An Introduction to Applied Topology

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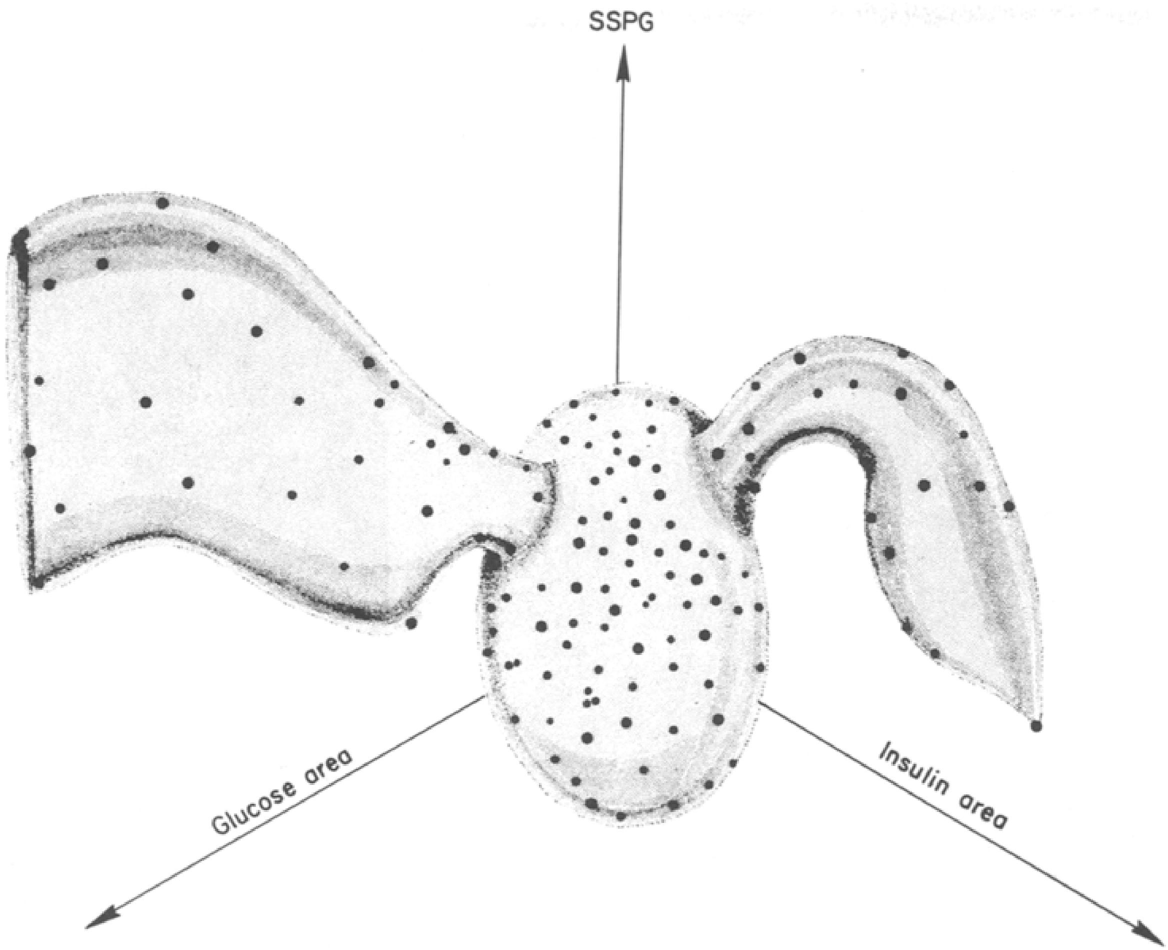
An Introduction to Applied Topology

Part I: Topology applied to data analysis
Part II: Real examples, machine learning, and dynamical systems
Part III: Topology applied to sensor networks

Datasets have shapes

Example: Diabetes study

145 points in 5-dimensional space

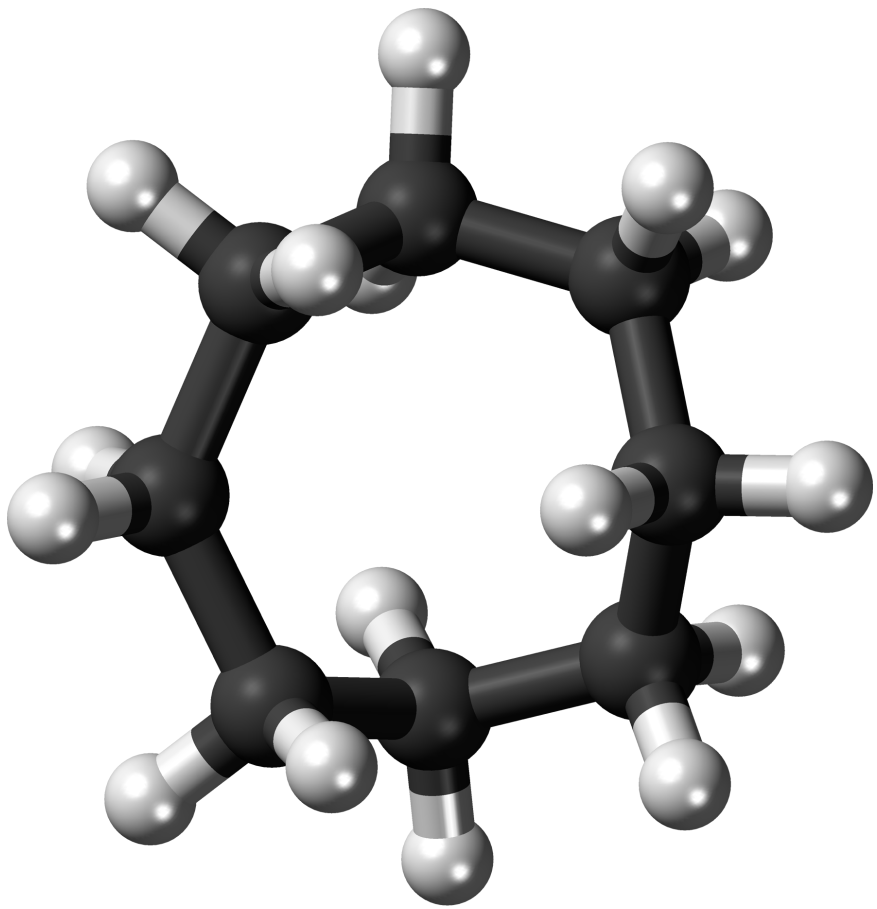


An attempt to define the nature of chemical diabetes using a multidimensional analysis by G. M. Reaven and R. G. Miller, 1979

Datasets have shapes

Example: Cyclo-Octane (C_8H_{16}) data

1,000,000+ points in 24-dimensional space



Non-Manifold Surface Reconstruction from High Dimensional Point Cloud Data
by Shawn Martin and Jean-Paul Watson, 2010.

Datasets have shapes

Example: Cyclo-Octane (C_8H_{16}) data

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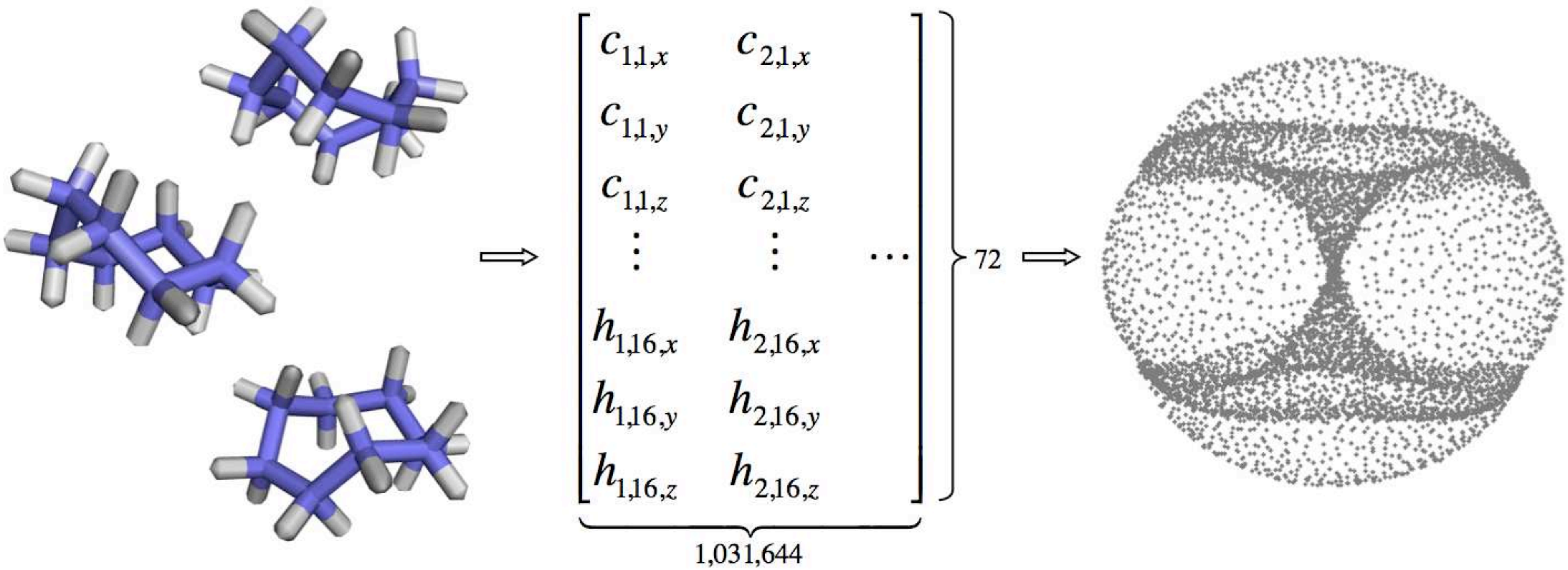

Non-Manifold Surface Reconstruction from High Dimensional Point Cloud Data
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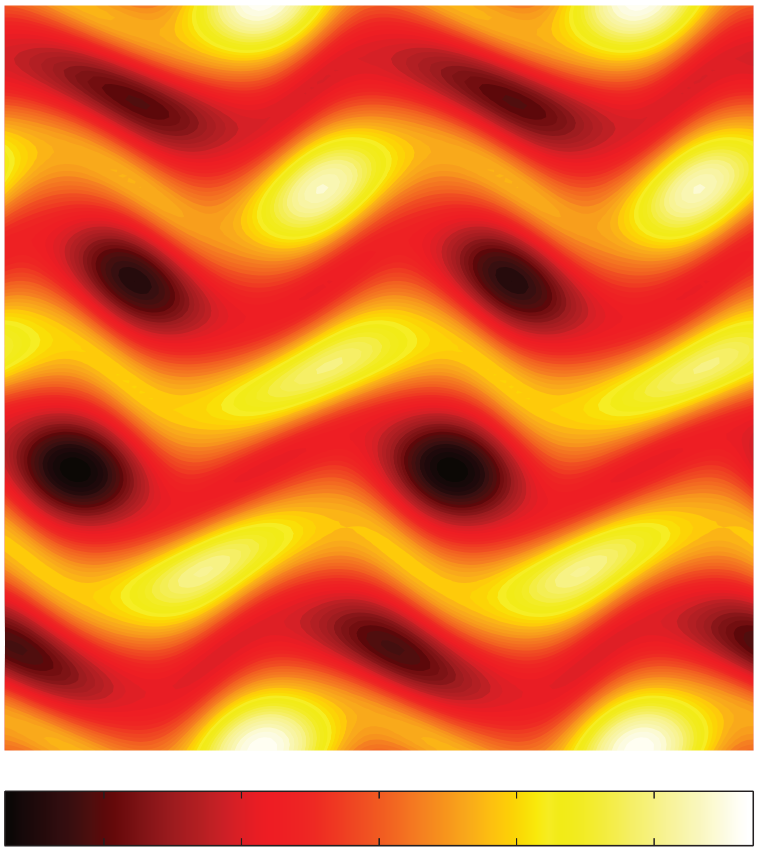
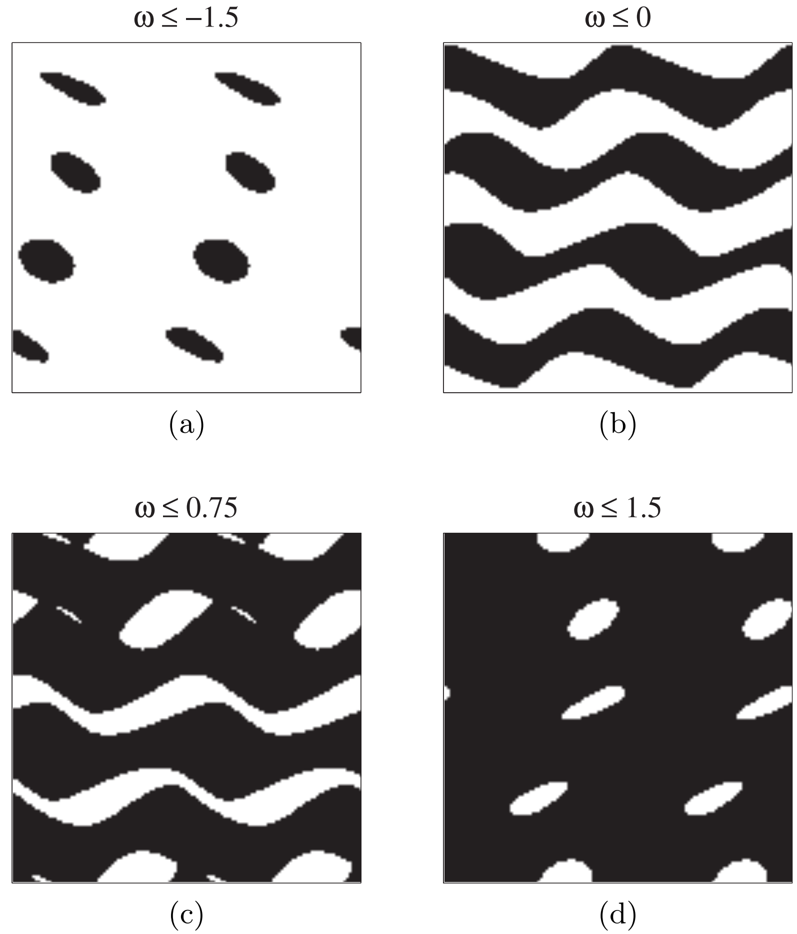
Datasets have shapes

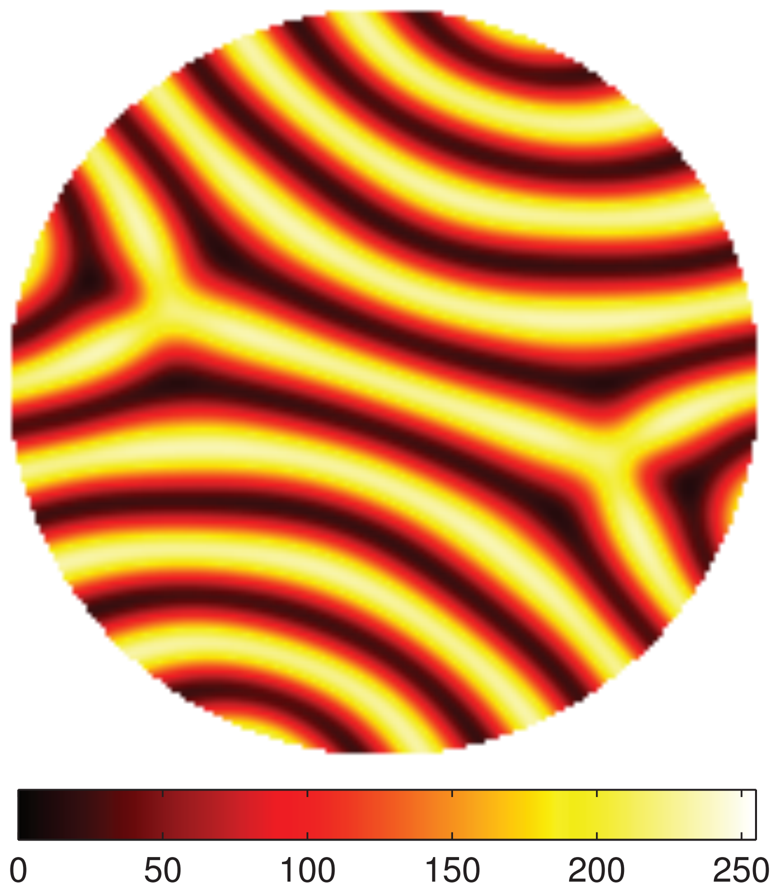
Persistent homology

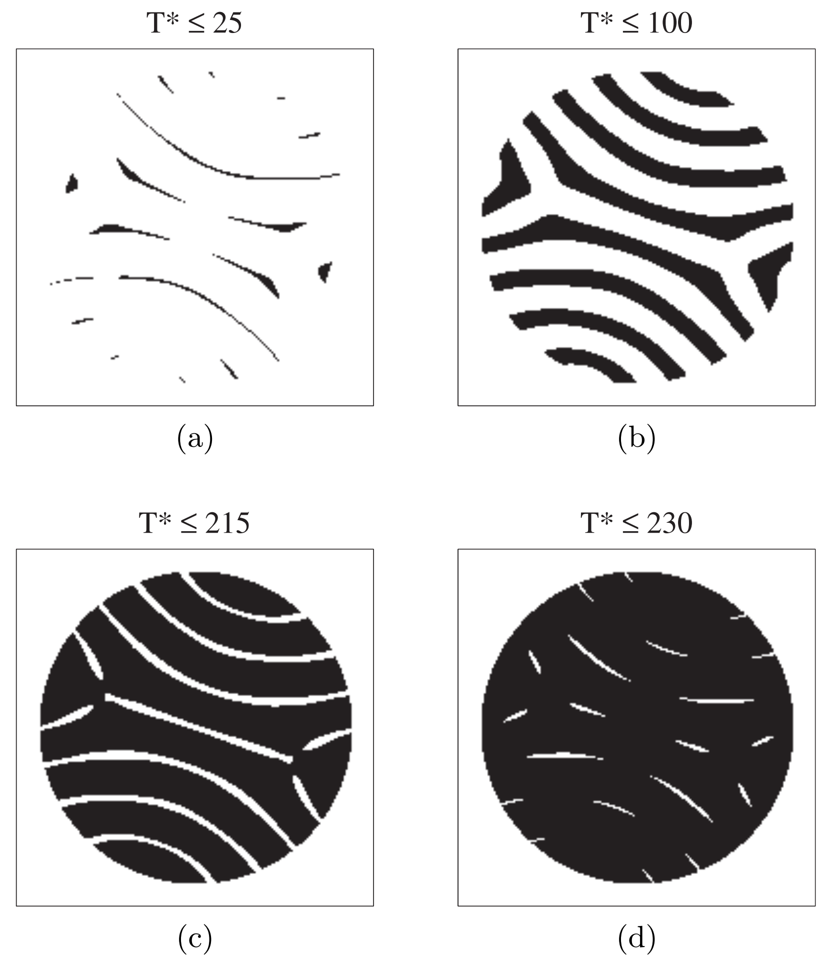




Analysis of Kolmogorov flow and Rayleigh–Bénard convection using persistent homology by Miroslav Kramár, Rachel Levanger, Jeffrey Tithof, Balachandra Suri, Mu Xu, Mark Paul, Michael F Schatz, Konstantin Mischaikow

