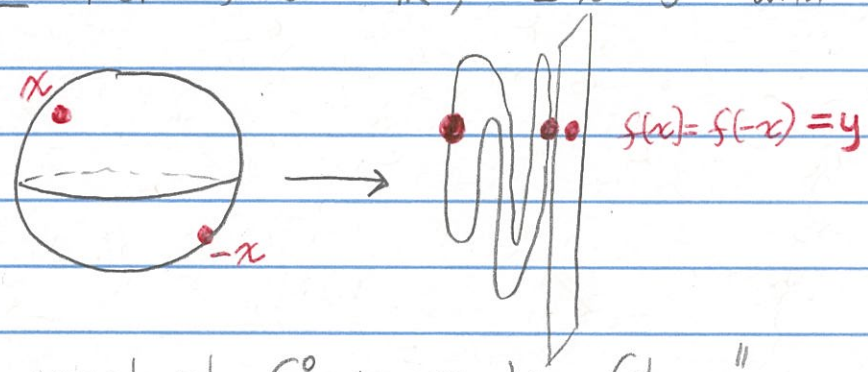
The waist inequality

See the survey paper "The waist inequality in Gromov's work" by Larry Guth.

↑ Kakeya needle problem 

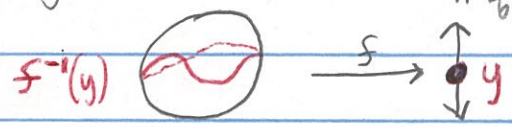
Borsuk-Ulam Thm For $f: S^n \to \mathbb{R}^n$, $\exists x \in S^n$ with $f(x) = f(-x)$



"Exactly equatorial S^0 in single fiber"

Waist Inequality Thm For $f: S^n \to \mathbb{R}^q$ with $q \leq n$, $\exists y \in \mathbb{R}^q$ with $\mathrm{Vol}_{n-q} f^{-1}(y) \geq \mathrm{Vol}_{n-q} S^{n-q}$

↑ equatorial $S^{n-q} \subseteq S^n$



Constant 1 is sharp Let $f: S^n \to \mathbb{R}^q$ be the restriction of a projection $\mathbb{R}^{n+1} \to \mathbb{R}^q$.
Fibers of f are $(n-q)$-spheres (or points), the largest of which is equatorial.

Proof for $q=1$ is easy A special case of the isoperimetric inequality on spheres (1919) says if $U \subseteq S^n$ has $\mathrm{vol}_n(U) = \frac{1}{2}\mathrm{vol}_n(S^n)$, then $\mathrm{vol}_{n-1}(\partial U) \geq \mathrm{vol}_{n-1}(S^{n-1})$.

Choose $y \in \mathbb{R}$ so that $\mathrm{vol}_n(\{x \in S^n \mid f(x) \leq y\}) = \frac{1}{2}\mathrm{vol}_n(S^n)$.
Then the boundary $f^{-1}(y)$ satisfies $\mathrm{vol}_{n-1} f^{-1}(y) \geq \mathrm{vol}_{n-1}(S^{n-1})$.

<u>Proof for $q \geq 2$ is hard</u>

- fibers no longer divide S^n into regions
- not clear which $y \in \mathbb{R}^q$ to look at
- an arbitrarily large "fraction" of the fibers can be arbitrarily small, which can't happen for $q=1$.

<u>Minimax proof</u> Almgren 1965, geometric measure theory, minimal surfaces, 100-200 pages

<u>Short proof non-sharp constant</u> Gromov 1983, isoperimetric inequality

<u>Borsuk-Ulam proof</u> Gromov 2003, sharp constant, even more topology
(hard generalizations of Borsuk-Ulam using characteristic classes) than other proofs (degree theory).

Rmk The waist inequality is more closely connected to topology than is its cousin, the isoperimetric inequality. (First rigorous proof: 1800's)

One small reason why is that the waist inequality implies <u>topological invariance of dimension</u>

$$\mathbb{R}^q \cong \mathbb{R}^{q'} \iff q = q'$$

<u>Typical proof</u>

$$S^{q-1} \simeq \mathbb{R}^q \setminus pt \cong \mathbb{R}^{q'} \setminus pt \simeq S^{q'-1}$$
$$\Longrightarrow H_*(S^{q-1}) \cong H_*(S^{q'-1})$$
$$\Longrightarrow q = q'.$$

<u>Waist inequality proof</u> Let $q' > q$.

$$\begin{tikzcd} S^n \arrow[r, "L \text{ linear}"'] \arrow[rr, bend right, "f = h \circ L"'] & \mathbb{R}^{q'} \arrow[r, "\cong", "h"'] & \mathbb{R}^q \end{tikzcd}$$

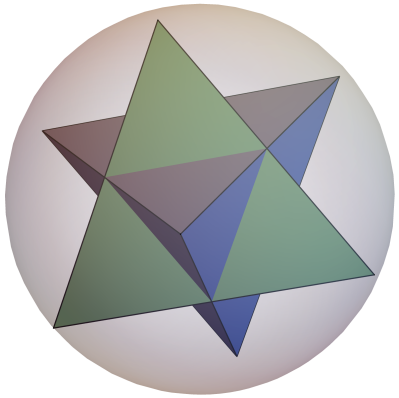
L linear $\Rightarrow$ fibers of L are $(n-q')$-dim'l spheres.
h injective $\Rightarrow$ fibers of f are also $(n-q')$-dim'l spheres with $\mathrm{vol}_{n-q} = 0$ (since $n-q' < n-q$).
This contradicts the waist inequality.

