An introduction to discrete Morse theory

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Abstract

This talk will be an introduction to discrete Morse theory. Whereas standard Morse theory studies smooth functions on a differentiable manifold, discrete Morse theory considers cell complexes equipped with a discrete function assigning a single value to each cell. Many results in Morse theory have discrete analogues, and I hope to explain these results by example. I will follow "A User's Guide to Discrete Morse Theory" by Robin Forman [1].

0.1 Morse Theory

Let M be a compact differentiable manifold and $f\colon M \to \mathbb{R}$ a Morse function (smooth with non-degenerate critical points, i.e. Hessions nonsingular) [2].

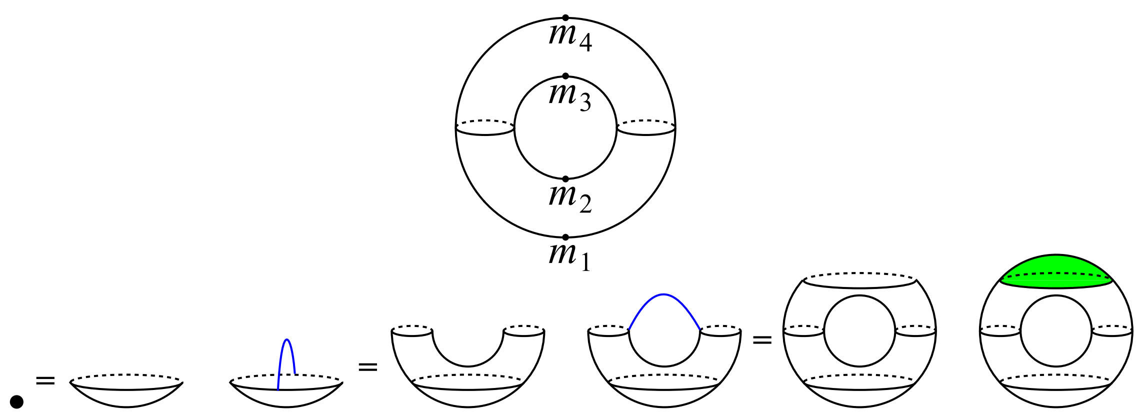

Figure 1: Torus with Morse function given by height. The critical points of index 0, 1, 1, 2 show the torus has a CW decomposition with one 0-cell, two 1-cells, and one 2-cell.

Theorem 1. M is homotopy equivalent ($\simeq$) to a CW complex with one d-cell for each critical point of index d (number of directions in which f decreases).

Proof. (Sketch). For $t \in \mathbb{R}$ let $M(t) = f^{-1}(\infty, t]$ be its sublevelset.

- No critical points with value in $(s, t] \Rightarrow M(t) \simeq M(s)$.
- Single critical point (index d) with value in $(s, t] \Rightarrow$

$$M(t) \simeq M(s) \text{ with a single } d\text{-cell added.}$$

□

0.2 Discrete Morse Theory

Let K be a simplicial complex (theory also holds for CW complexes).

Definition 1. Function $f : K \to \mathbb{R}$ is *discrete Morse* if for every $\alpha^{(d)} \in K$,

- at most one $\beta^{(d+1)} \supset \alpha$ satisfies $f(\beta) \leq f(\alpha)$
- at most one $\gamma^{(d-1)} \subset \alpha$ satisfies $f(\gamma) \geq f(\alpha)$.

Figure 2: Not discrete Morse; discrete Morse.

Lemma 1. For each α there are either no such β or no such γ.

Proof. (Sketch). □

Figure 3: $d = 1$; $d = 2$.

Definition 2. If there are no such β and no such γ, then α is *critical*.

Let $f : K \to \mathbb{R}$ be discrete Morse.

Theorem 2. $K \simeq$ a CW complex with one d-cell for each critical simplex of dimension d.

Proof. (Sketch). For $t \in \mathbb{R}$ let $K(t)$ be its sublevelset subcomplex (include faces of all simplices with $f(\alpha) \leq t$).