Persistent homology





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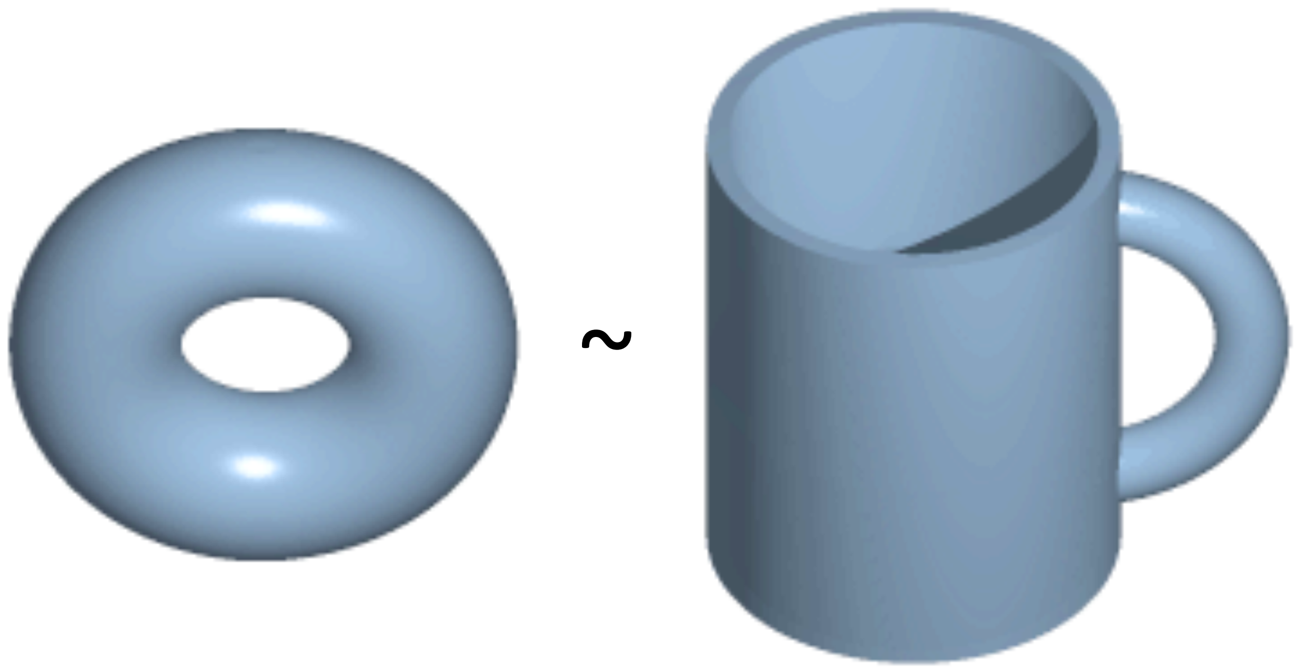
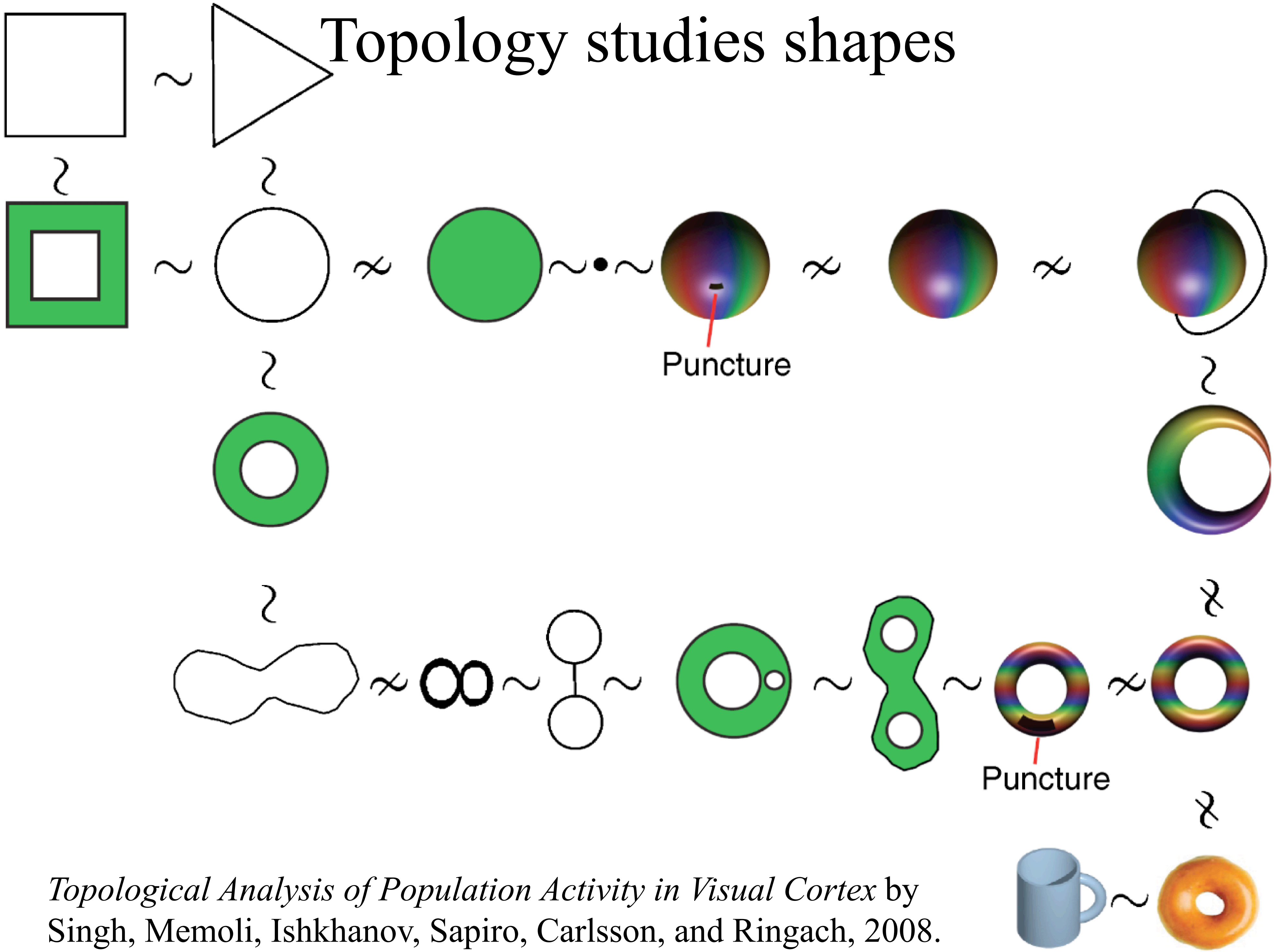
Topology studies shapes

A donut and coffee mug are "homotopy equivalent", and considered to be the same shape. You can bend and stretch (but not tear) one to get the other.



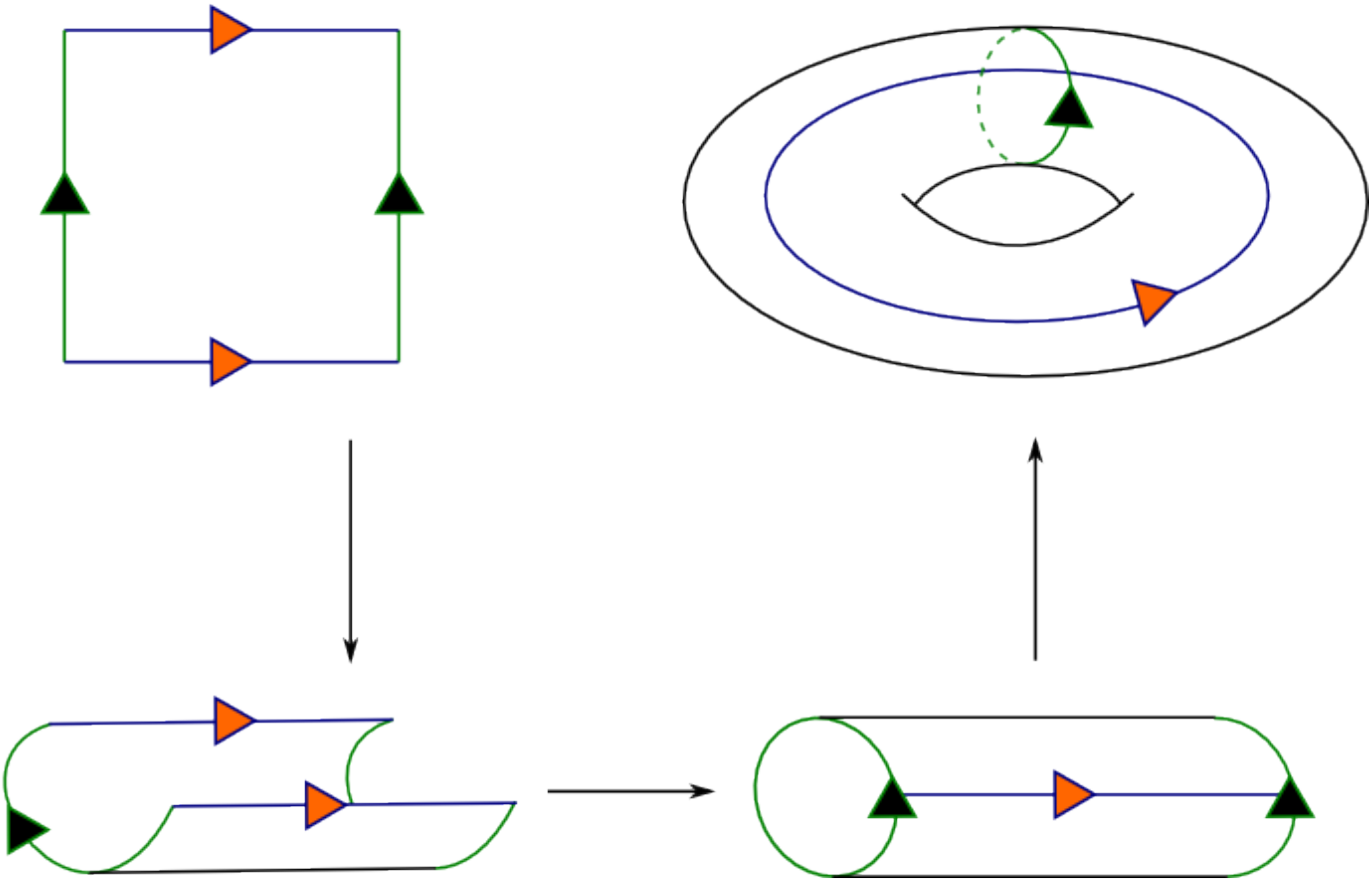
Topology studies shapes



Topological Analysis of Population Activity in Visual Cortex by Singh, Memoli, Ishkhanov, Sapiro, Carlsson, and Ringach, 2008.

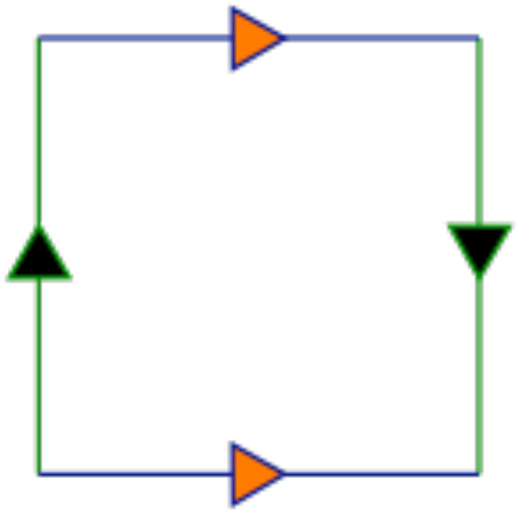
Topology studies shapes

Torus



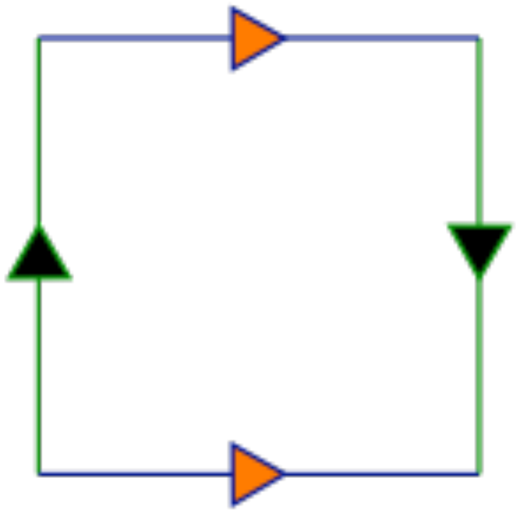
Topology studies shapes

Klein bottle



Topology studies shapes

Klein bottle



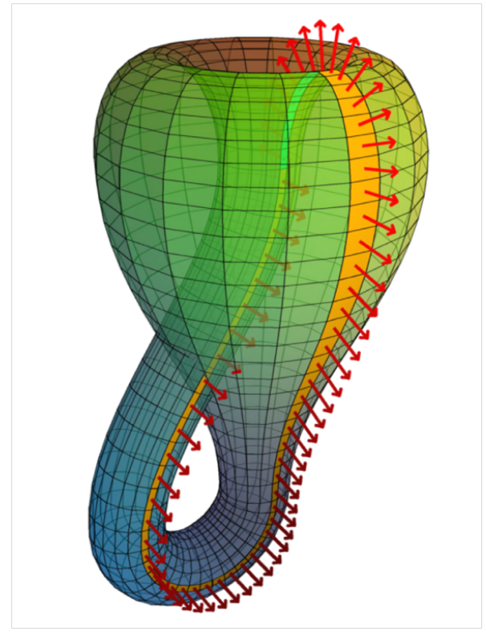
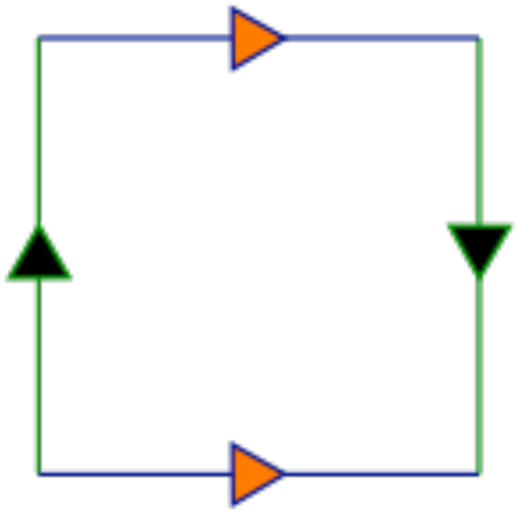
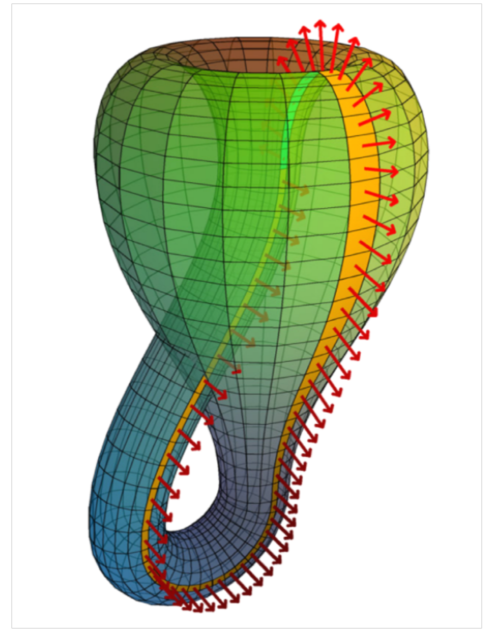


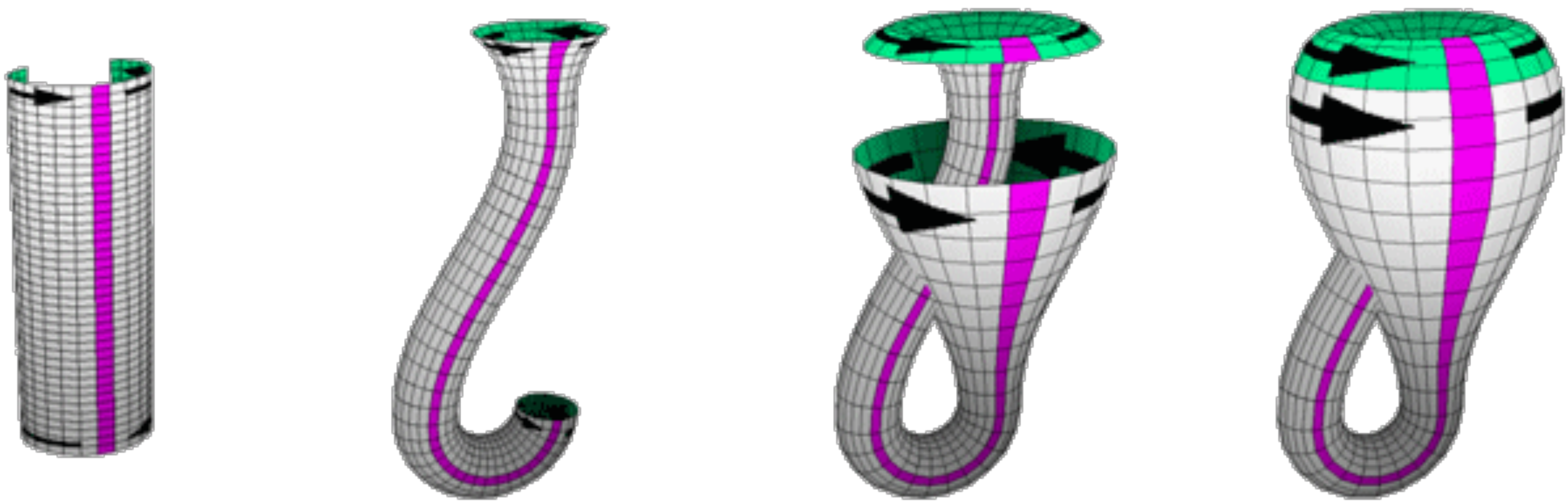
Image credit: https://plus.maths.org/content/imaging-maths-inside-klein-bottle

Topology studies shapes

Klein bottle







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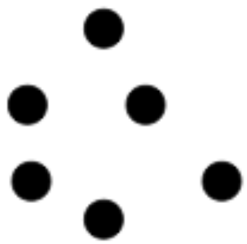
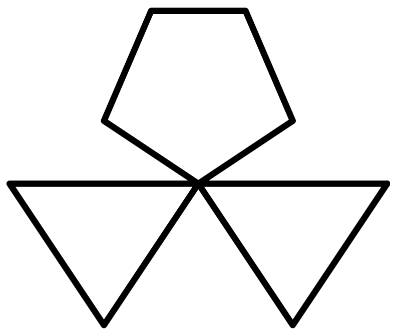
Homology

- i-dimensional homology H_i "counts the number of i-dimensional holes"
- i-dimensional homology H_i actually has the structure of a vector space!



