Metric Reconstruction Via Optimal Transport

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Main point A simplicial complex whose vertex set is a metric space should often be equipped with an ______________ instead of the simplicial complex topology.
$X$ metric space, $r \geq 0$.

**Def** The Vietoris-Rips simplicial complex has
- vertex set $X$
- finite simplex $\sigma \subseteq X$ when

**History**
- Cohomology theory for metric spaces
- Geometric group theory
- Applied topology
Thm (Hausmann 1995)

$M$ compact Riemannian manifold. Then $\exists \, r_0 > 0$ such that $VR(M; r) \cong M \ \forall r < r_0$.

Proof Sketch

$VR(M; r) \xrightarrow{c \text{ depends on}} M$

- Not canonical
- $M \xrightarrow{\times} VR(M; r)$ not continuous.

Thm (Latschev 2001)

$M$, $r_0$ as above. $\forall r < r_0 \ \exists \delta > 0$ such that if $d_{GH}(x, M) < \delta$, then $x \notin M$.

Ex
Cyclo-octane molecule $C_8H_{16}$
(Martin, Thompson, Coutsias, Watson 2010)

Stability $\text{PH}_1(VR(M; r))$

$\text{PH}_1(VR(X; r))$
**Metric Reconstruction**

\[ \text{Def} \quad X \text{ metric space, } r \geq 0. \]

The **Vietoris–Rips metric thickening** is

\[
VR^m(X; r) = \left\{ \frac{1}{k} \sum_{i=0}^{k} \lambda_i \left| \begin{array}{c}
\lambda_i \geq 0 \\
\sum \lambda_i = 1
\end{array} \right. \right\}
\]

equipped with the $p$-Wasserstein metric.

**Prop** \( VR^m(X; r) \) is an

**Rmk** Can do this for any simplicial complex \( K \) whose vertex set is a metric space, yielding the
Thm. $M$ complete Riemannian manifold, $r_0 > 0$ small enough so that measures of diameter $\leq r_0$ have unique Karcher means. Then

\[
Pf \text{ Sketch} \quad VR^n(M; r) \quad \sum_{i \in I} \lambda_i x_i
\]
$S'$ is circle with geodesic metric, unit circumference.

Theorem:

$VR(S'; r) \simeq \begin{cases} 
 & \text{if } \frac{k}{2k+1} < r < \frac{k+1}{2k+3} \quad k \in \mathbb{N} \\
& \text{if } r = \frac{k}{2k+1} 
\end{cases}$

By contrast, $VR^m(S'; \frac{1}{3}) \simeq S^3$. Why?
More generally,

$$f(r, \sigma) = \begin{cases} V_{n}(S^{n}) & \text{if } n \leq r \leq \sigma \\ 1 & \text{if } r = n \\ r = r' \end{cases}$$
Questions
(1) $VR^m(S^n;r)$ for larger $r$?
(2) Čech$^m(S^n;r)$?
(3) Other manifolds?
(4) $VR_\varepsilon(X;r) \approx VR_\varepsilon(X;r)$?
(5) Stability of $VR^m(X;r)$ for $X$ infinite?
(6) Morse and Morse-Bott theories (Mirth PhD thesis)
(7) Borsuk-Ulam theorems $S^n \to \mathbb{R}^k$ for $k \geq n$
   (A., Bush, Frick, Mathematika 2020)
(8) Categorical framework – infinite support?
   (A., Bush, Mirth, Applied Category Theory 2020)
(9) Connections to quantitative topology, especially Kuratowski embeddings and filling radii.
   (Lim, Mémoli, Okutan 2020, Katz 1983-1991)