Vietoris–Rips and Restricted Čech Complexes of Circular Points

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Main Theorem

Theorem. A Vietoris–Rips complex or a restricted Čech complex on a finite set of points from the circle is homotopy equivalent to either a point, an odd sphere, or a wedge sum of spheres of the same even dimension.

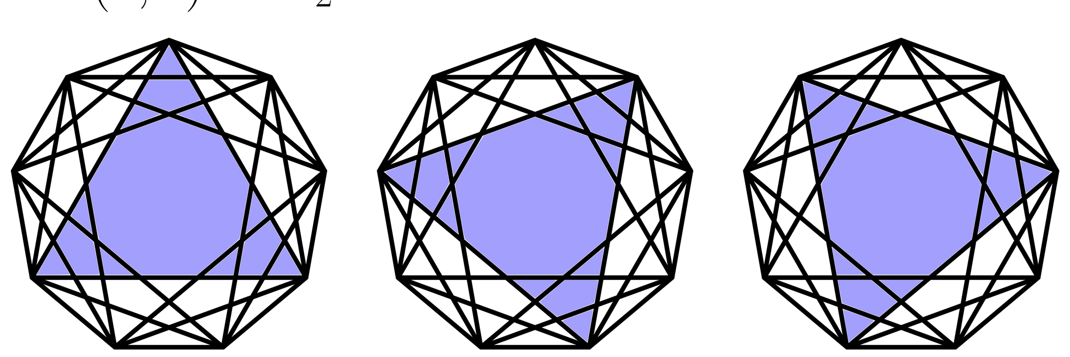
Remark: The homotopy types of Vietoris–Rips complexes of evenly-spaced circular points are proven in [1].

Remark: The proof for arbitrary circular points relies on the evenly-spaced case, which we consider first.

Notation for evenly-spaced points

Definition. Let $\mathrm{VR}(n,k)$ be the Vietoris–Rips complex on n evenly-spaced circular vertices with connectivity parameter $2\pi k/n$. That is, simplex σ is in $\mathrm{VR}(n,k)$ when $\mathrm{diam}(\sigma) \leq 2\pi k/n$.

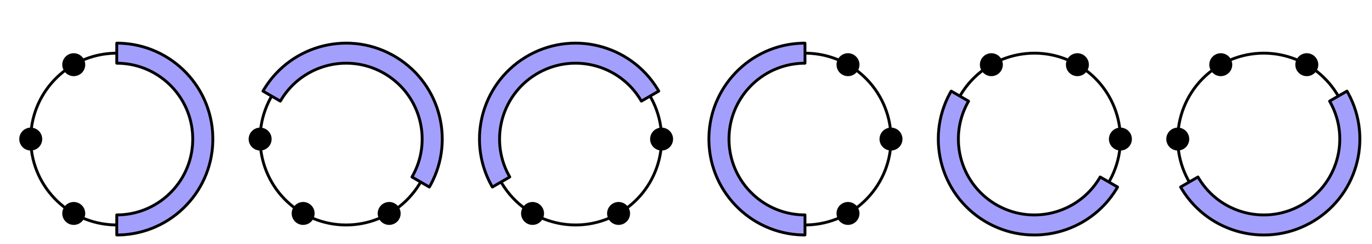
Example. $\mathrm{VR}(9,3)$ is the clique complex of the graph below, giving $\mathrm{VR}(9,3) \simeq \vee_2 S^2$.