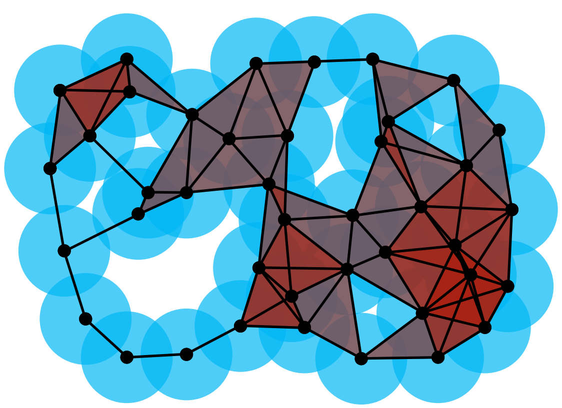
0-dimensional homology H_0: rank 6

1-dimensional homology H_1: rank 0



0-dimensional homology H_0: rank 1

1-dimensional homology H_1: rank 3

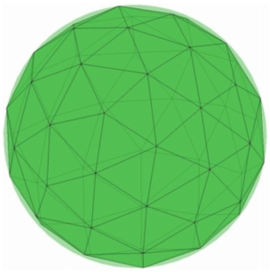


0-dimensional homology H_0: rank 1

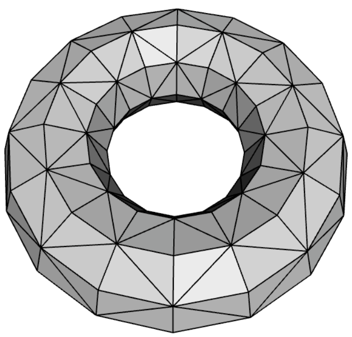
1-dimensional homology H_1: rank 6

Homology

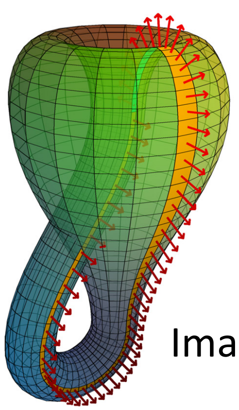
- i-dimensional homology "counts the number of i-dimensional holes"
- i-dimensional homology actually has the structure of a vector space!



0-dimensional homology H_0: rank 1
1-dimensional homology H_1: rank 0
2-dimensional homology H_2: rank 1



0-dimensional homology H_0: rank 1
1-dimensional homology H_1: rank 2
2-dimensional homology H_2: rank 1



Be careful! (Same as torus over $\mathbb{Z}/2\mathbb{Z}$)

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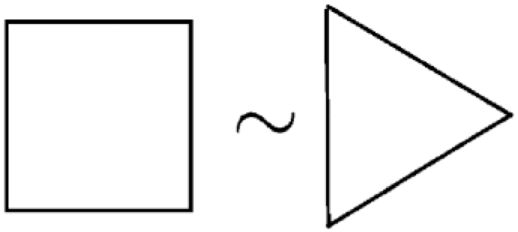
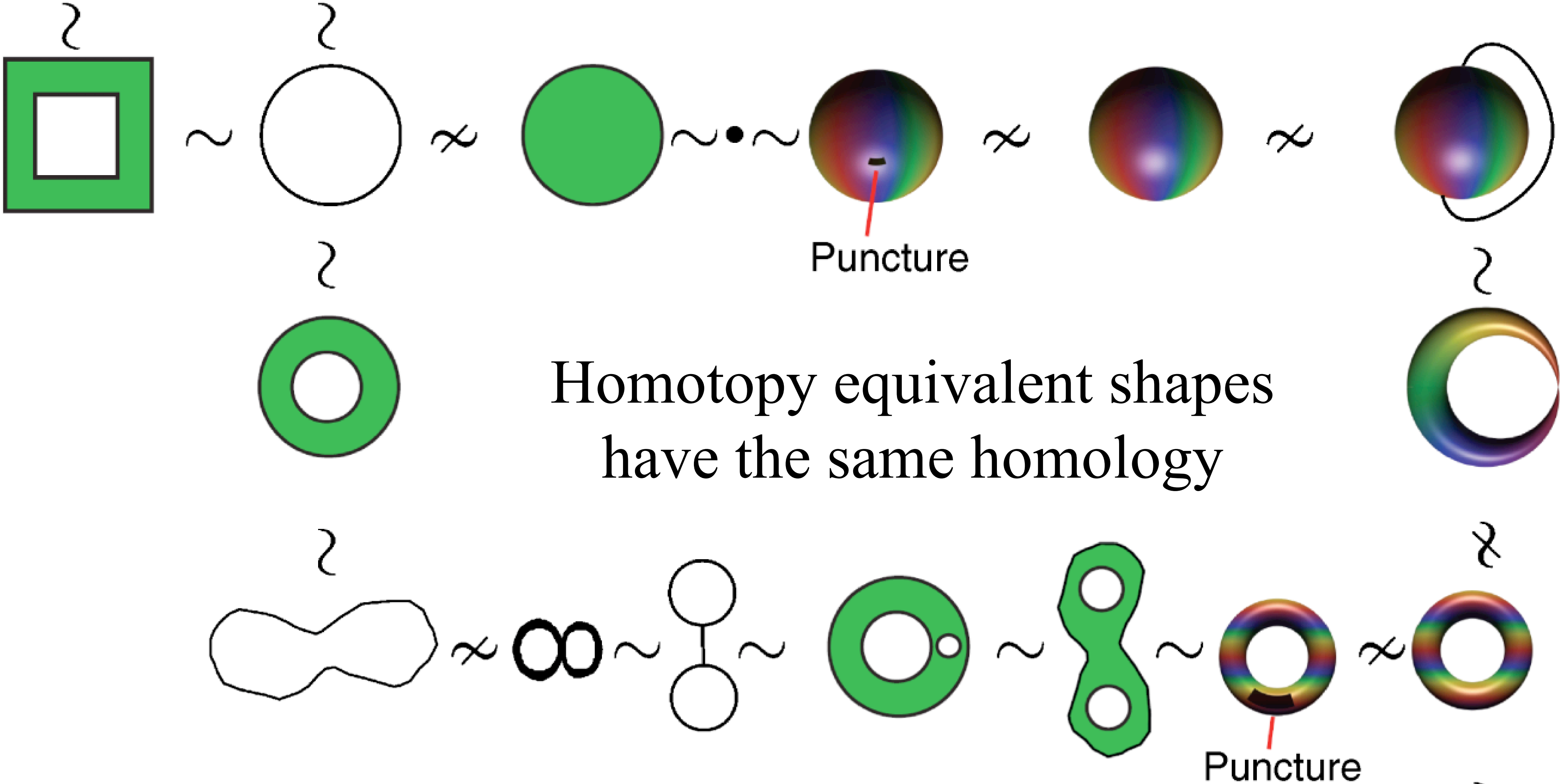
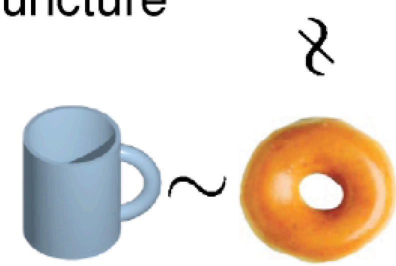
Homology





Homotopy equivalent shapes have the same homology



Topological Analysis of Population Activity in Visual Cortex by Singh, Memoli, Ishkhanov, Sapiro, Carlsson, and Ringach, 2008.

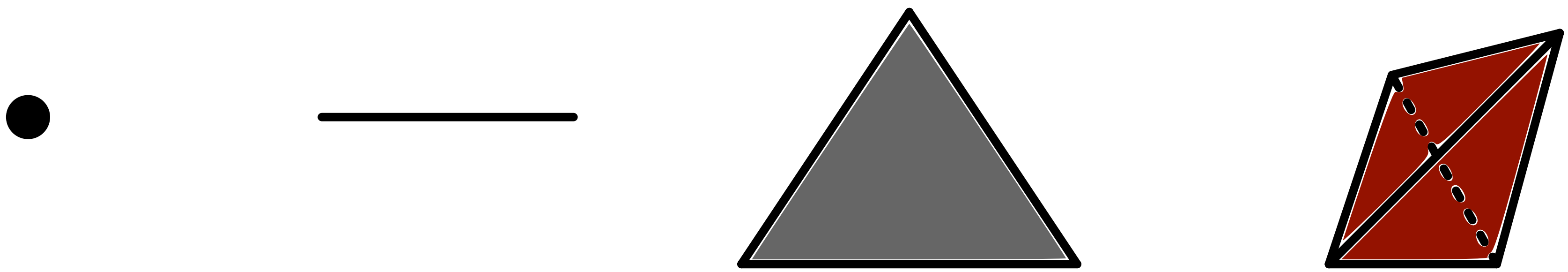
Homology

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & A_{n+1} \arrow[r, "\alpha_{n+1}"] \arrow[d, "\partial_{n+1}"'] & B_{n+1} \arrow[r, "\beta_{n+1}"] \arrow[d, "\partial'_{n+1}"'] & C_{n+1} \arrow[r] \arrow[d, "\partial''_{n+1}"'] & 0 \\
0 \arrow[r] & A_n \arrow[r, "\alpha_n"] \arrow[d, "\partial_n"'] & B_n \arrow[r, "\beta_n"] \arrow[d, "\partial'_n"'] & C_n \arrow[r] \arrow[d, "\partial''_n"'] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "\alpha_{n-1}"] \arrow[d] & B_{n-1} \arrow[r, "\beta_{n-1}"] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & \vdots \arrow[dll] \\
H_{n+1}(\mathcal{A}) \arrow[r, "\alpha_*"'] & H_{n+1}(\mathcal{B}) \arrow[r, "\beta_*"] & H_{n+1}(\mathcal{C}) \arrow[dll, "\delta_{n+1}"] \\
H_n(\mathcal{A}) \arrow[r, "\alpha_*"'] & H_n(\mathcal{B}) \arrow[r, "\beta_*"] & H_n(\mathcal{C}) \arrow[dll, "\delta_n"] \\
H_{n-1}(\mathcal{A}) \arrow[r, "\alpha_*"'] & H_{n-1}(\mathcal{B}) \arrow[r, "\beta_*"] & H_{n-1}(\mathcal{C}) \arrow[dl] \\
 & \vdots &
\end{tikzcd}
$$

Homology

0-simplex 1-simplex 2-simplex 3-simplex



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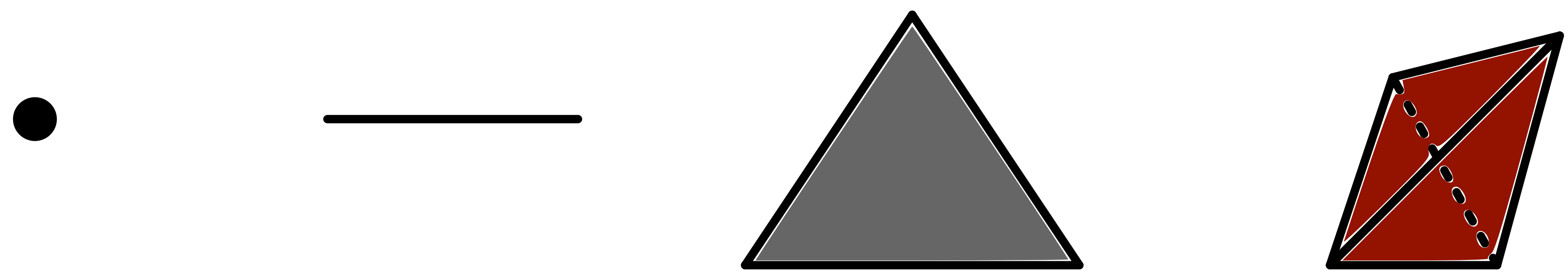
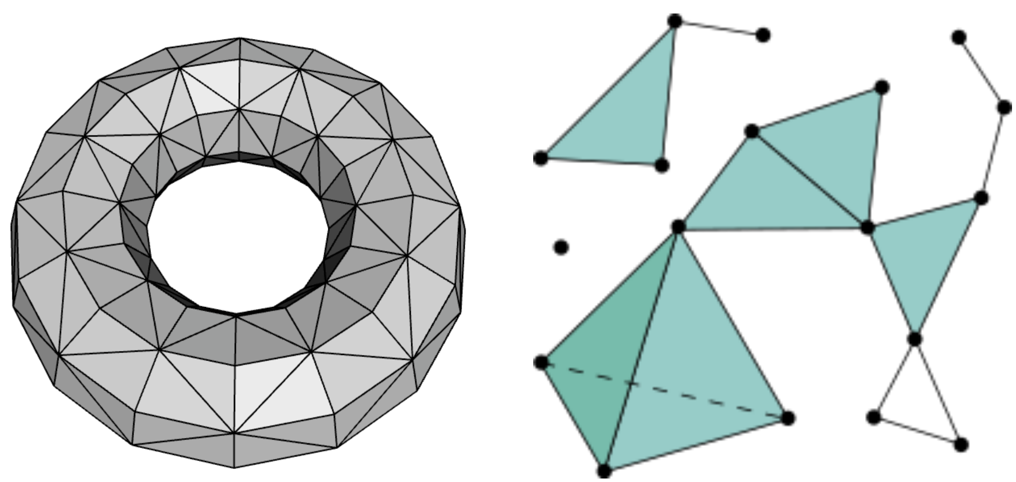


Simplicial complexes



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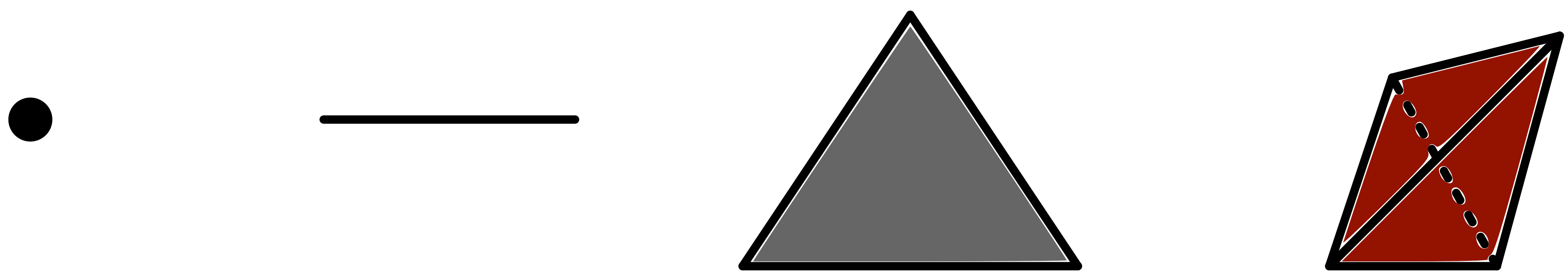
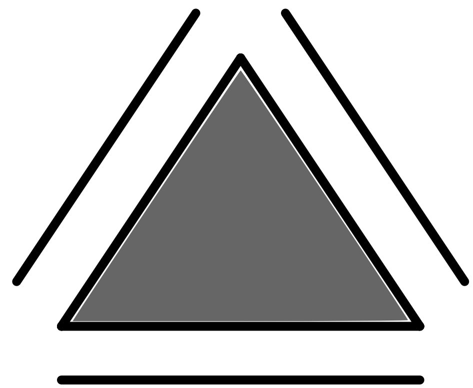