<u>Quantitative topology</u> is a "hot" area.
Considers not only dimensions, but also sizes (volumes, diameters, lengths, radii)
Many of its foundational tools invented by Gromov (waist inequality).
Feels related to persistent homology & applied topology.

Metric thickenings, Borsuk–Ulam theorems, and orbitopes

Henry Adams (Colorado State University)
Johnathan Bush (Colorado State University)
Florian Frick (Carnegie Mellon)

Paper in preparation



Theorem (Borsuk–Ulam)

For $f : S^n \to \mathbb{R}^n$, there exists a point $x \in S^n$ with $f(x) = f(-x)$.

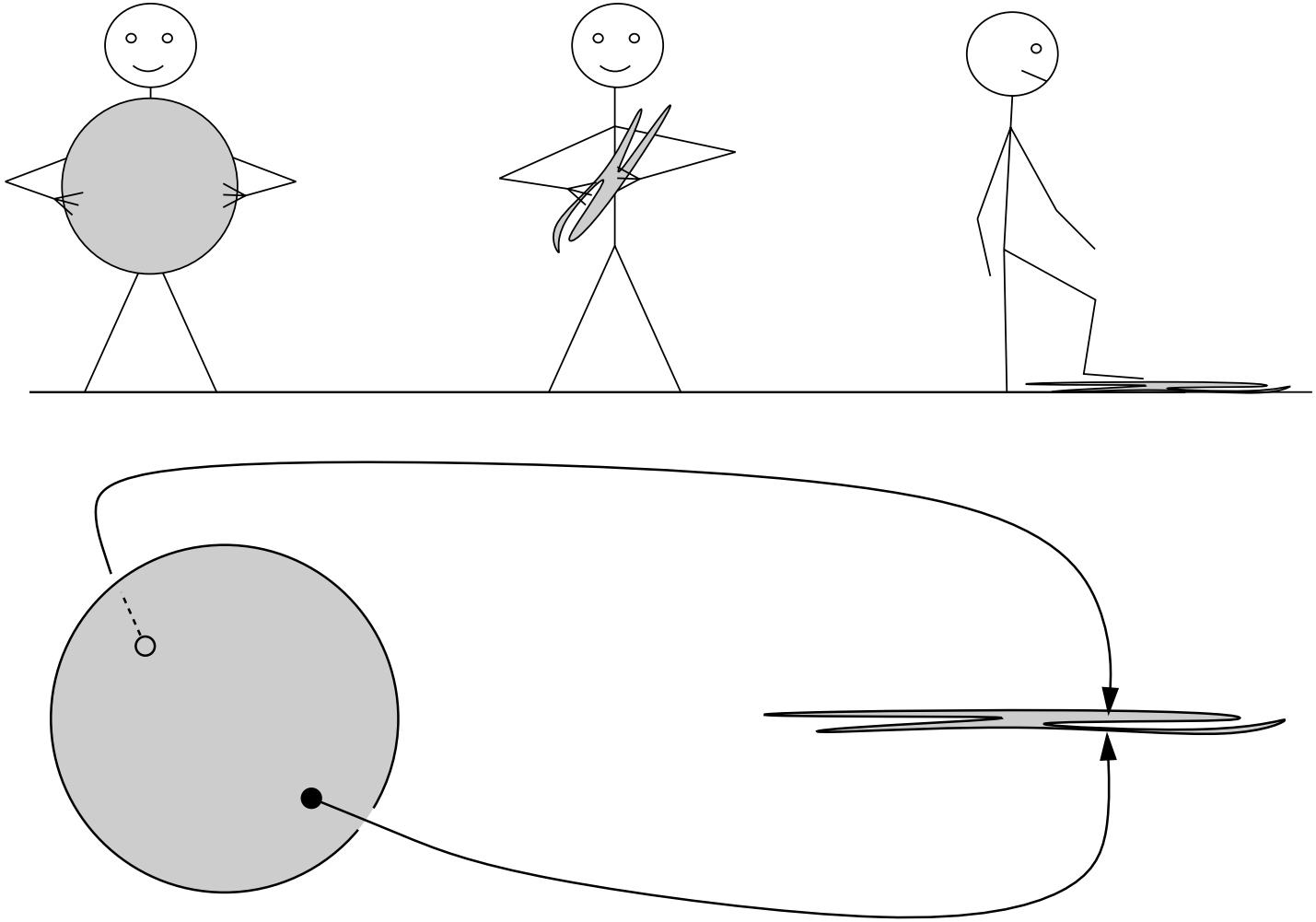


Figure credit: Jiří Matoušek, *Using the Borsuk–Ulam theorem*

Theorem (Borsuk–Ulam)

For $f : S^n \to \mathbb{R}^n$, there exists a point $x \in S^n$ with $f(x) = f(-x)$.

Theorem (Gromov's "waist" theorem)

For $f : S^n \to \mathbb{R}^n$, there exists some $y \in \mathbb{R}^n$ with $\mathrm{vol}_0(f^{-1}(y)) \geq \mathrm{vol}_0(S^0 \subseteq S^n)$.