- No critical simplices with value in $(s, t] \Rightarrow K(t) \simeq K(s)$ (simplicial collapse).
- Single critical simplex $\alpha^{(d)}$ with $f(\alpha) \in (s, t] \Rightarrow$

$$K(t) \simeq K(s) \text{ with a single } d\text{-cell added.}$$

Figure 4: $K(0) \subset K(1) = K(2) \subset K(3) = K(4) \subset K$

For $\sigma \in K$, defining $f(\sigma) = \dim(\sigma)$ gives a "trivial" Morse function with all simplices critical. Often one wants to find a Morse function with as few critical simplices as possible. □

0.3 Gradient Vector Fields

Definition 3. A *discrete vector field* V on K is a collection of pairs $\alpha^{(d)} \subset \beta^{(d+1)}$ such that each simplex is in at most one pair.

Figure 5: A discrete vector field on the projective plane $\mathbb{P}^2$.

From a discrete Morse function $f\colon K \to \mathbb{R}$ we can produce a (negative) *gradient vector field*. Think of the arrows as simplicial collapses.

Figure 6: Gradient vector fields

When is a discrete vector field the gradient vector field of some discrete Morse function?

Theorem 3. A discrete vector field V is the gradient vector field of a discrete Morse function $\Leftrightarrow$ there are no non-trivial closed V-paths.

Definition 4. A *V-path* is a sequence

$$\alpha_0^{(d)} \subset \beta_0^{(d+1)} \supset \alpha_1^{(d)} \subset \beta_1^{(d+1)} \supset \ldots \subset \beta_r^{(d+1)} \supset \alpha_{r+1}^{(d)}$$

with $\alpha_i \subset \beta_i$ in V and with $\alpha_i \neq \alpha_{i+1}$.

Figure 7: A V-path

Fact. For V the gradient vector field of a discrete Morse function f, we have

$$f(\alpha_0) \geq f(\beta_0) > f(\alpha_1) \geq f(\beta_1) > \ldots \geq f(\beta_r) > f(\alpha_{r+1}).$$

For our $\mathbb{P}^2$ example, there are no closed V-paths since all V-paths go to the boundary and there are no closed V-paths on the boundary. Hence

$$\mathbb{P}^2 \simeq \text{CW complex with one 0-cell, one 1-cell, one 2-cell.}$$

Which one?

0.4 The Morse Complex

K a simplicial complex with Morse function f. Let $C_d = C_d(K, Z)$ (free abelian group generated by the d-simplices of K), and let $M_d \subset C_d$ be the span of the critical d-simplices.

The chain complex

$$\ldots \xrightarrow{\partial_{d+1}} C_d \xrightarrow{\partial_d} C_{d-1} \xrightarrow{\partial_{d-1}} \ldots$$

gives $\ker(\partial_d)/\operatorname{im}(\partial_{d+1}) =: H_d(K; \mathbb{Z})$.

Theorem 4. The chain complex

$$\dots \xrightarrow{\tilde{\partial}_{d+1}} M_d \xrightarrow{\tilde{\partial}_d} M_{d-1} \xrightarrow{\tilde{\partial}_{d-1}} \dots$$

gives $\ker(\tilde{\partial}_d)/\operatorname{im}(\tilde{\partial}_{d+1}) \cong H_d(K;\mathbb{Z})$. For $\beta^{(d+1)}$ critical,

$$\tilde{\partial}\beta = \sum_{\text{critical } \alpha^{(d)}} c_{\alpha,\beta}\alpha,$$

where $c_{\alpha,\beta}$ is the sum of signs (± 1) of all gradient paths from the boundary of β to α.

Figure 8: Sign of a V-path from the boundary of β to α

Proof. (Sketch). Since homotopy equivalent spaces have isomorphic homology, Theorem 2 gives all parts of this theorem except for the definition of $\tilde{\partial}$. □

Figure 9: A discrete vector field on the projective plane $\mathbb{P}^2$.