Homology

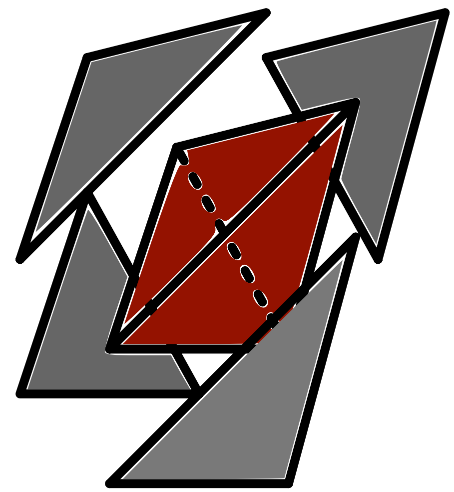
0-simplex

1-simplex

2-simplex

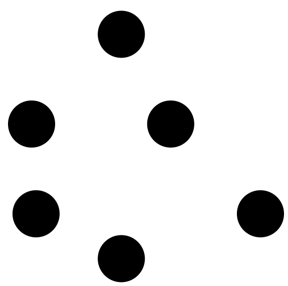


3-simplex

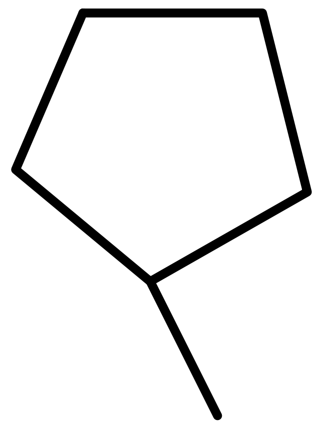


Homology

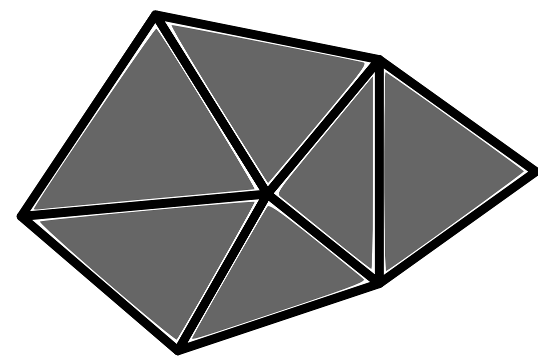
0-simplices



1-simplices



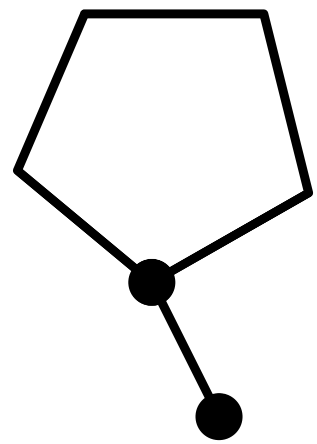
2-simplices



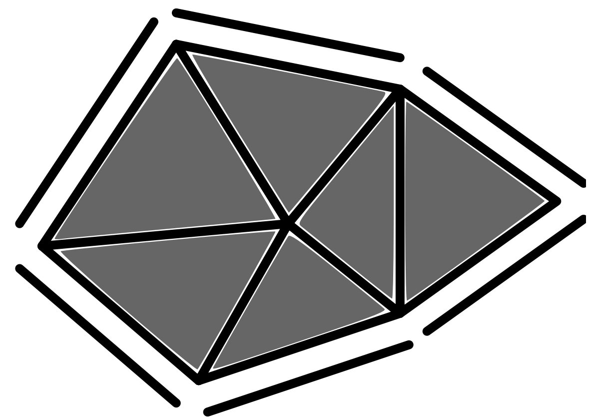
Homology

0-simplices

1-simplices

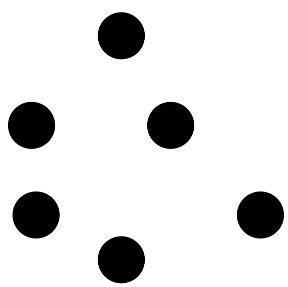


2-simplices

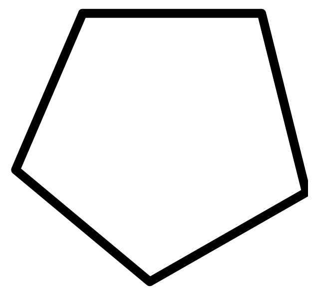


Homology

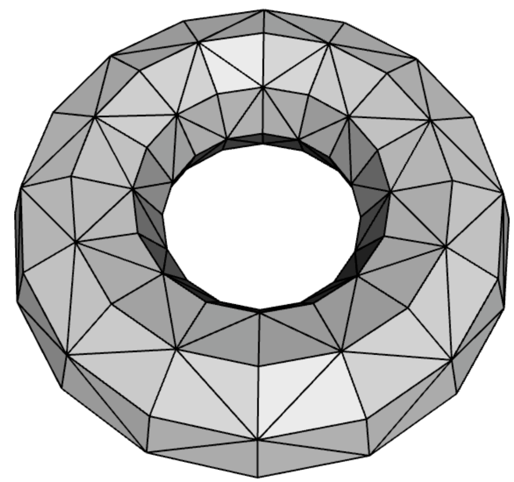
0-cycle



1-cycle



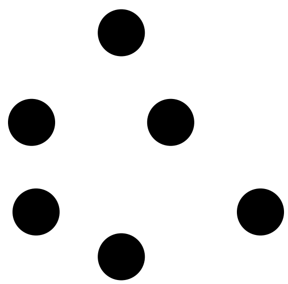
2-cycle



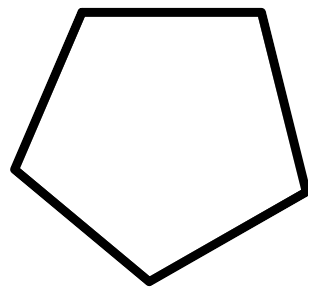
A cycle has no boundary.

Homology

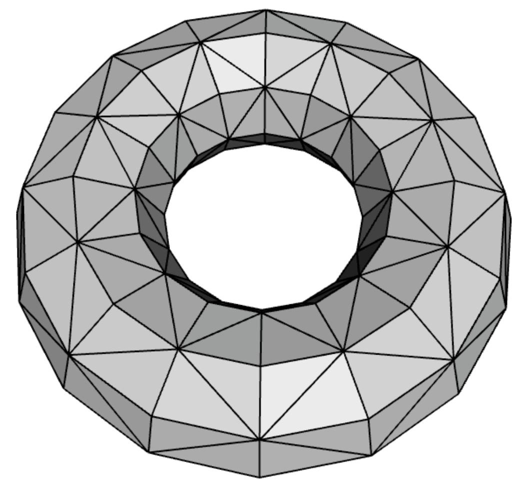
0-cycle



1-cycle



2-cycle



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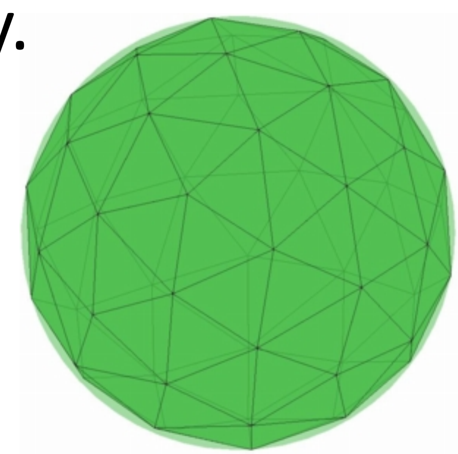
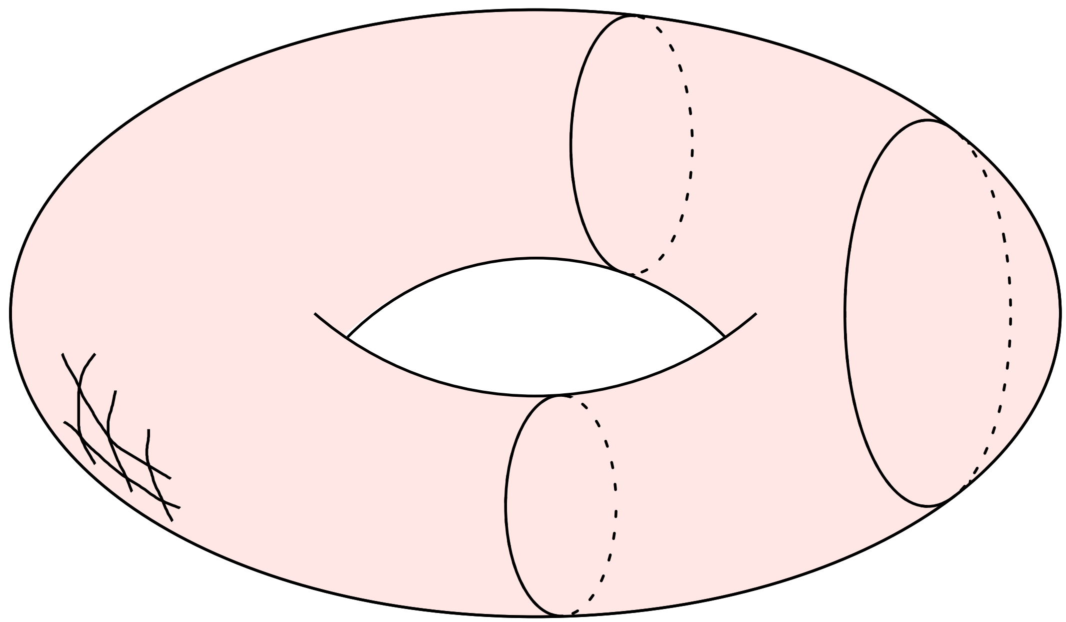


Homology

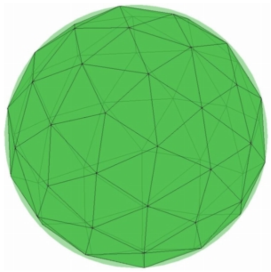


Two cycles are equivalent if they differ by a boundary.

H_i measures equivalence classes of i-cycles.

Homology

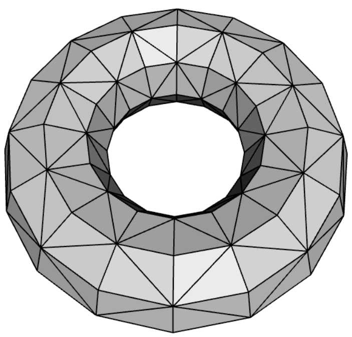
- *i*-dimensional homology "counts the number of *i*-dimensional holes"
- *i*-dimensional homology actually has the structure of a vector space!



0-dimensional homology H_0: rank 1

1-dimensional homology H_1: rank 0

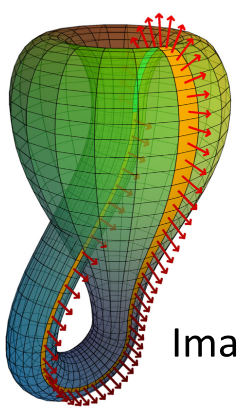
2-dimensional homology H_2: rank 1



0-dimensional homology H_0: rank 1

1-dimensional homology H_1: rank 2

2-dimensional homology H_2: rank 1

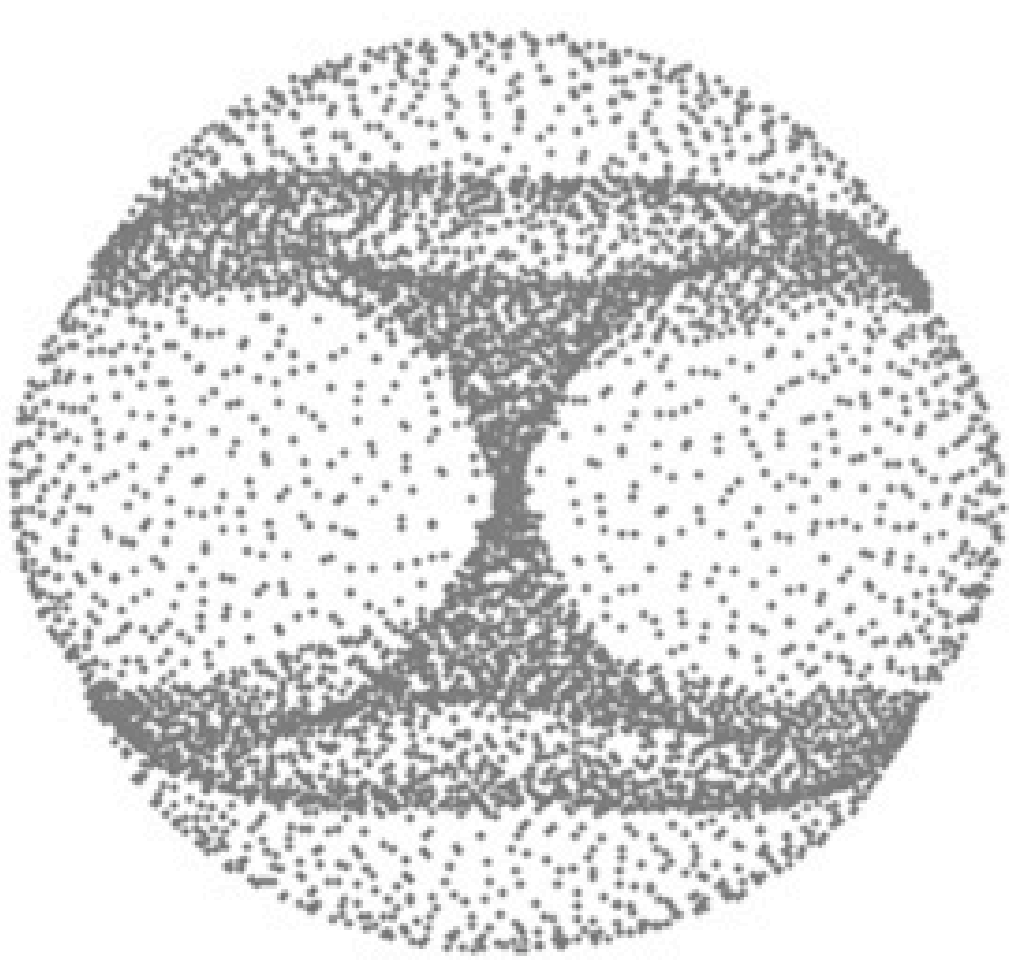


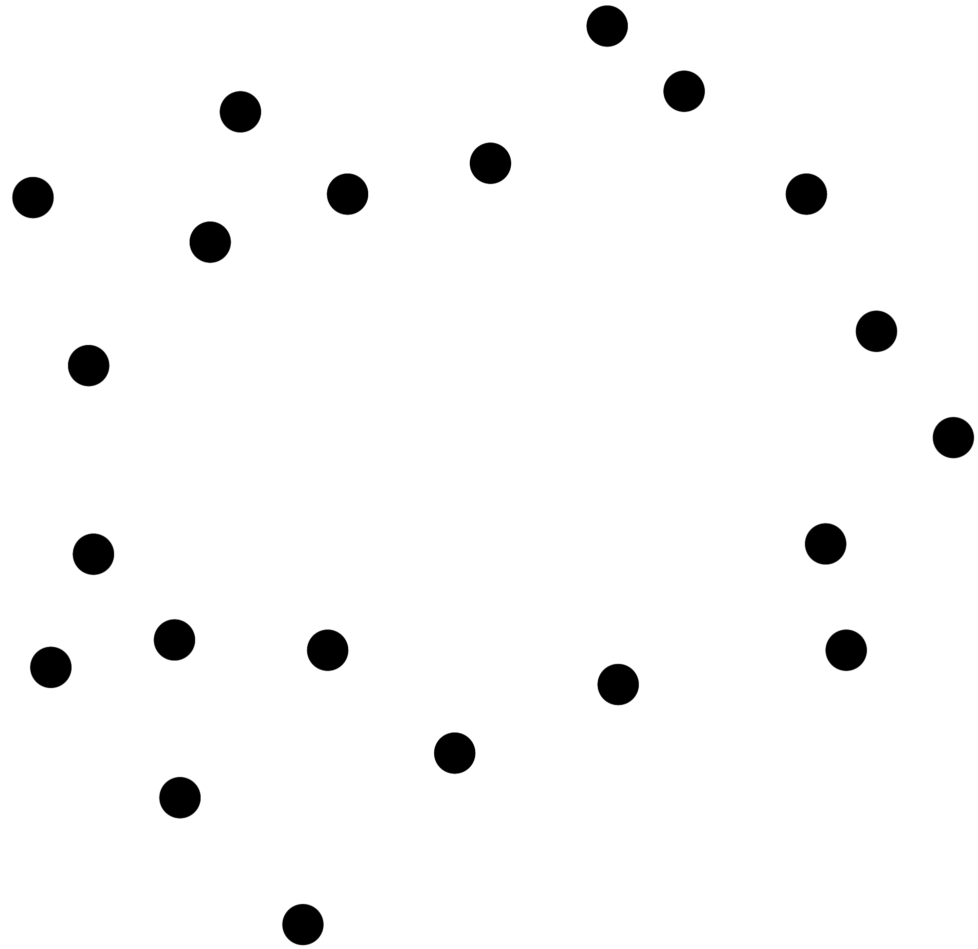
Be careful! (Same as torus over $\mathbb{Z}/2\mathbb{Z}$)

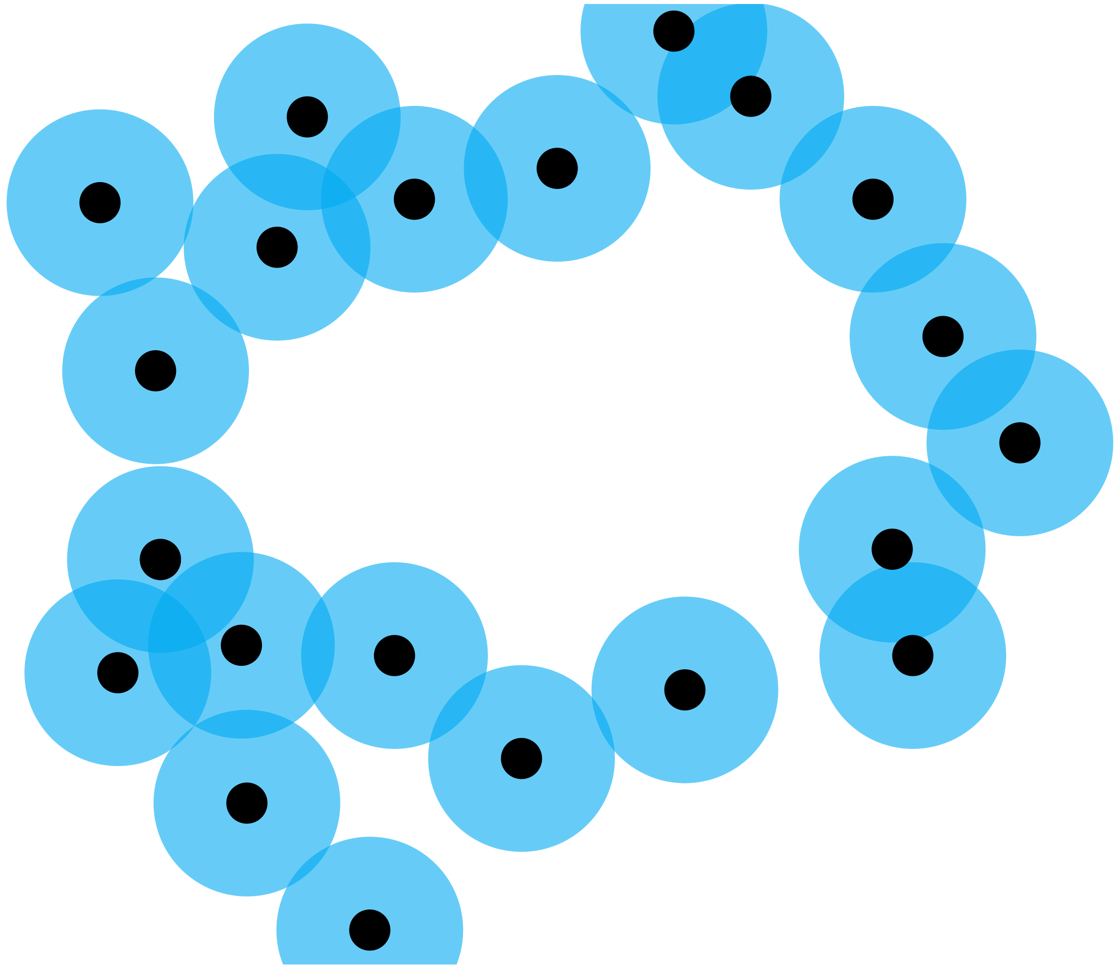
Image credit: https://plus.maths.org/content/imaging-maths-inside-klein-bottle