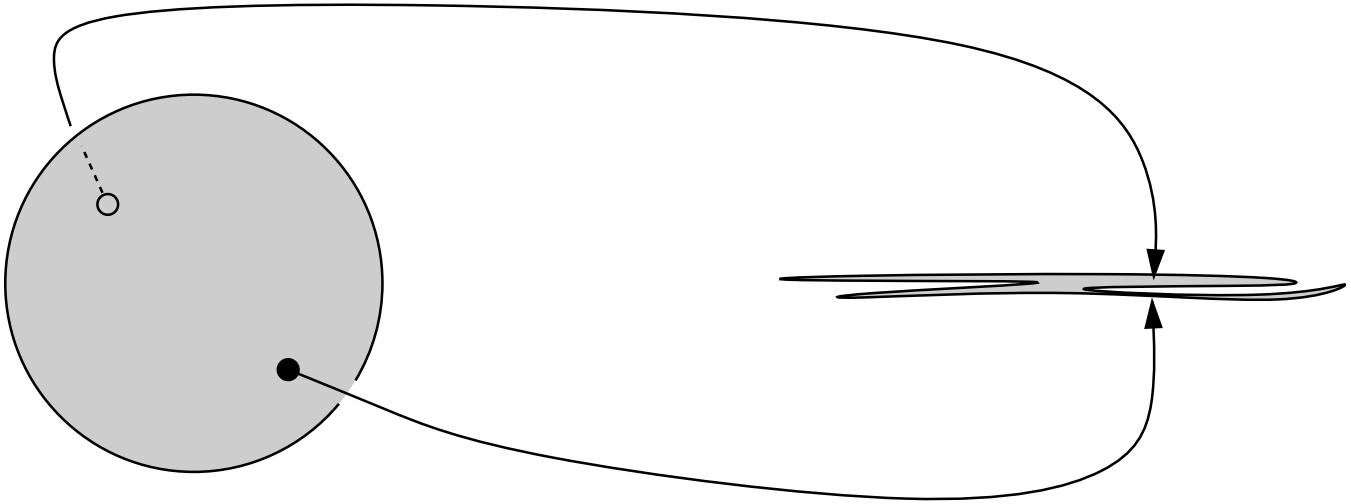


Figure credit: Jiří Matoušek, *Using the Borsuk–Ulam theorem*

Theorem (Borsuk–Ulam)

For $f : S^n \to \mathbb{R}^n$, there exists a point $x \in S^n$ with $f(x) = f(-x)$.

Theorem (Gromov's "waist" theorem)

For $f : S^n \to \mathbb{R}^k$ with $k \leq n$, there exists some $y \in \mathbb{R}^n$ with $\mathrm{vol}_{n-k}(f^{-1}(y)) \geq \mathrm{vol}_{n-k}(S^{n-k} \subseteq S^n)$.

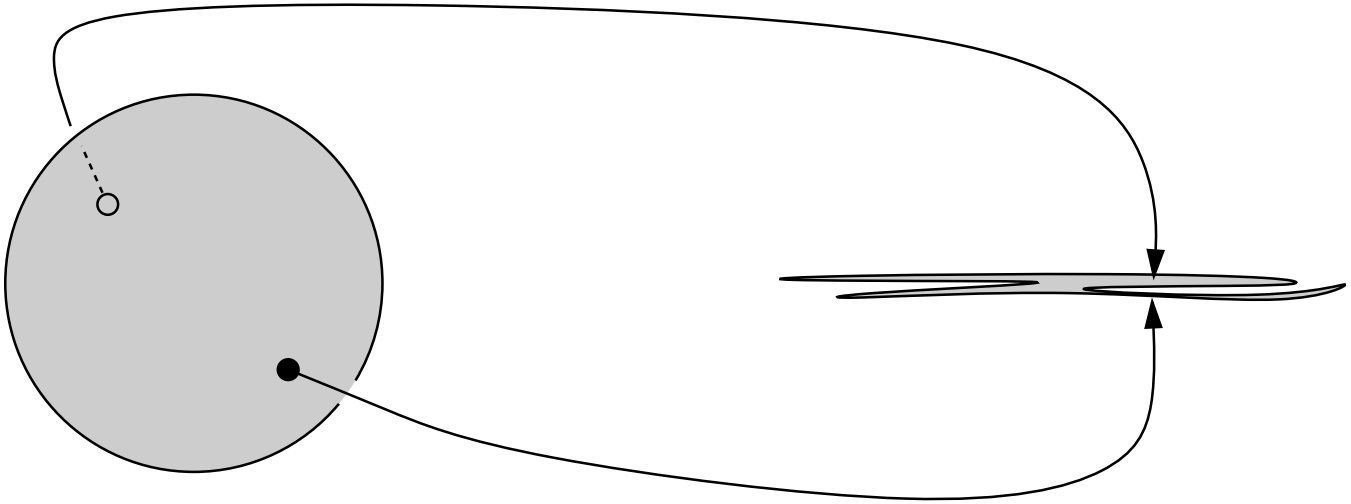


Figure credit: Jiří Matoušek, *Using the Borsuk–Ulam theorem*

Theorem (Borsuk–Ulam)

For $f : S^n \to \mathbb{R}^n$, there exists a point $x \in S^n$ with $f(x) = f(-x)$.

Theorem (Gromov's "waist" theorem)

For $f : S^n \to \mathbb{R}^k$ with $k \leq n$, there exists some $y \in \mathbb{R}^n$ with $\mathrm{vol}_{n-k}(f^{-1}(y)) \geq \mathrm{vol}_{n-k}(S^{n-k} \subseteq S^n)$.

