The waist inequality

See the survey paper "The waist inequality in Gromov's work" by Larry Guth.

Borsuk-Ulam Thm For \( f: S^n \to \mathbb{R}^n \), \( \exists x \in S^n \) with \( f(x) = f(-x) \)

"Exactly equatorial \( S^0 \) in single fiber"

Waist Inequality Thm For \( f: S^n \to \mathbb{R}^q \) with \( q \leq n \),
\( \exists y \in \mathbb{R}^q \) with \( \text{Vol}_{n-q} f^{-1}(y) = \text{Vol}_{n-q} S^{n-q} \)

Constant 1 is sharp. Let \( f: S^n \to \mathbb{R}^q \) be the restriction of a projection \( \mathbb{R}^{n+1} \to \mathbb{R}^q \).
Fibers of \( f \) are \((n-q)\)-spheres (or points), the largest of which is equatorial.

Proof for \( q=1 \) is easy. A special case of the isoperimetric inequality on spheres says that if \( U \subseteq S^n \) has \( \text{vol}_n(U) = \frac{1}{2} \text{vol}_n(S^n) \), then \( \text{vol}_{n-1}(\partial U) \geq \text{vol}_{n-1}(S^{n-1}) \).
Choose $y \in \mathbb{R}$ so that $\text{vol}_n(\{x \in S^n \mid f(x) = y\}^3) = \frac{1}{2} \text{vol}_n(S^n)$.

Then the boundary $f^{-1}(y)$ satisfies $\text{vol}_{n-1}(f^{-1}(y)) \geq \text{vol}_{n-1}(S^{n-1})$.

Proof for $q \geq 2$ is hard

- fibers no longer divide $S^n$ into regions
- not clear which $y \in \mathbb{R}^q$ to look at
- an arbitrarily large "fraction" of the fibers can be arbitrarily small, which can't happen for $q = 1$.

Minimax proof Almgren 1965, geometric measure theory, minimal surfaces, 100-200 pages

Short proof non-sharp constant Gromov 1983, isoperimetric inequality

Borsuk-Ulam proof Gromov 2003, sharp constant, even more topology

(hard generalizations of Borsuk-Ulam using characteristic classes)

than other proofs (degree theory).
The waist inequality is more closely connected to topology than is its cousin, the isoperimetric inequality.

One small reason why is that the waist inequality implies topological invariance of dimension:

\[ \mathbb{R}^q \cong \mathbb{R}^{q'} \iff q = q' \]

Typical proof:

\[ S^{q-1} \cong \mathbb{R}^q \setminus pt \cong \mathbb{R}^{q'} \setminus pt \cong S^{q'-1} \]

\[ \Rightarrow H_*(S^{q-1}) \cong H_*(S^{q'-1}) \]

\[ \Rightarrow q = q' \]

Waist inequality proof:

Let \( q' > q \).

\[ S^n \xrightarrow{L} \mathbb{R}^{q'} \xrightarrow{h} \mathbb{R}^q \]

\[ f = h \circ L \]

\( L \) linear \( \Rightarrow \) fibers of \( L \) are \((n-q')\)-dim'l spheres.

\( h \) injective \( \Rightarrow \) fibers of \( f \) are also \((n-q')\)-dim'l spheres with \( \text{vol}_{n-q'} = 0 \) (since \( n - q' < n - q \)).

This contradicts the waist inequality.

Quantitative topology is a "hot" area.

Considers not only dimensions, but also sizes (volumes, diameters, lengths, radii).

Many of its foundational tools invented by Gromov (waist inequality).

Feels related to persistent homology & applied topology.
Metric thickenings, Borsuk–Ulam theorems, and orbitopes

Henry Adams (Colorado State University)
Johnathan Bush (Colorado State University)
Florian Frick (Carnegie Mellon)

Paper in preparation
Theorem (Borsuk–Ulam)

For \( f : S^n \to \mathbb{R}^n \), there exists a point \( x \in S^n \) with \( f(x) = f(-x) \).

Figure credit: Jiří Matoušek, Using the Borsuk–Ulam theorem
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**Theorem (Gromov’s “waist” theorem)**

For $f : S^n \to \mathbb{R}^k$ with $k \leq n$, there exists some $y \in \mathbb{R}^n$ with

$$\text{vol}_{n-k}(f^{-1}(y)) \geq \text{vol}_{n-k}(S^{n-k} \subseteq S^n).$$

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For $f : S^1 \to \mathbb{R}^{2k+1}$, there exists a set $\{x_0, \ldots, x_{2k+1}\}$ of diameter at most $\frac{k}{2k+1}$ such that $\sum \lambda_i f(x_i) = \sum \lambda_i f(-x_i)$. 
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**Proof:** $S^{2k+1} \simeq VR^m(S^1; \frac{k}{2k+1}) \xrightarrow{f} \mathbb{R}^{2k+1}$

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Theorem (A, Bush, Frick)

For $f : S^n \to \mathbb{R}^{n+2}$, there exists a set $\{x_0, \ldots, x_{n+2}\}$ of diameter at most $r_n$ such that $\sum \lambda_i f(x_i) = \sum \lambda_i f(-x_i)$.

Proof: $S^{n+2} \subset \text{VR}^m(S^n; r_n) \xrightarrow{f} \mathbb{R}^{n+2}$

$r_n$ is the side-length of an inscribed simplex
References


Thank you!