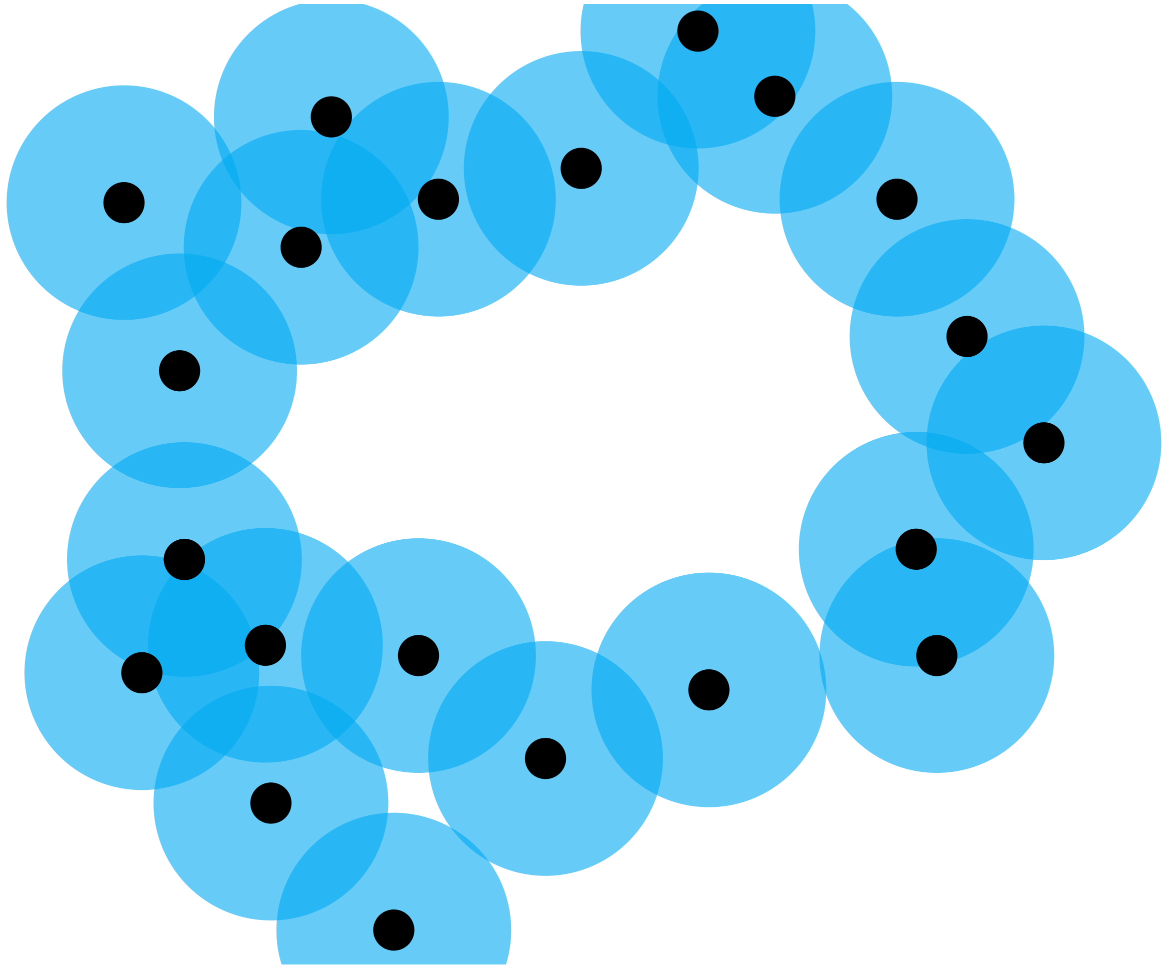
Topology studies shapes

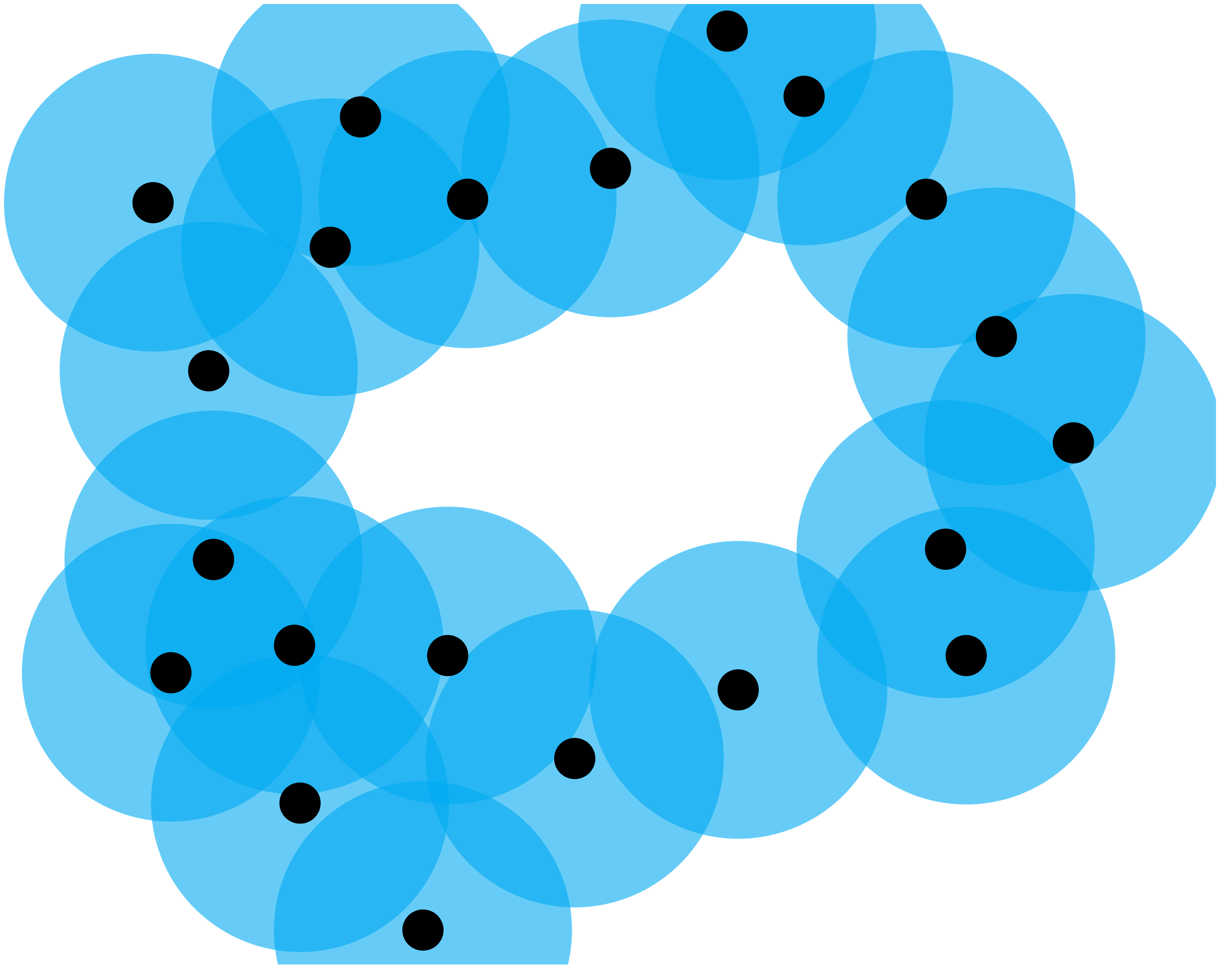
What shape is this?