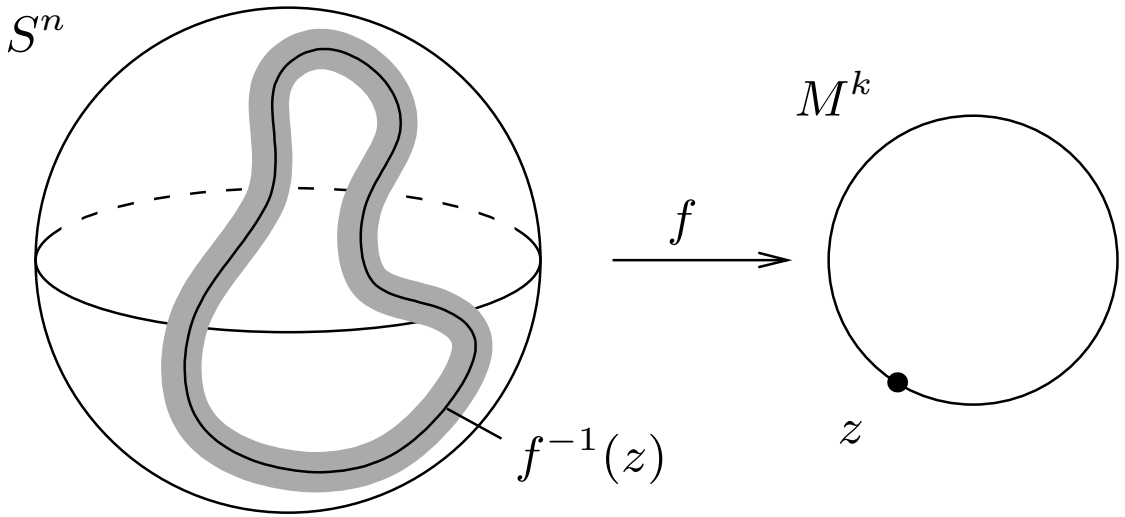


Figure credit: Benjamin Matschke, *Journal of Topology & Analysis*

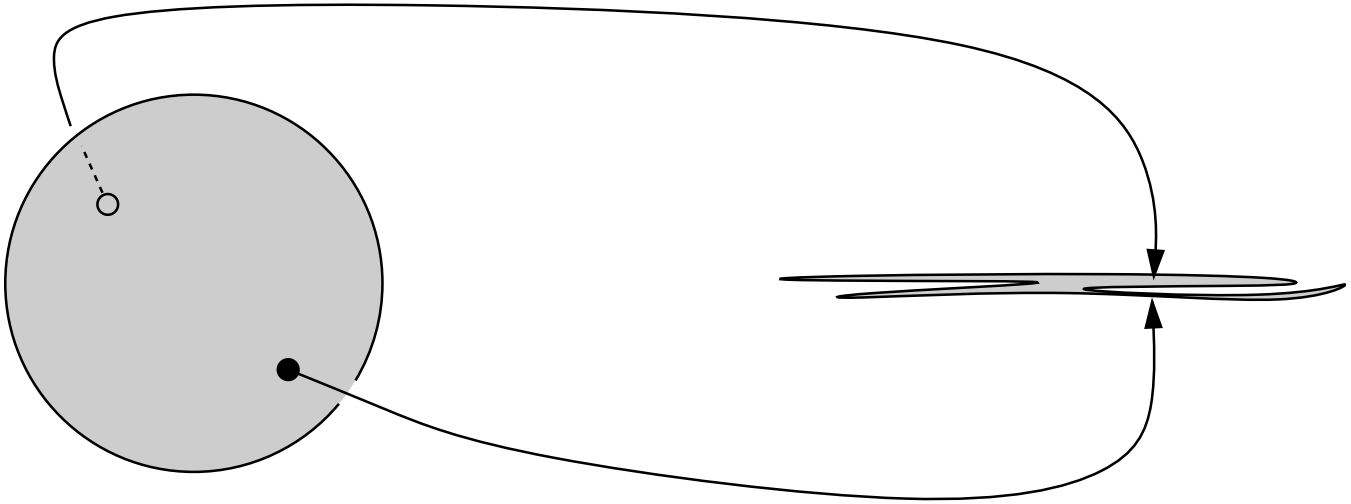
Theorem (Borsuk–Ulam)

For $f\colon S^n \to \mathbb{R}^n$, there exists a point $x \in S^n$ with $f(x) = f(-x)$.

Theorem (Gromov's "waist" theorem)

For $f\colon S^n \to \mathbb{R}^k$ with $k \leq n$, there exists some $y \in \mathbb{R}^n$ with $\mathrm{vol}_{n-k}(f^{-1}(y)) \geq \mathrm{vol}_{n-k}(S^{n-k} \subseteq S^n)$.

What about $f\colon S^n \to \mathbb{R}^k$ with $k \geq n$?



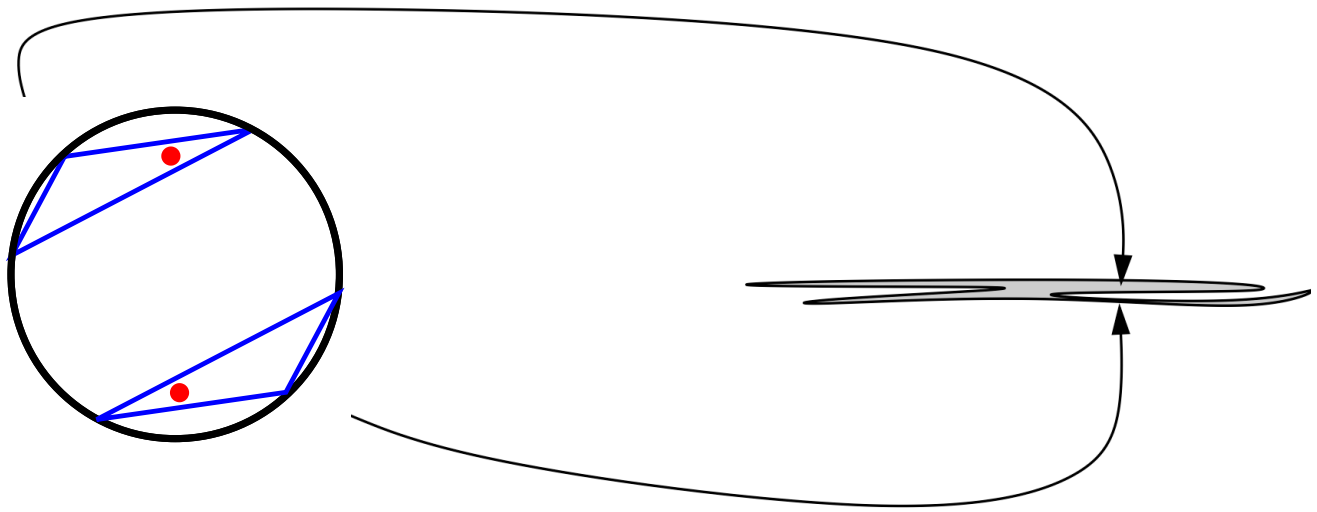
Theorem (Borsuk–Ulam)