For our $\mathbb{P}^2$ example, we have

- $M_0 = M_1 = M_2 = \mathbb{Z}$
- $\tilde{\partial}(e) = (1-1)v_0 = 0$
- $\tilde{\partial}(t) = (1+1)e = 2e$, giving

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0.$$

Hence

$$H_*(\mathbb{P}^2;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}/2\mathbb{Z} & * = 1 \\ 0 & \text{otherwise.} \end{cases}$$

0.5 Evasiveness

Suppose $K \subset \Delta^n = [v_0, v_1, \dots v_n]$ is known and $\sigma \in \Delta^n$ is not.

Figure 10: $K \subset \Delta^2$

- Goal is to determine if $\sigma \in K$ by asking "Is vertex $v_i \in \sigma$?"
- Answers can affect future questions.
- Win if you ask $< n+1$ questions.

Figure 11: Evaders are $\sigma = v_2, [v_0, v_2]$.

Definition 5. K is *nonevasive* if there is a guessing algorithm deciding if $\sigma \in K$ in $< n+1$ questions for all σ. Else *evasive*.

Theorem 5. The number of evaders in any guessing algorithm $\geq 2 \dim \tilde{H}_*(K)$.

Proof. (Sketch).

Figure 12: Vector field $V = \{\emptyset \subset v_0, v_0 \subset [v_0, v_1], v_2 \subset [v_0, v_2], [v_1, v_2] \subset [v_0, v_1, v_2]\}$ on Δ^n induced by guessing algorithm. One critical simplex.

Fact. V is a gradient vector field.

Restrict V to K to get $V|_K = \{v_0 \subset [v_0, v_1]\}$.

- Still no closed orbits.
- A pair of evaders for each extra critical simplex in $V|_K$.
- # evaders $= 2(\#\text{critical simplices} - 1) \geq 2 \dim \tilde{H}_*(K)$.

□

Theorem 6. If K is nonevasive then K simplicially collapses to a point.

Proof. (Sketch). No evaders $\Rightarrow$ only one critical 0-simplex. Apply simplicial collapse part of Theorem 2. □

0.6 Cancelling Critical Points

Finding Morse function with fewest critical points is hard.

Smooth case:

- Contains Poincaré conjecture (smooth manifold $M^d \simeq S^d \Rightarrow M \cong S^d$) since spheres are those spaces with 2 critical points.

- Milnor presented Smale's proof for dim ≥ 5 using Morse theory. Roughly, let $M \simeq S^d$. Pick smooth Morse function f. Cancel critical points in pairs until only two remain, implying $M \cong S^d$.

Theorem 7. Suppose f is a discrete Morse function on K with $\alpha^{(d)}$, $\beta^{(d+1)}$ critical and exactly one gradient path from $\partial\beta$ to α. Then reversing the direction of this gradient path produces a discrete Morse function with α, β no longer critical.

Figure 13: Cancelling critical points.

Proof. Unique gradient path $\Rightarrow$ resulting gradient field has no closed orbits, hence corresponds to Morse function. Note α, β no longer critical while other simplices are unchanged. □

In smooth analogue, one must also smoothly adjust vectors near the gradient path (without creating closed orbits).

0.7 Homeomorphism type

Definition 6. K is a *combinatorial d-ball* [*resp.* $(d-1)$*-sphere*] if K and Δ^d [resp. $\partial\Delta^d$] have isomorphic subdivisions.

Theorem 8. A combinatorial d-manifold (link of every vertex is a combinatorial $(d-1)-$sphere or ball) with a discrete Morse function with two critical simplices is a combinatorial d-sphere.

0.8 Conclusion

References

[1] Robin Forman. A user's guide to discrete Morse theory. *Sém. Lothar. Combin*, 48:B48c, 2002.

[2] J. Milnor. *Morse Theory*. Princeton University Press, Princeton, 1965.