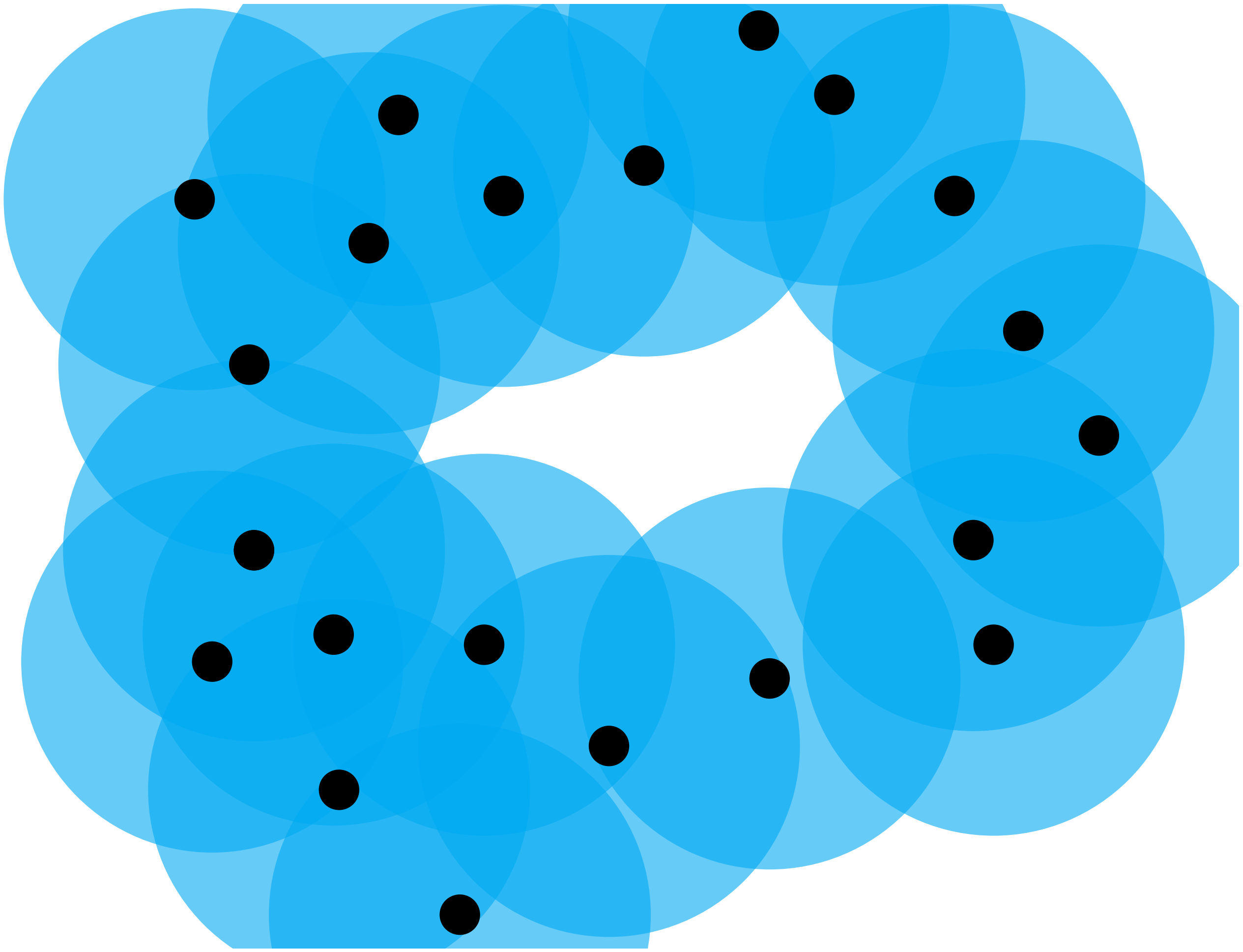


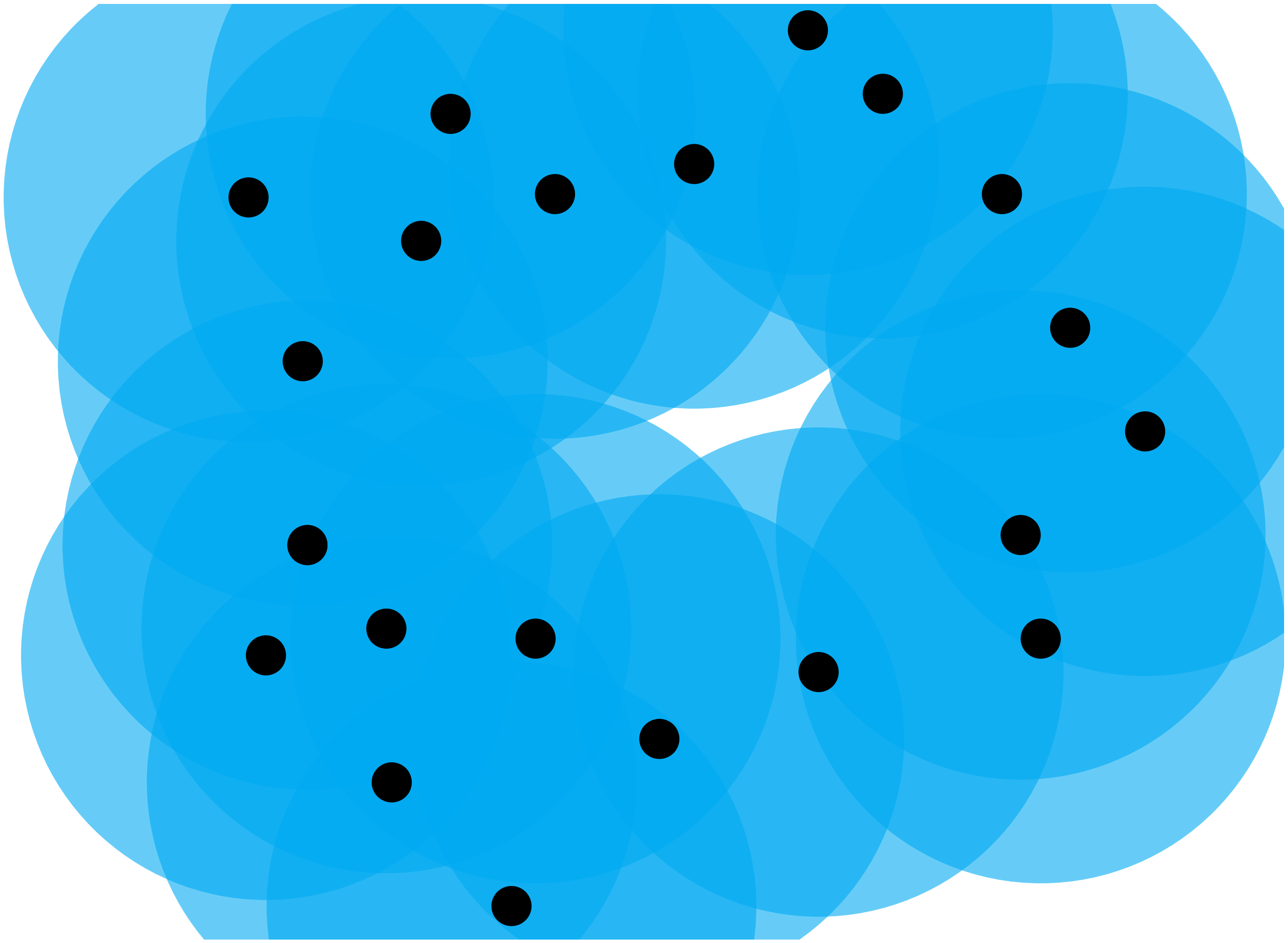


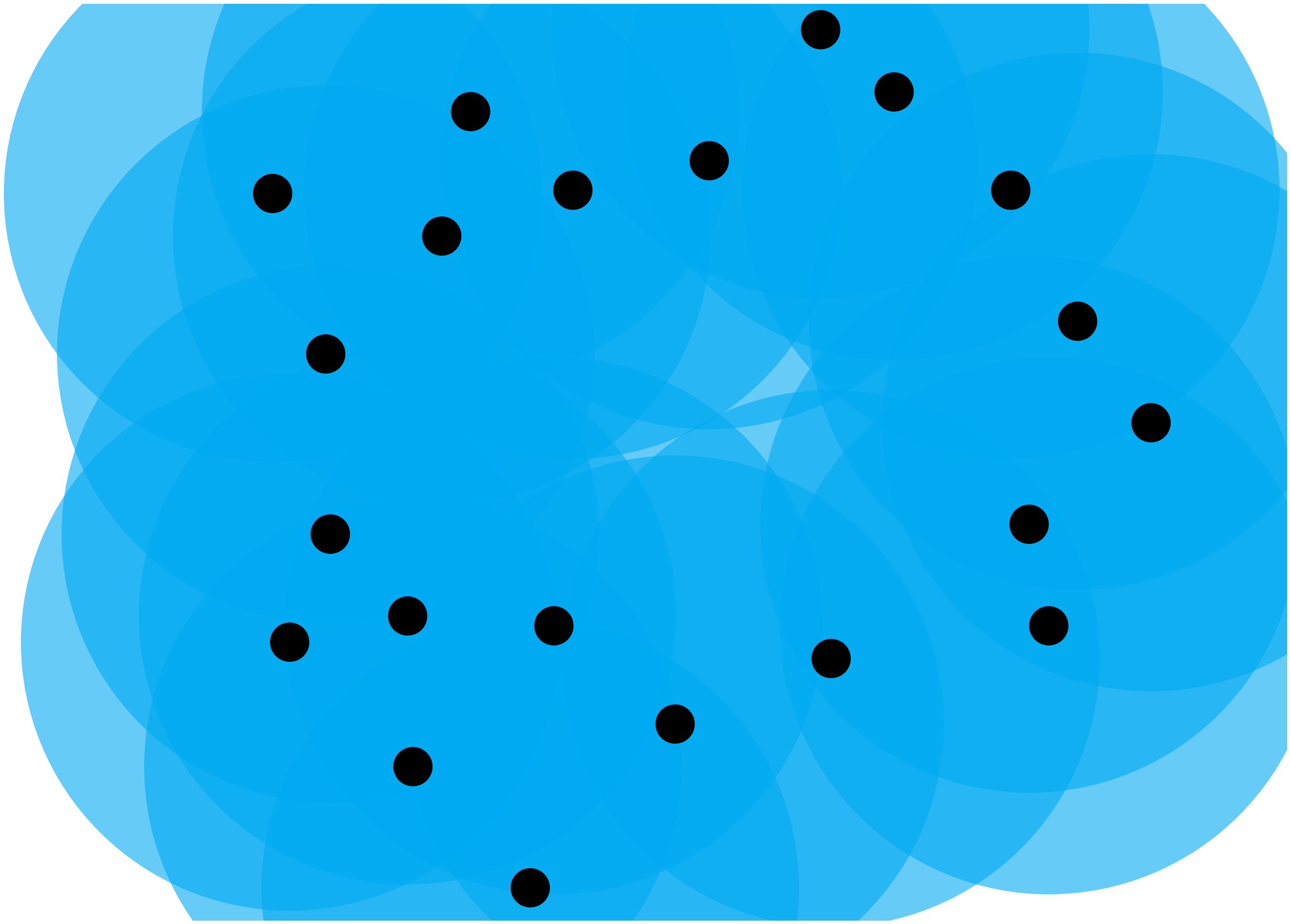


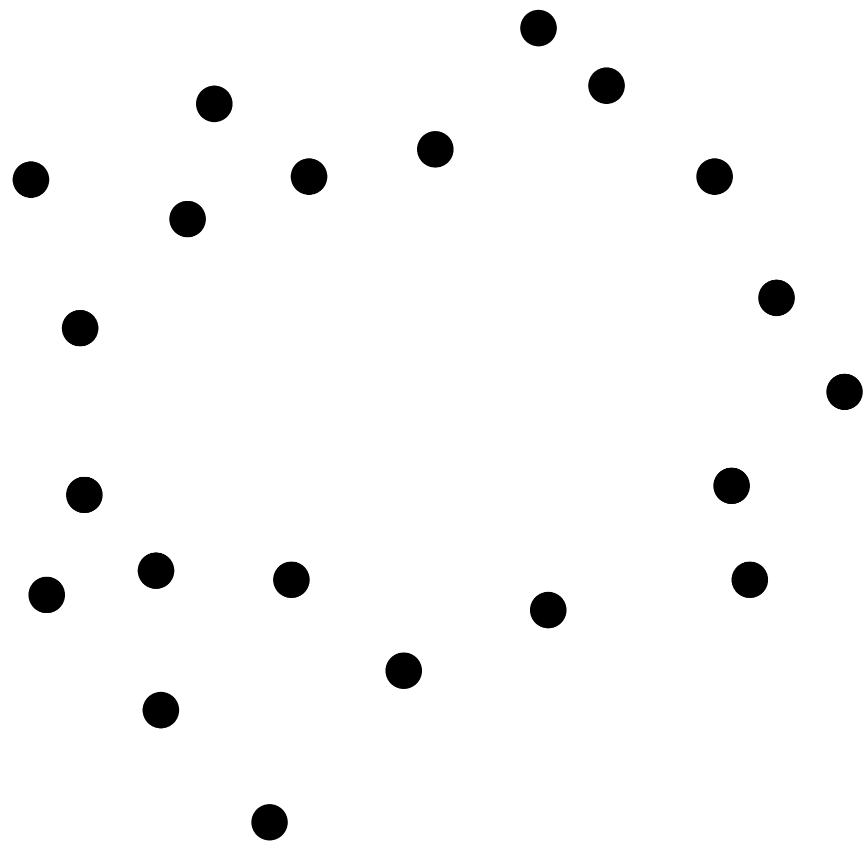








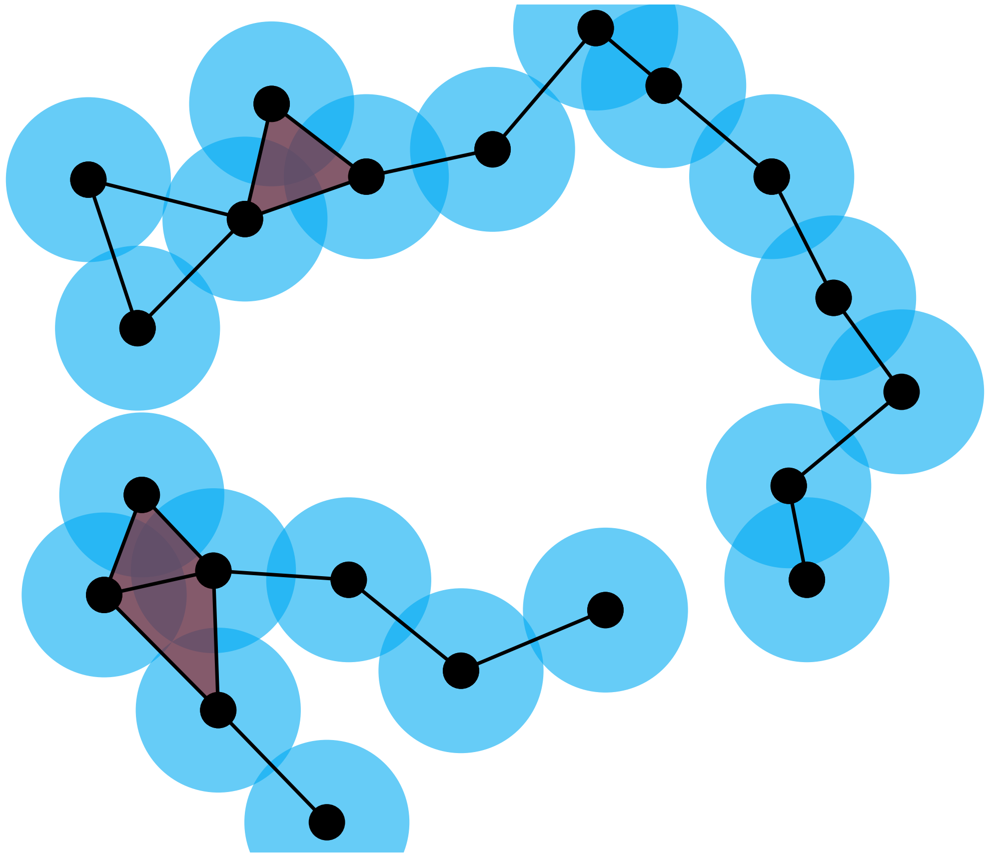




Definition

For a data set $X \subseteq \mathbb{R}^n$ and scale $r \geq 0$, the *Čech simplicial complex* $\text{Čech}(X; r)$ has

- vertex set X
- finite simplex $\{x_0, x_1, \ldots, x_k\}$ when $\cap_{i=0}^{k} B(x_i, r) \neq \emptyset$.



Definition

For a data set $X \subseteq \mathbb{R}^n$ and scale $r \geq 0$, the *Čech simplicial complex* $\check{\mathrm{C}}\mathrm{ech}(X; r)$ has

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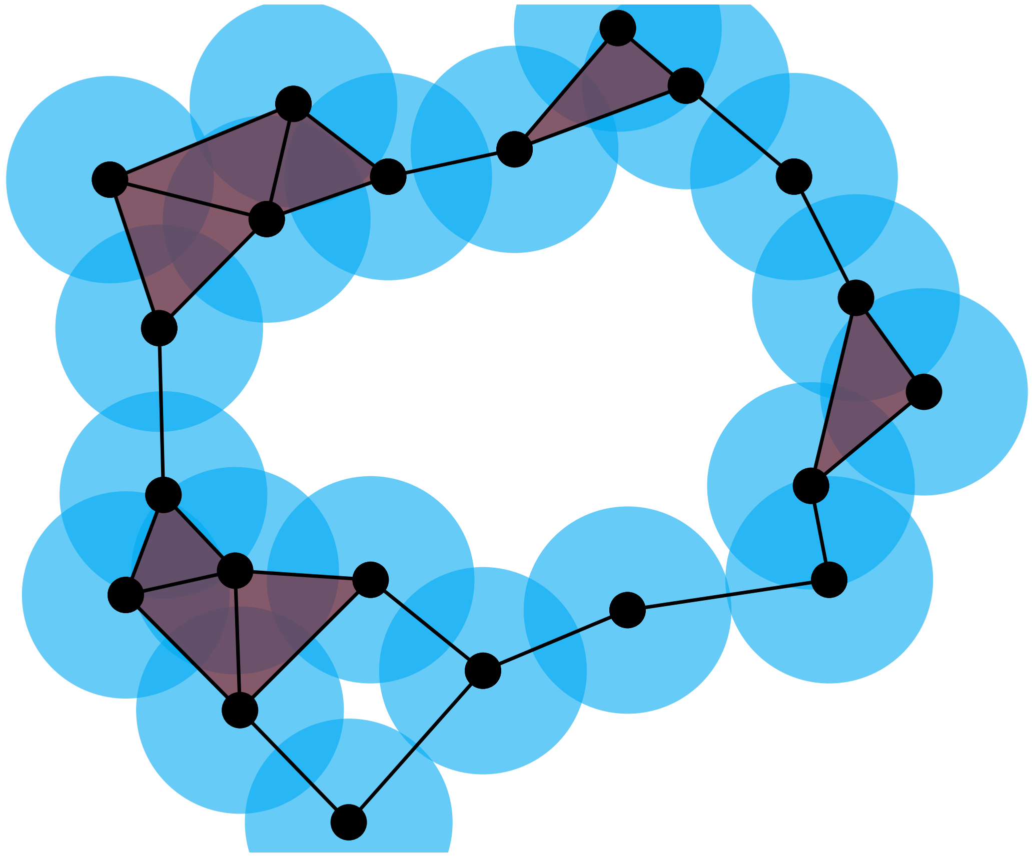

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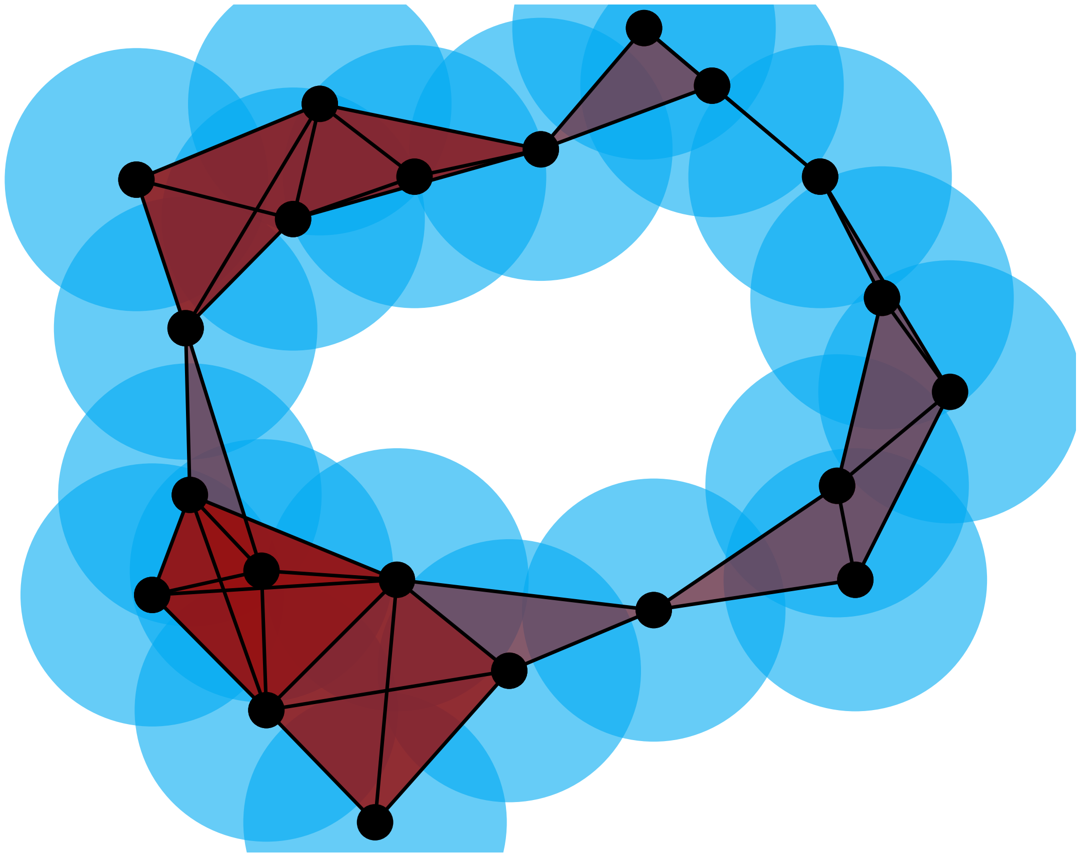

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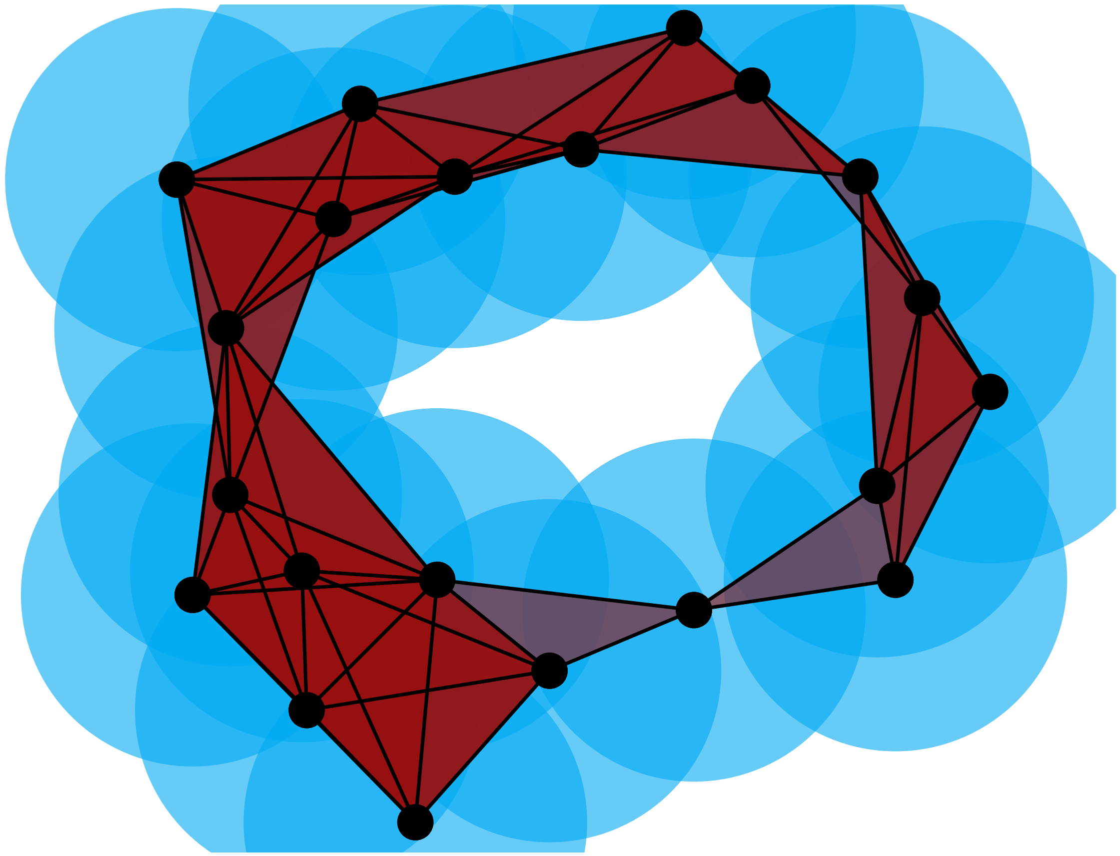

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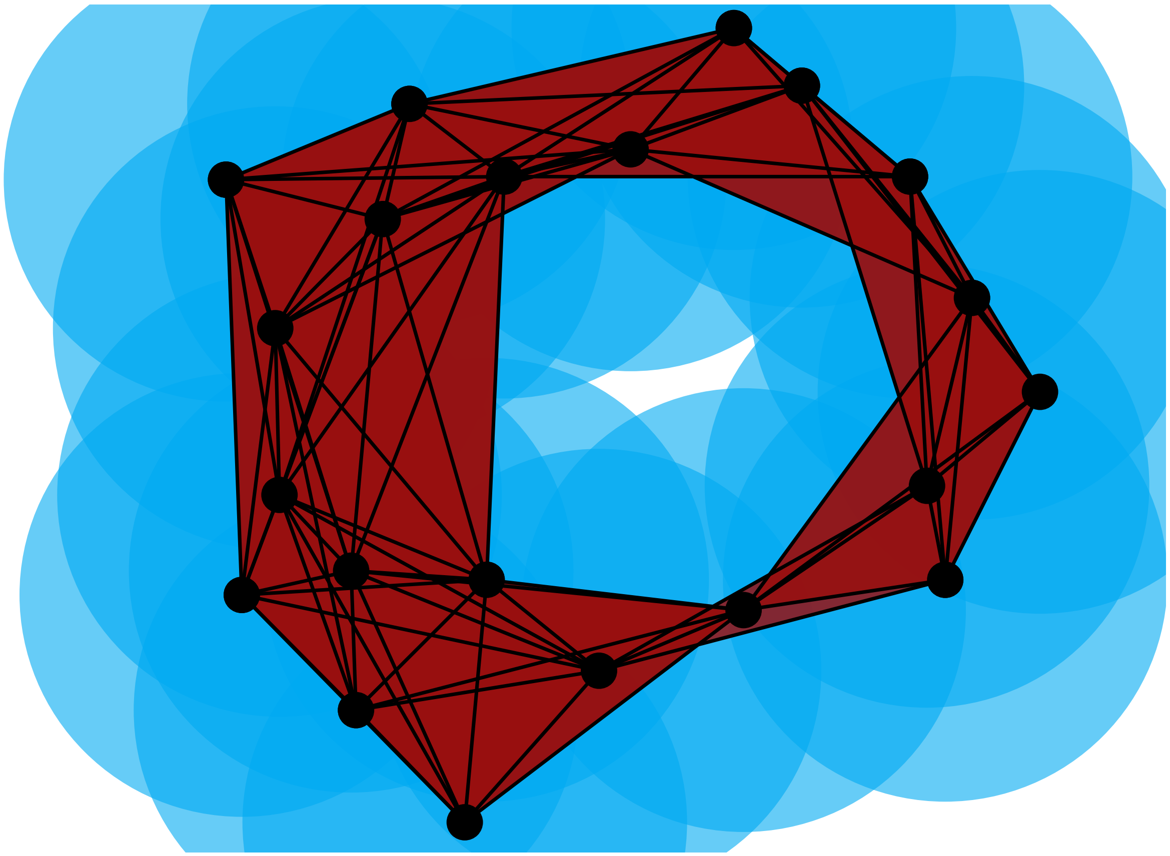