For $f : S^n \to \mathbb{R}^n$, there exists a point $x \in S^n$ with $f(x) = f(-x)$.

Theorem (Gromov's "waist" theorem)

For $f : S^n \to \mathbb{R}^k$ with $k \leq n$, there exists some $y \in \mathbb{R}^n$ with $\mathrm{vol}_{n-k}(f^{-1}(y)) \geq \mathrm{vol}_{n-k}(S^{n-k} \subseteq S^n)$.

What about $f : S^n \to \mathbb{R}^k$ with $k \geq n$?

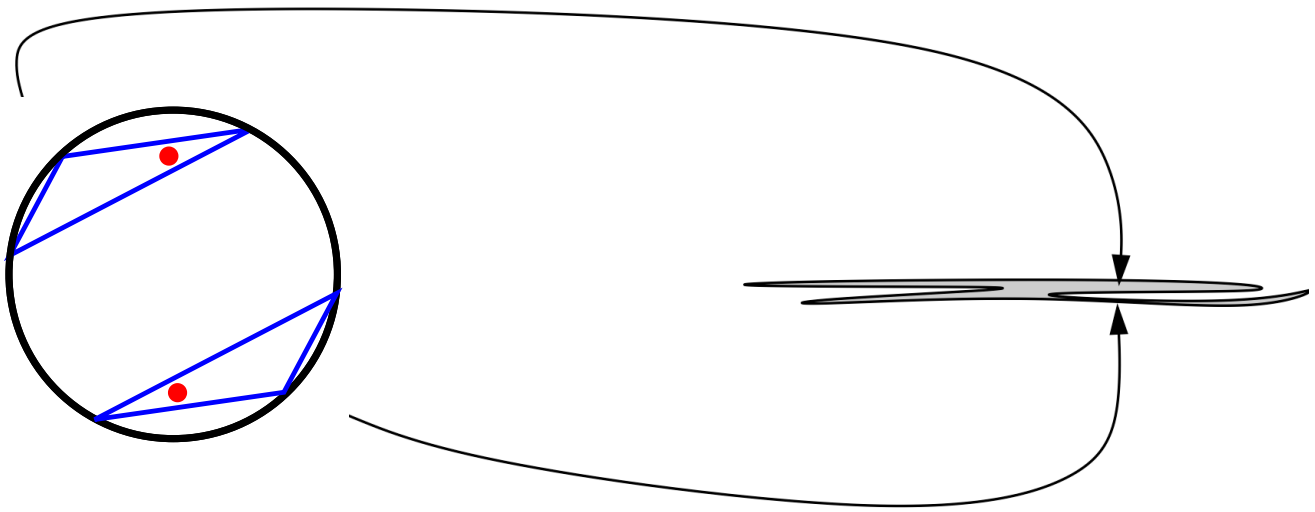


Theorem (Borsuk–Ulam)

For $f : S^n \to \mathbb{R}^n$, there exists a point $x \in S^n$ with $f(x) = f(-x)$.

Theorem (A, Bush, Frick)

For $f : S^1 \to \mathbb{R}^{2k+1}$, there exists a set $\{x_0, \ldots, x_{2k+1}\}$ of diameter at most $\frac{k}{2k+1}$ such that $\sum \lambda_i f(x_i) = \sum \lambda_i f(-x_i)$.

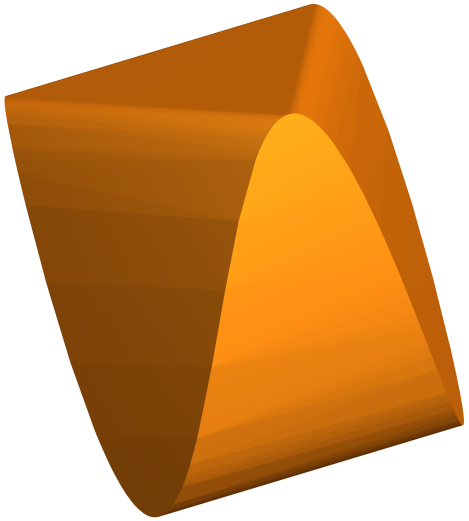


Theorem (Borsuk–Ulam)

For $f : S^n \to \mathbb{R}^n$, there exists a point $x \in S^n$ with $f(x) = f(-x)$.

Theorem (A, Bush, Frick)

For $f : S^1 \to \mathbb{R}^{2k+1}$, there exists a set $\{x_0, \ldots, x_{2k+1}\}$ of diameter at most $\frac{k}{2k+1}$ such that $\sum \lambda_i f(x_i) = \sum \lambda_i f(-x_i)$.