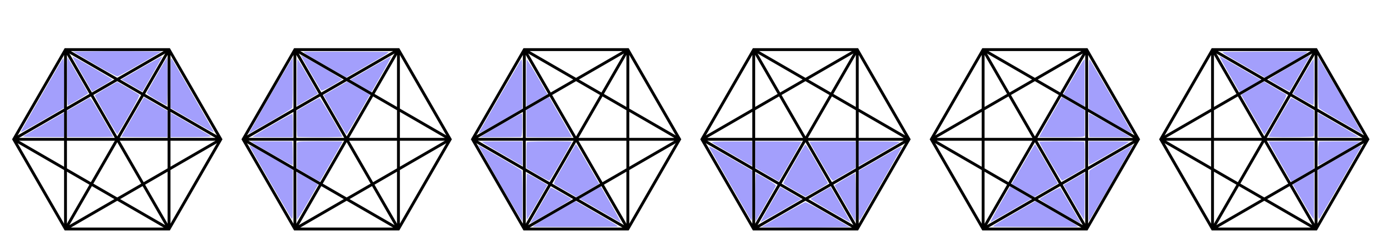
To visualize this, note that $\mathrm{VR}(9,3)$ has three maximal 2-simplices.

Definition. Let restricted Čech complex $\check{\mathrm{C}}(n,k)$ be the nerve of the covering of S^1 by n evenly-spaced closed arcs of arc length $2\pi k/n$.

Example. $\check{\mathrm{C}}(6,3)$ is the nerve of the 6 circular arcs below, giving $\check{\mathrm{C}}(6,3) \simeq \vee_2 S^2$.



Alternatively, $\check{\mathrm{C}}(6,3)$ has 6 maximal 3-simplices, glued together to form $\check{\mathrm{C}}(6,3) \simeq \vee_2 S^2$.



Case of evenly-spaced points

Corollary 6.7 from [1].
Let $k < n/2$ and write $n-k = q(n-2k)+r$ with $0 \leq r < n-2k$.

$$\text{Then } \mathrm{VR}(n,k) \simeq \begin{cases} \vee_{n-2k-1} S^{2q-2} & \text{if } r=0 \\ S^{2q-1} & \text{otherwise.} \end{cases}$$

	$k=1$	2	3	4	5	6	7	8	9	10	11	12
$n=6$	S^1	S^2	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
7	S^1	S^1	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
8	S^1	S^1	S^3	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
9	S^1	S^1	$\vee_2 S^2$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
10	S^1	S^1	S^1	S^4	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
11	S^1	S^1	S^1	S^3	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
12	S^1	S^1	S^1	$\vee_3 S^2$	S^5	$*$	$*$	$*$	$*$	$*$	$*$	$*$
13	S^1	S^1	S^1	S^1	S^3	$*$	$*$	$*$	$*$	$*$	$*$	$*$
14	S^1	S^1	S^1	S^1	S^3	S^6	$*$	$*$	$*$	$*$	$*$	$*$
15	S^1	S^1	S^1	S^1	$\vee_4 S^2$	$\vee_2 S^4$	$*$	$*$	$*$	$*$	$*$	$*$
16	S^1	S^1	S^1	S^1	S^1	S^3	S^7	$*$	$*$	$*$	$*$	$*$
17	S^1	S^1	S^1	S^1	S^1	S^3	S^5	$*$	$*$	$*$	$*$	$*$
18	S^1	S^1	S^1	S^1	S^1	$\vee_5 S^2$	S^3	S^8	$*$	$*$	$*$	$*$
19	S^1	S^1	S^1	S^1	S^1	S^1	S^3	S^5	$*$	$*$	$*$	$*$
20	S^1	S^1	S^1	S^1	S^1	S^1	S^3	$\vee_3 S^4$	S^9	$*$	$*$	$*$
21	S^1	S^1	S^1	S^1	S^1	S^1	$\vee_6 S^2$	S^3	$\vee_2 S^6$	$*$	$*$	$*$
22	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^3	S^5	S^{10}	$*$	$*$
23	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^3	S^3	S^7	$*$	$*$
24	S^1	S^1	S^1	S^1	S^1	S^1	S^1	$\vee_7 S^2$	S^3	S^5	S^{11}	$*$
25	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^3	$\vee_4 S^4$	S^7	$*$
26	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^3	S^3	S^5	S^{12}
27	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^1	$\vee_8 S^2$	S^3	S^5	$\vee_2 S^8$
28	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^3	S^3	$\vee_3 S^6$

Proposition.
Let $k < n-1$ and write $n = q(n-k)+r$ with $0 \leq r < n-k$.

$$\text{Then } \check{\mathrm{C}}(n,k) \simeq \begin{cases} \vee_{n-k-1} S^{2q-2} & \text{if } r=0 \\ S^{2q-1} & \text{otherwise.} \end{cases}$$

The proof uses [1, 2, 3].

