5$	*	*	*	*	*	*
18	$S^1$	$S^1$	$S^1$	$S^1$	$S^1$	$\vee_5 S^2$	$S^3$	$S^8$	*	*	*	*	*
19	$S^1$	$S^1$	$S^1$	$S^1$	$S^1$	$S^1$	$S^3$	$S^5$	*	*	*	*	*
20	$S^1$	$S^1$	$S^1$	$S^1$	$S^1$	$S^1$	$S^3$	$\vee_3 S^4$	$S^9$	*	*	*	*
21	$S^1$	$S^1$	$S^1$	$S^1$	$S^1$	$S^1$	$\vee_6 S^2$	$S^3$	$\vee_2 S^6$	*	*	*	*
22	$S^1$	$S^1$	$S^1$	$S^1$	$S^1$	$S^1$	$S^1$	$S^3$	$S^5$	$S^{10}$	*	*	*
23	$S^1$	$S^1$	$S^1$	$S^1$	$S^1$	$S^1$	$S^1$	$S^3$	$S^3$	$S^7$	*	*	*
24	$S^1$	$S^1$	$S^1$	$S^1$	$S^1$	$S^1$	$S^1$	$\vee_7 S^2$	$S^3$	$S^5$	$S^{11}$	*	*
25	$S^1$	$S^1$	$S^1$	$S^1$	$S^1$	$S^1$	$S^1$	$S^1$	$S^3$	$\vee_4 S^4$	$S^7$	*	*
26	$S^1$	$S^1$	$S^1$	$S^1$	$S^1$	$S^1$	$S^1$	$S^1$	$S^3$	$S^3$	$S^5$	$S^{12}$	*
27	$S^1$	$S^1$	$S^1$	$S^1$	$S^1$	$S^1$	$S^1$	$S^1$	$\vee_8 S^2$	$S^3$	$S^5$	$\vee_2 S^8$	*
28	$S^1$	$S^1$	$S^1$	$S^1$	$S^1$	$S^1$	$S^1$	$S^1$	$S^1$	$S^3$	$S^3$	$\vee_3 S^6$	$S^{13}$