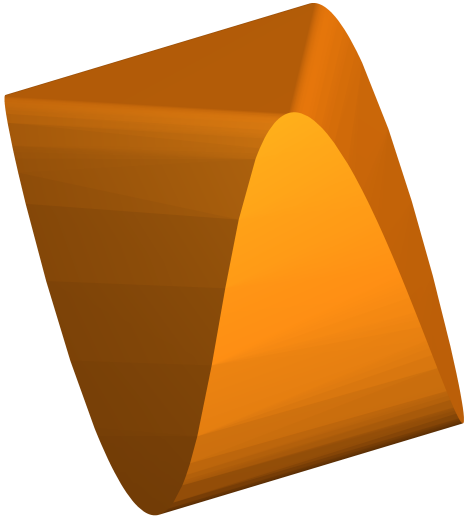
Theorem (Borsuk–Ulam)

For $f\colon S^n \to \mathbb{R}^n$, there exists a point $x \in S^n$ with $f(x) = f(-x)$.

Theorem (A, Bush, Frick)

For $f\colon S^1 \to \mathbb{R}^{2k+1}$, there exists a set $\{x_0, \ldots, x_{2k+1}\}$ of diameter at most $\frac{k}{2k+1}$ such that $\sum \lambda_i f(x_i) = \sum \lambda_i f(-x_i)$.

Proof: $S^{2k+1} \simeq \mathrm{VR}^m(S^1; \frac{k}{2k+1}) \xrightarrow{f} \mathbb{R}^{2k+1}$



Sharpness: $f = (\cos\theta, \sin\theta, \cos 3\theta, \sin 3\theta, \cos 5\theta, \sin 5\theta, \ldots)$

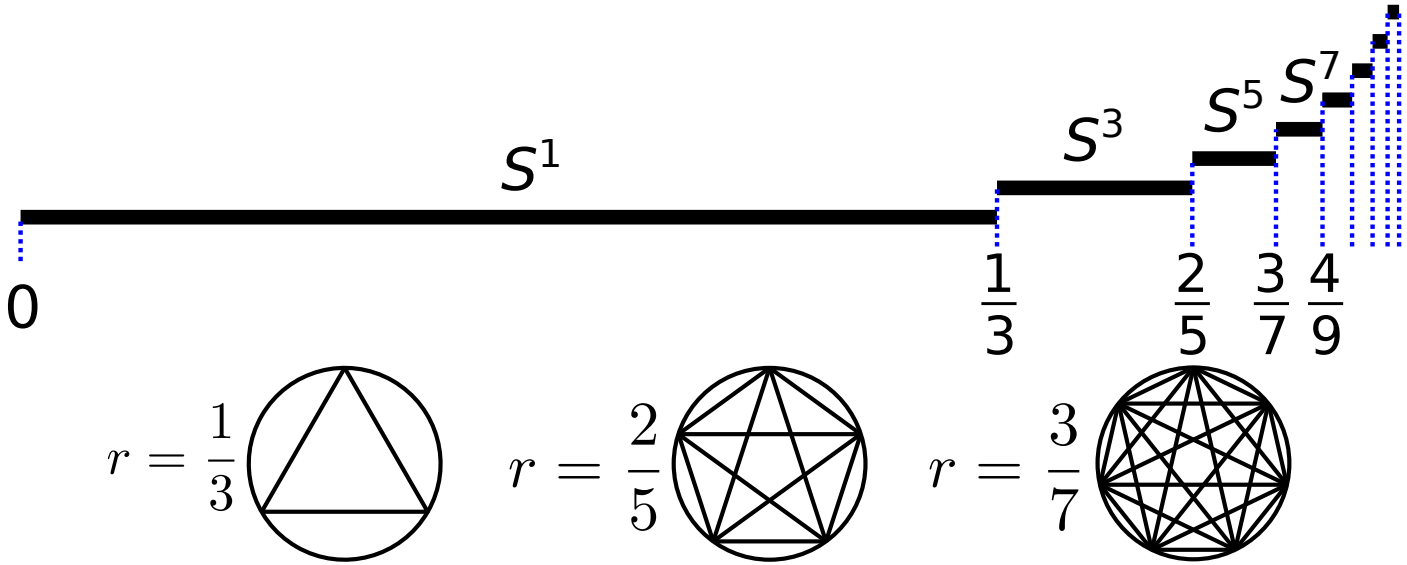
Theorem (Borsuk–Ulam)

For $f\colon S^n \to \mathbb{R}^n$, there exists a point $x \in S^n$ with $f(x) = f(-x)$.

Theorem (A, Bush, Frick)

For $f\colon S^1 \to \mathbb{R}^{2k+1}$, there exists a set $\{x_0, \ldots, x_{2k+1}\}$ of diameter at most $\frac{k}{2k+1}$ such that $\sum \lambda_i f(x_i) = \sum \lambda_i f(-x_i)$.

Proof: $S^{2k+1} \simeq \mathrm{VR}^m(S^1; \frac{k}{2k+1}) \xrightarrow{f} \mathbb{R}^{2k+1}$



Sharpness: $f = (\cos\theta, \sin\theta, \cos 3\theta, \sin 3\theta, \cos 5\theta, \sin 5\theta, \ldots)$

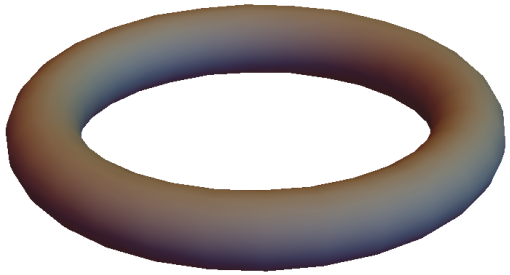
Theorem (Borsuk–Ulam)

For $f\colon S^n \to \mathbb{R}^n$, there exists a point $x \in S^n$ with $f(x) = f(-x)$.