	$k=1$	2	3	4	5	6	7	8	9	10	11	12
$n=3$	S^1	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
4	S^1	S^2	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
5	S^1	S^1	S^3	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
6	S^1	S^1	$\vee_2 S^2$	S^4	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$
7	S^1	S^1	S^1	S^3	S^5	$*$	$*$	$*$	$*$	$*$	$*$	$*$
8	S^1	S^1	S^1	$\vee_3 S^2$	S^3	S^6	$*$	$*$	$*$	$*$	$*$	$*$
9	S^1	S^1	S^1	S^1	S^3	$\vee_2 S^4$	S^7	$*$	$*$	$*$	$*$	$*$
10	S^1	S^1	S^1	S^1	$\vee_4 S^2$	S^3	S^5	S^8	$*$	$*$	$*$	$*$
11	S^1	S^1	S^1	S^1	S^1	S^3	S^3	S^5	S^9	$*$	$*$	$*$
12	S^1	S^1	S^1	S^1	S^1	$\vee_5 S^2$	S^3	$\vee_3 S^4$	$\vee_2 S^6$	S^{10}	$*$	$*$
13	S^1	S^1	S^1	S^1	S^1	S^1	S^3	S^3	S^5	S^7	S^{11}	$*$
14	S^1	S^1	S^1	S^1	S^1	S^1	$\vee_6 S^2$	S^3	S^3	S^5	S^7	S^{12}
15	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^3	S^3	$\vee_4 S^4$	S^5	$\vee_2 S^8$
16	S^1	S^1	S^1	S^1	S^1	S^1	S^1	$\vee_7 S^2$	S^3	S^3	S^5	$\vee_3 S^6$
17	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^3	S^3	S^3	S^5
18	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^1	$\vee_8 S^2$	S^3	S^3	$\vee_5 S^4$
19	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^3	S^3	S^3
20	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^1	$\vee_9 S^2$	S^3	S^3
21	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^3	S^3
22	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^1	$\vee_{10} S^2$	S^3
23	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^3
24	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^1	S^1	$\vee_{11} S^2$

Remark: Note $\mathrm{VR}(n+k,k) \simeq \check{\mathrm{C}}(n,k)$. For example, $\mathrm{VR}(9,3) \simeq \vee_2 S^2 \simeq \check{\mathrm{C}}(6,3)$.

Arbitrary circular points

When built on an arbitrary finite set of circular points, a Vietoris–Rips or restricted Čech complex is still homotopy equivalent to either a point, an odd sphere, or a wedge sum of spheres of the same even dimension.

Proof idea: If K is a simplicial complex and u and v are two distinct vertices with $\mathrm{st}(u) \subseteq \mathrm{st}(v)$ (we say u is *dominated* by v), then $K \simeq K \setminus u$. We show how to remove dominated vertices until we are left with a complex equivalent to some $\mathrm{VR}(n,k)$ or $\check{\mathrm{C}}(n,k)$.

References

[1] Michał Adamaszek, *Clique complexes and graph powers*, Israel Journal of Mathematics 196 (2013), 295–319.

[2] Jonathan Barmak, *Star clusters in independence complexes of graphs*, Advances in Mathematics 241 (2013), 33–57.

[3] Jakob Jonsson *On the topology of independence complexes of triangle-free graphs*, unpublished manuscript.