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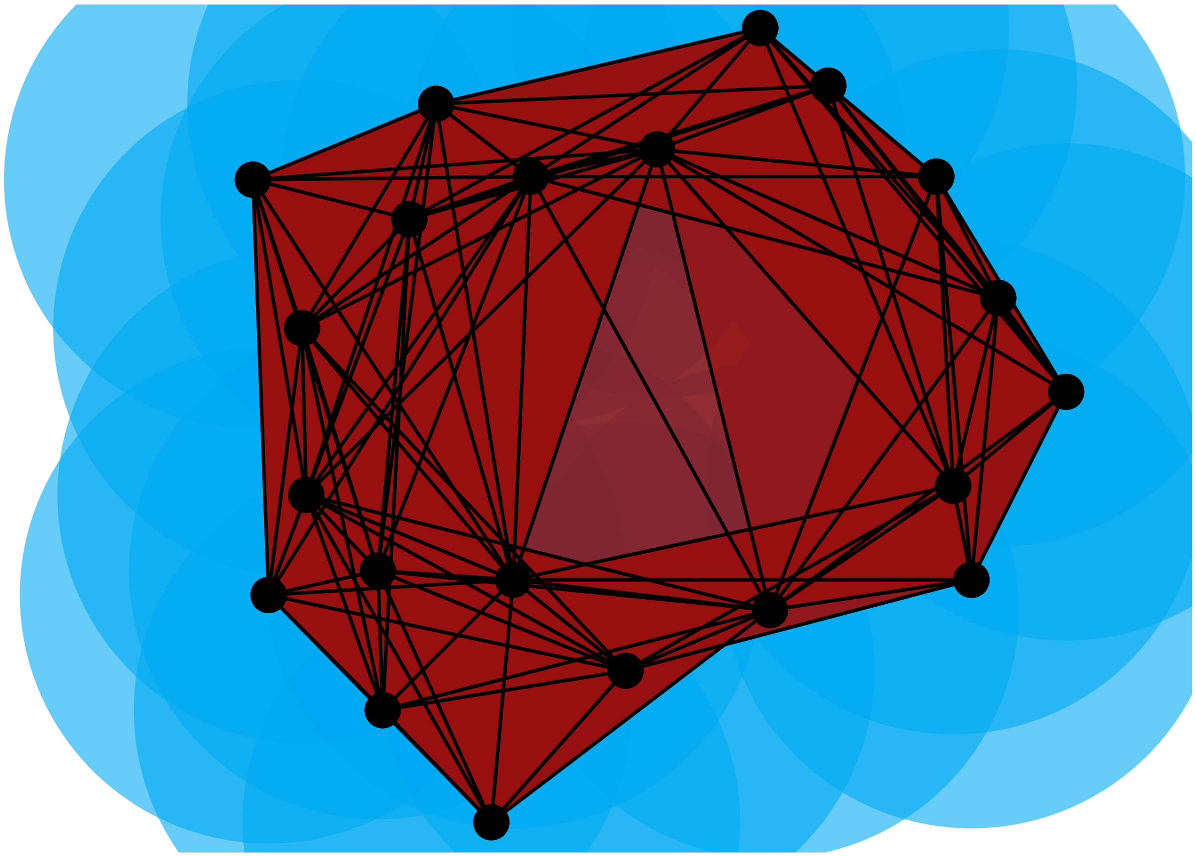

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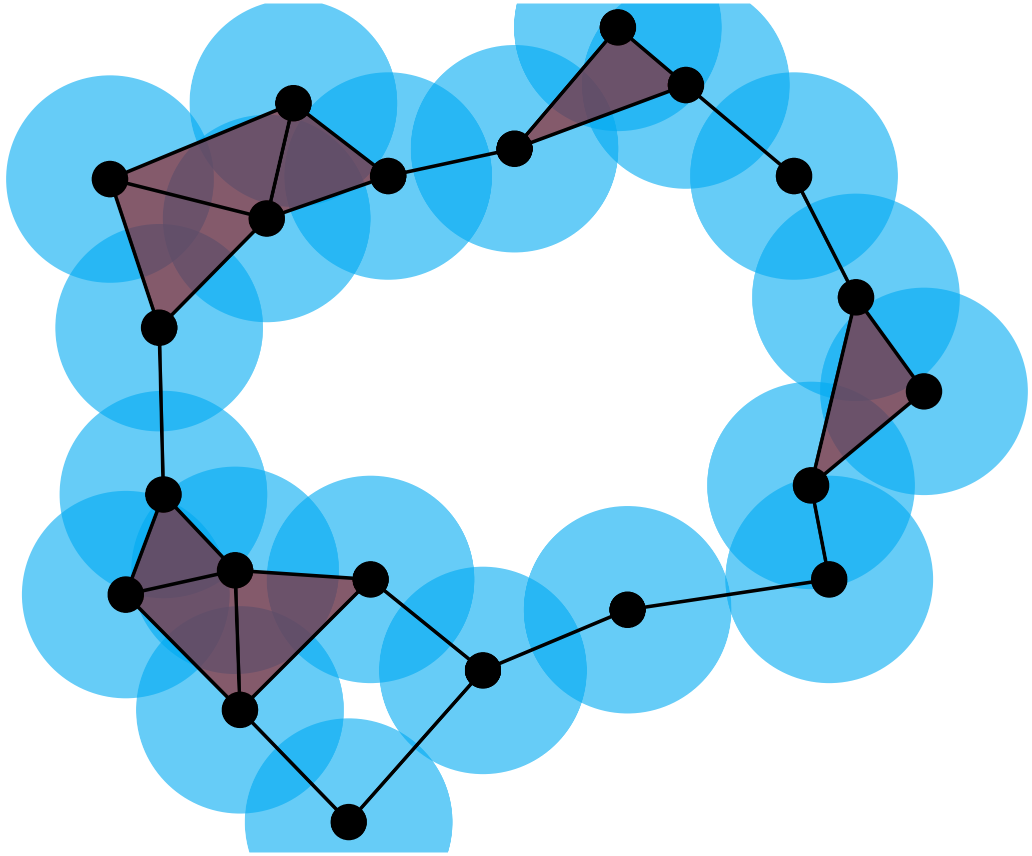

Nerve Lemma. $\check{\mathrm{C}}\mathrm{ech}(X; r) \simeq$ union of balls

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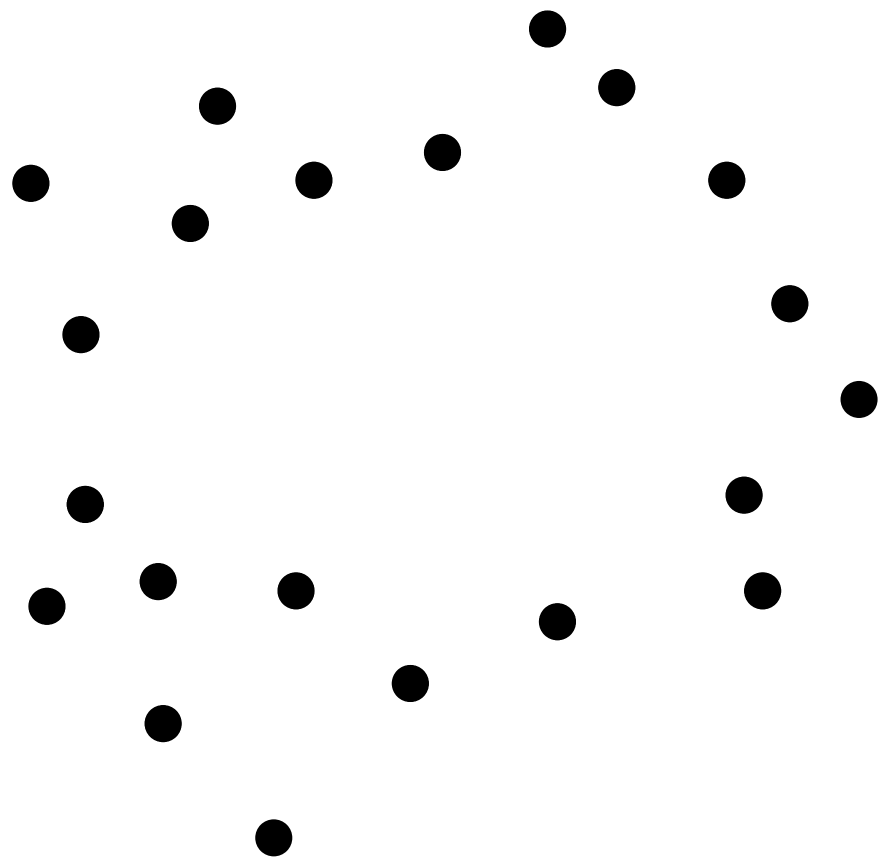

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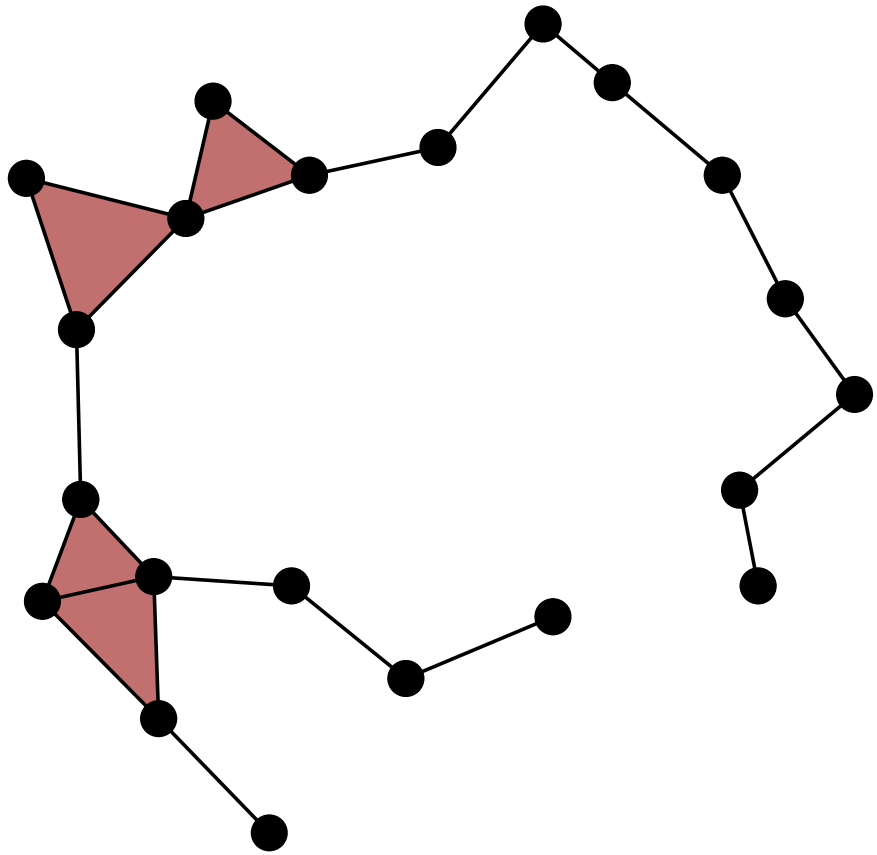

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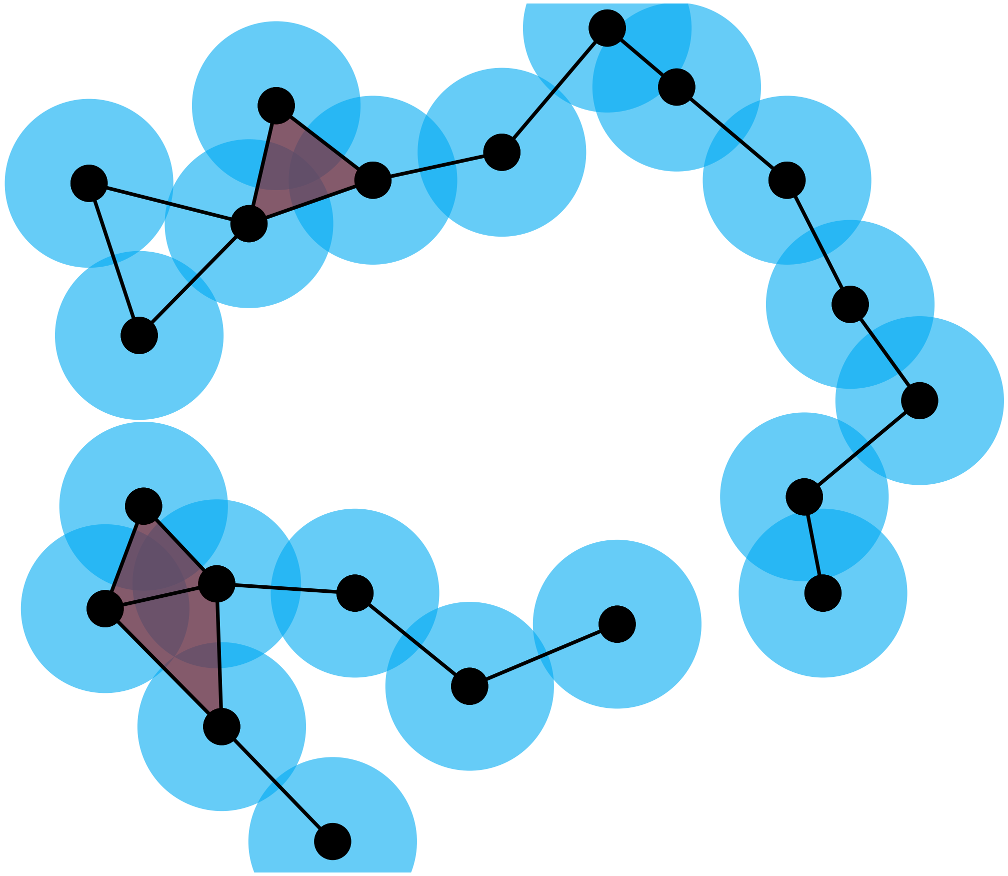

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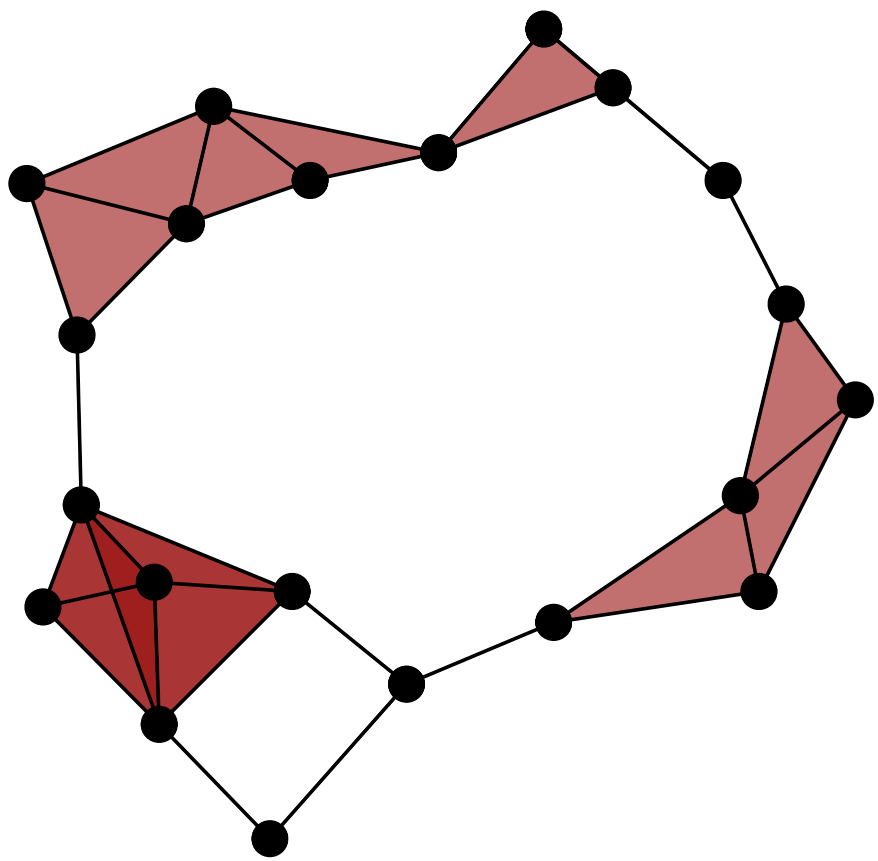

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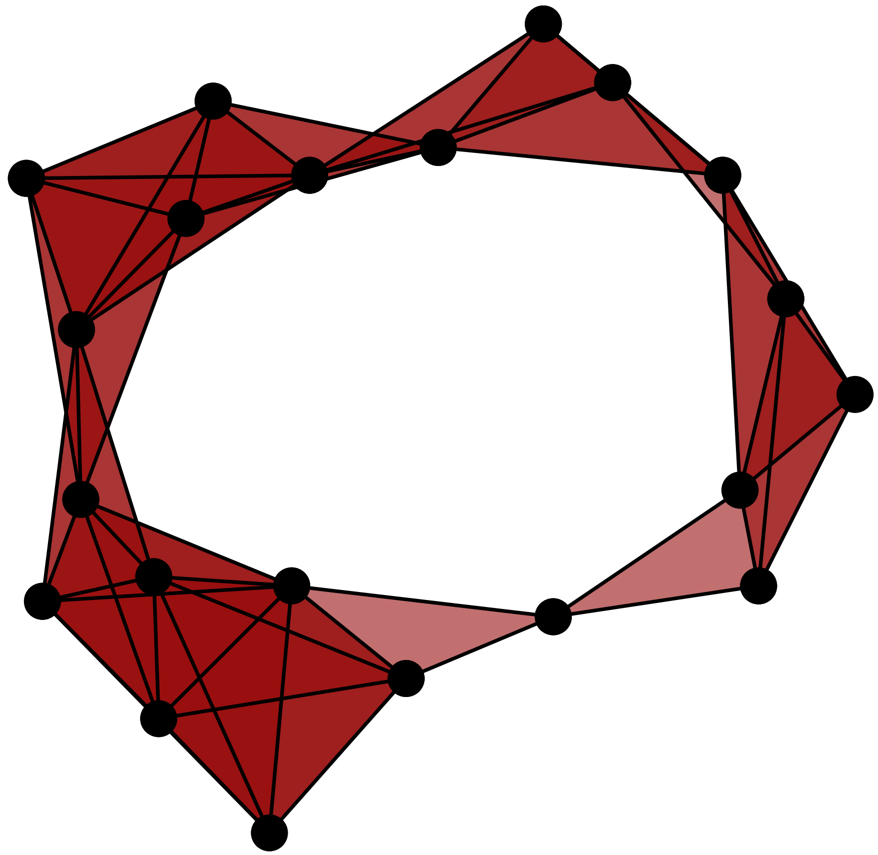

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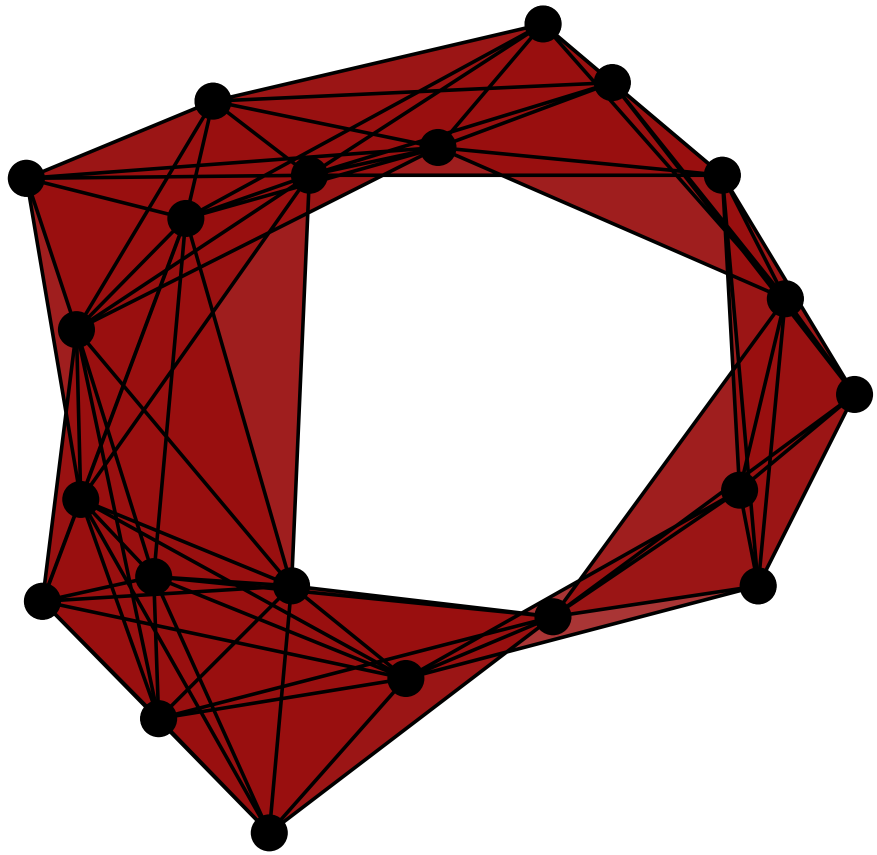