Theorem (A, Bush, Frick)

For $f\colon S^1 \to \mathbb{R}^{2k+1}$, there exists a set $\{x_0, \ldots, x_{2k+1}\}$ of diameter at most $\frac{k}{2k+1}$ such that $\sum \lambda_i f(x_i) = \sum \lambda_i f(-x_i)$.

Proof: $S^{2k+1} \simeq \mathrm{VR}^m(S^1; \frac{k}{2k+1}) \xrightarrow{f} \mathbb{R}^{2k+1}$



Sharpness: $f = (\cos\theta, \sin\theta, \cos 3\theta, \sin 3\theta, \cos 5\theta, \sin 5\theta, \ldots)$

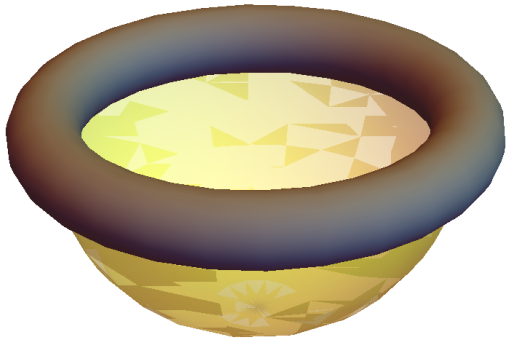
Theorem (Borsuk–Ulam)

For $f\colon S^n \to \mathbb{R}^n$, there exists a point $x \in S^n$ with $f(x) = f(-x)$.

Theorem (A, Bush, Frick)

For $f\colon S^1 \to \mathbb{R}^{2k+1}$, there exists a set $\{x_0, \ldots, x_{2k+1}\}$ of diameter at most $\frac{k}{2k+1}$ such that $\sum \lambda_i f(x_i) = \sum \lambda_i f(-x_i)$.

Proof: $S^{2k+1} \simeq \mathrm{VR}^m(S^1; \frac{k}{2k+1}) \xrightarrow{f} \mathbb{R}^{2k+1}$



Sharpness: $f = (\cos\theta, \sin\theta, \cos 3\theta, \sin 3\theta, \cos 5\theta, \sin 5\theta, \ldots)$

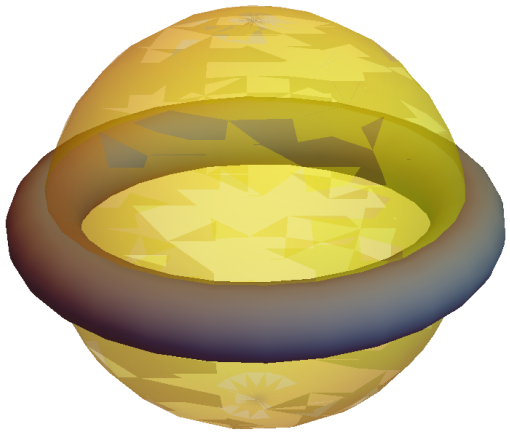
Theorem (Borsuk–Ulam)

For $f\colon S^n \to \mathbb{R}^n$, there exists a point $x \in S^n$ with $f(x) = f(-x)$.

Theorem (A, Bush, Frick)

For $f\colon S^1 \to \mathbb{R}^{2k+1}$, there exists a set $\{x_0, \ldots, x_{2k+1}\}$ of diameter at most $\frac{k}{2k+1}$ such that $\sum \lambda_i f(x_i) = \sum \lambda_i f(-x_i)$.

Proof: $S^{2k+1} \simeq \mathrm{VR}^m(S^1; \frac{k}{2k+1}) \xrightarrow{f} \mathbb{R}^{2k+1}$



Sharpness: $f = (\cos\theta, \sin\theta, \cos 3\theta, \sin 3\theta, \cos 5\theta, \sin 5\theta, \ldots)$

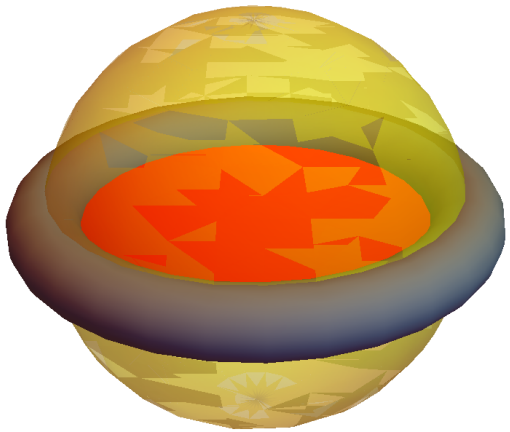
Theorem (Borsuk–Ulam)

For $f\colon S^n \to \mathbb{R}^n$, there exists a point $x \in S^n$ with $f(x) = f(-x)$.

Theorem (A, Bush, Frick)

For $f\colon S^1 \to \mathbb{R}^{2k+1}$, there exists a set $\{x_0, \ldots, x_{2k+1}\}$ of diameter at most $\frac{k}{2k+1}$ such that $\sum \lambda_i f(x_i) = \sum \lambda_i f(-x_i)$.

Proof: $S^{2k+1} \simeq \mathrm{VR}^m(S^1; \frac{k}{2k+1}) \xrightarrow{f} \mathbb{R}^{2k+1}$



Sharpness: $f = (\cos\theta, \sin\theta, \cos 3\theta, \sin 3\theta, \cos 5\theta, \sin 5\theta, \ldots)$

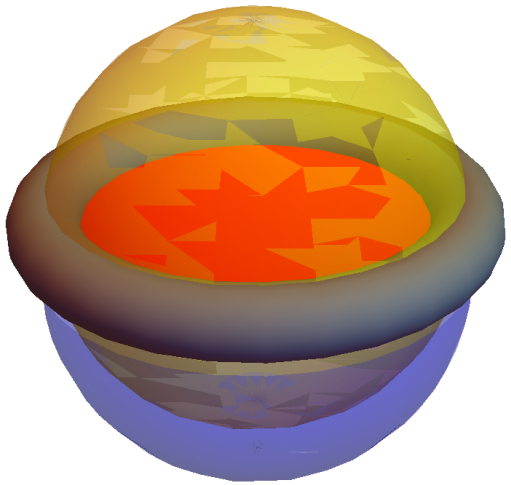
Theorem (Borsuk–Ulam)

For $f\colon S^n \to \mathbb{R}^n$, there exists a point $x \in S^n$ with $f(x) = f(-x)$.

Theorem (A, Bush, Frick)

For $f\colon S^1 \to \mathbb{R}^{2k+1}$, there exists a set $\{x_0, \ldots, x_{2k+1}\}$ of diameter at most $\frac{k}{2k+1}$ such that $\sum \lambda_i f(x_i) = \sum \lambda_i f(-x_i)$.

Proof: $S^{2k+1} \simeq \mathrm{VR}^m(S^1; \frac{k}{2k+1}) \xrightarrow{f} \mathbb{R}^{2k+1}$



Sharpness: $f = (\cos\theta, \sin\theta, \cos 3\theta, \sin 3\theta, \cos 5\theta, \sin 5\theta, \ldots)$

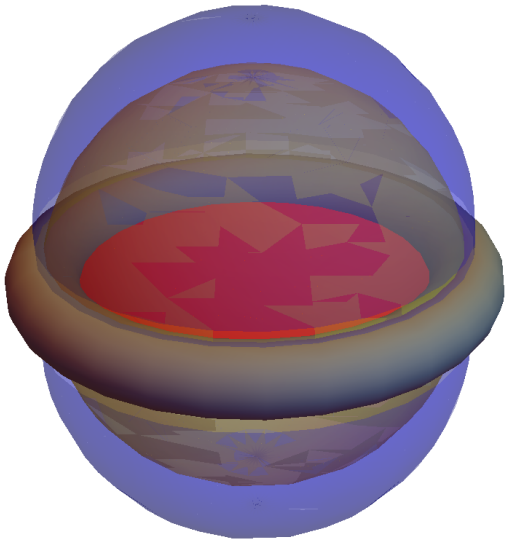
Theorem (Borsuk–Ulam)

For $f\colon S^n \to \mathbb{R}^n$, there exists a point $x \in S^n$ with $f(x) = f(-x)$.

Theorem (A, Bush, Frick)

For $f\colon S^1 \to \mathbb{R}^{2k+1}$, there exists a set $\{x_0, \ldots, x_{2k+1}\}$ of diameter at most $\frac{k}{2k+1}$ such that $\sum \lambda_i f(x_i) = \sum \lambda_i f(-x_i)$.

Proof: $S^{2k+1} \simeq \mathrm{VR}^m(S^1; \frac{k}{2k+1}) \xrightarrow{f} \mathbb{R}^{2k+1}$



Sharpness: $f = (\cos\theta, \sin\theta, \cos 3\theta, \sin 3\theta, \cos 5\theta, \sin 5\theta, \ldots)$

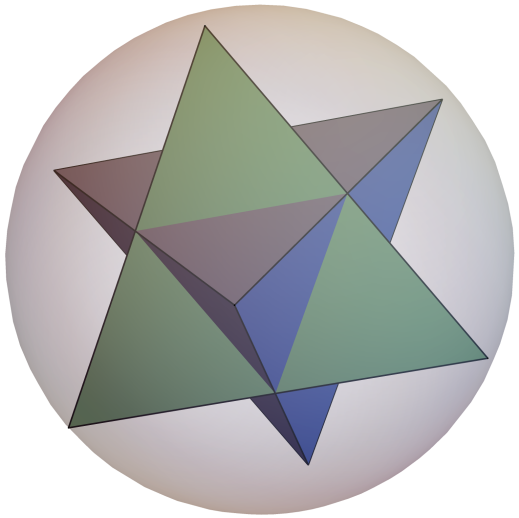
Theorem (Borsuk–Ulam)

For $f : S^n \to \mathbb{R}^n$, there exists a point $x \in S^n$ with $f(x) = f(-x)$.

Theorem (A, Bush, Frick)

For $f : S^n \to \mathbb{R}^{n+2}$, there exists a set $\{x_0, \ldots, x_{n+2}\}$ of diameter at most r_n such that $\sum \lambda_i f(x_i) = \sum \lambda_i f(-x_i)$.

Proof: S^{n+2} " $\subseteq$ " $\mathrm{VR}^m(S^n; r_n) \xrightarrow{f} \mathbb{R}^{n+2}$



r_n is the side-length of an inscribed simplex

References

- Michał Adamaszek, Henry Adams, Florian Frick, *Metric reconstruction via optimal transport*, SIAM Journal on Applied Algebra and Geometry 2 (2018).
- Michał Adamaszek and Henry Adams, *The Vietoris–Rips complexes of a circle*, Pacific Journal of Mathematics 290 (2017), 1–40.
- Henry Adams, Johnathan Bush, and Florian Frick, *Metric thickenings, Borsuk–Ulam theorems, and orbitopes* (2019), in preparation.
- Jean-Claude Hausmann, *On the Vietoris–Rips complexes and a cohomology theory for metric spaces*, Annals of Mathematics Studies 138 (1995), 175–188.

Thank you!