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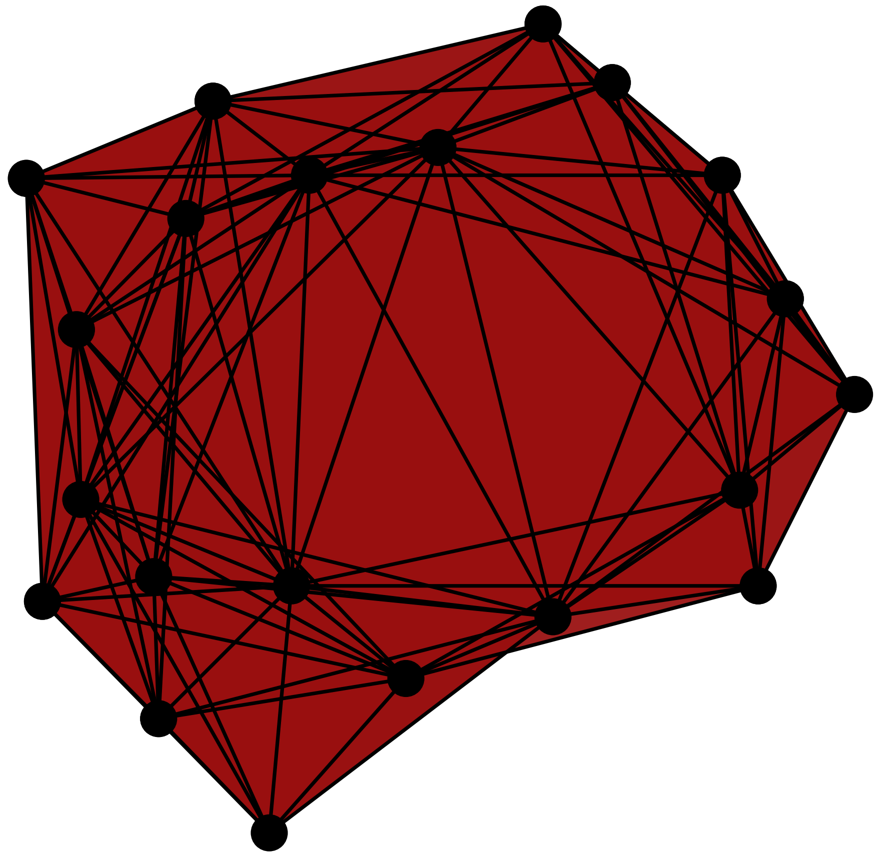

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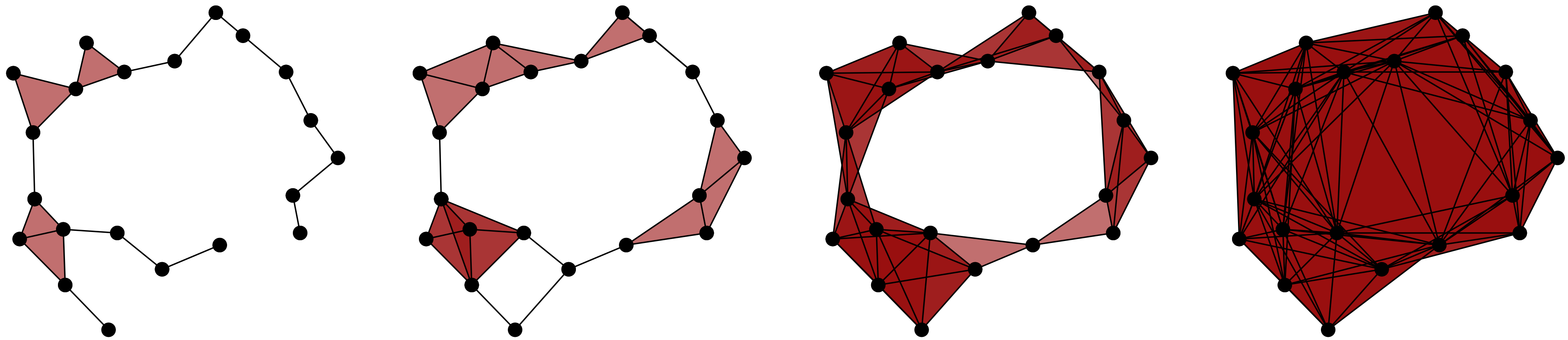
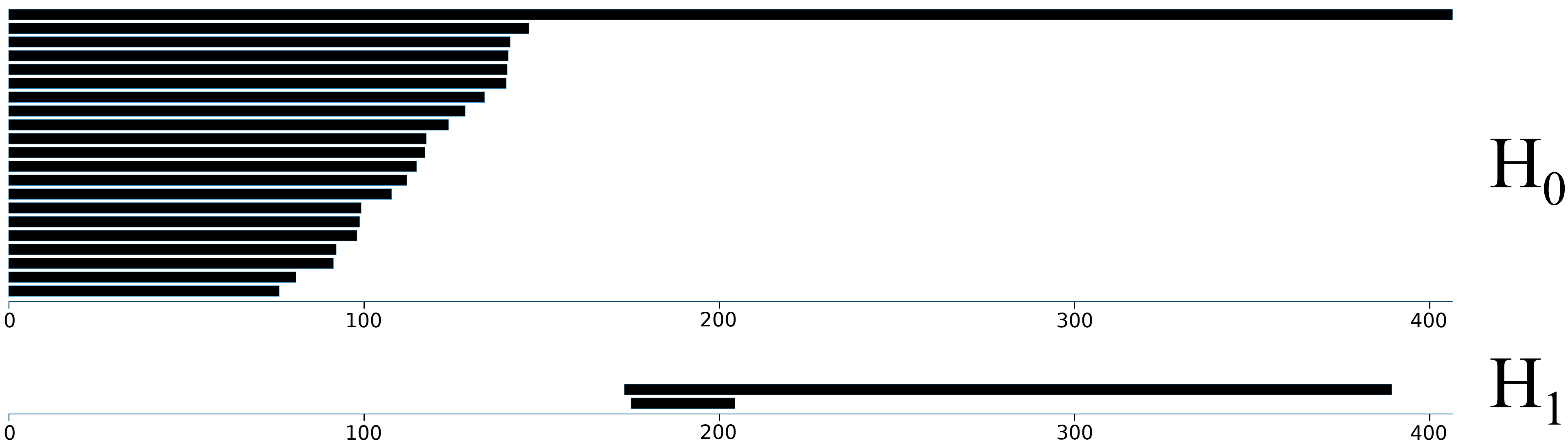
Persistent homology





- Input: Increasing spaces. Output: barcode.
- Significant features persist.
- Cubic computation time in the number of simplices.

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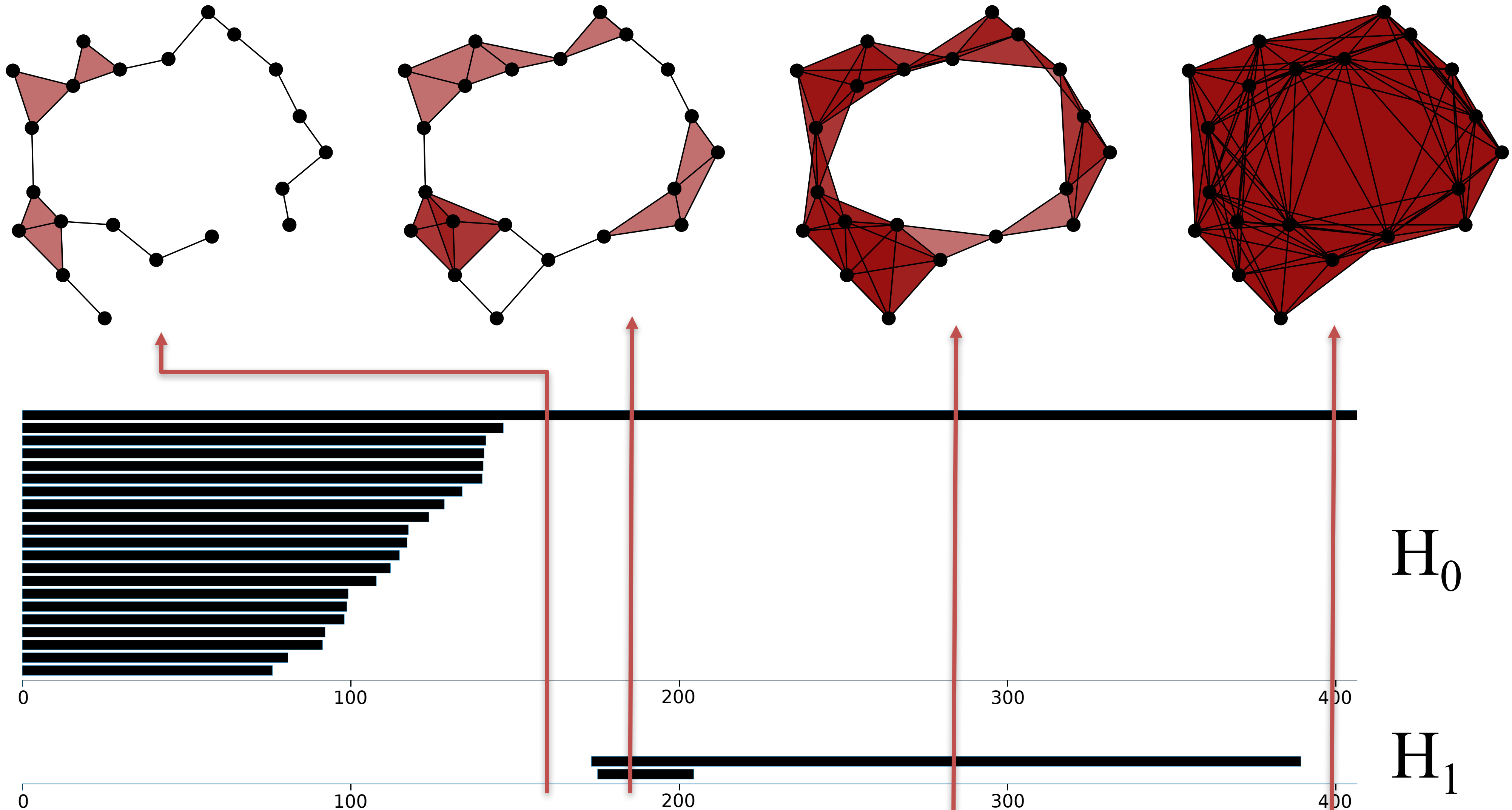

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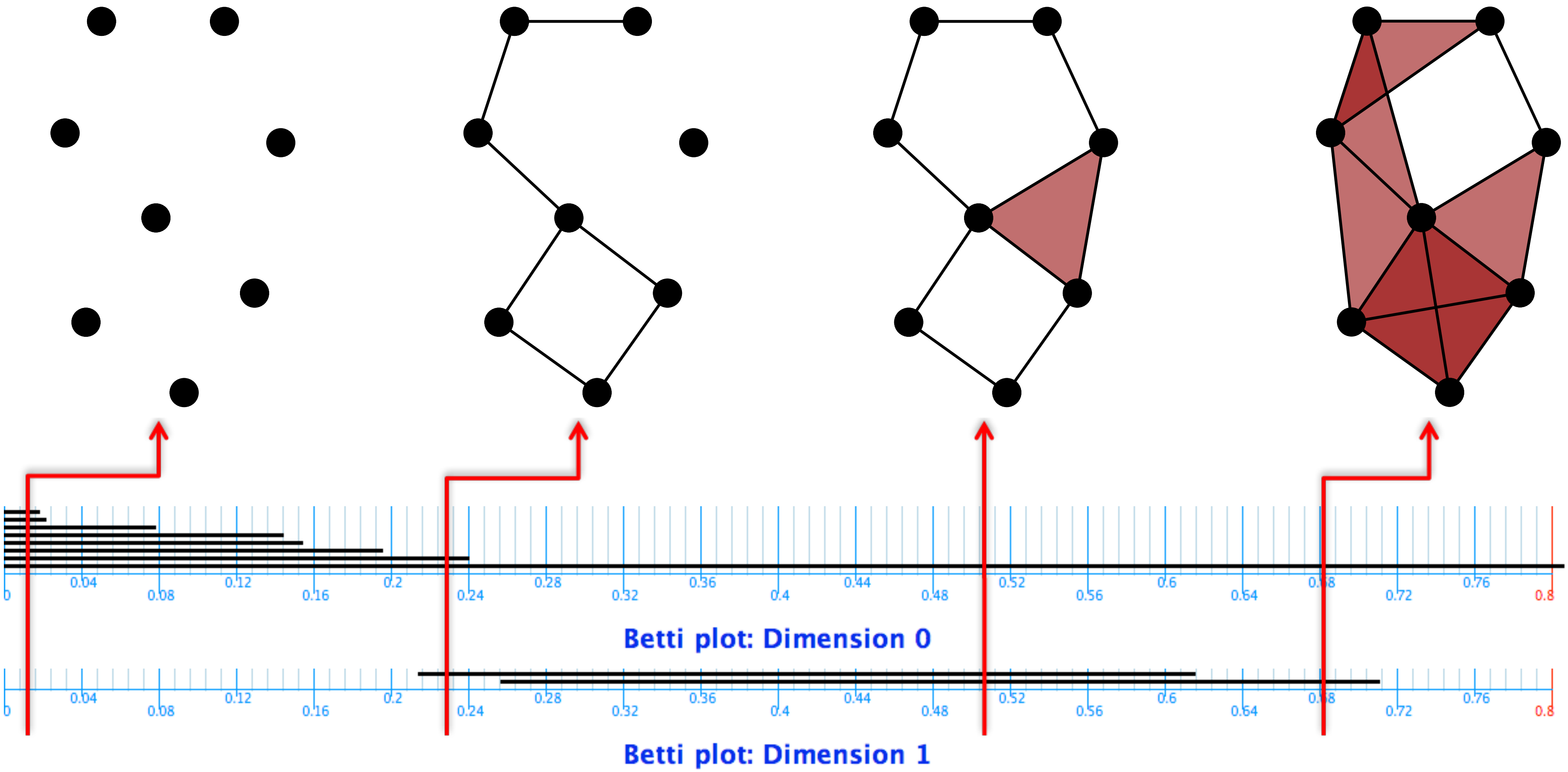

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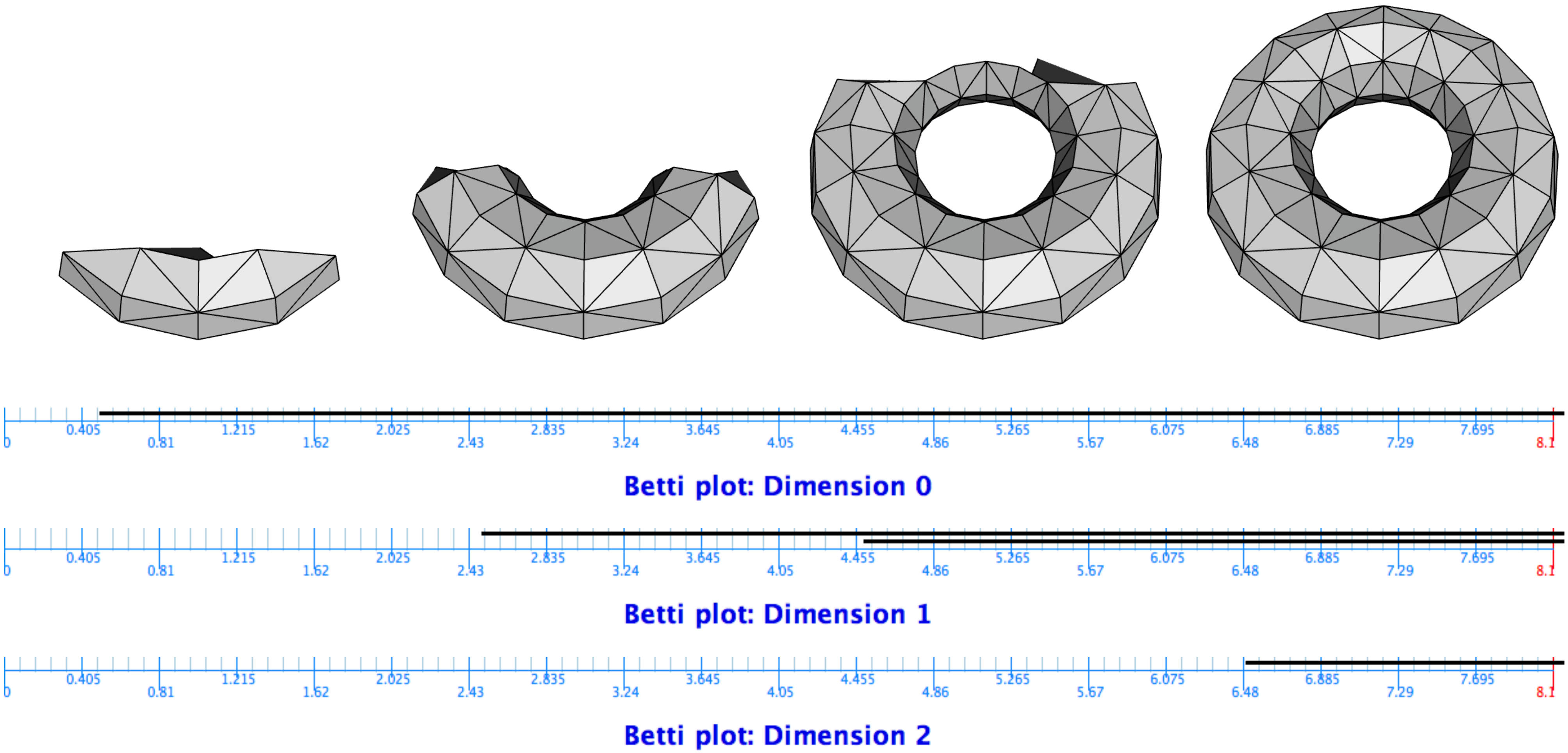

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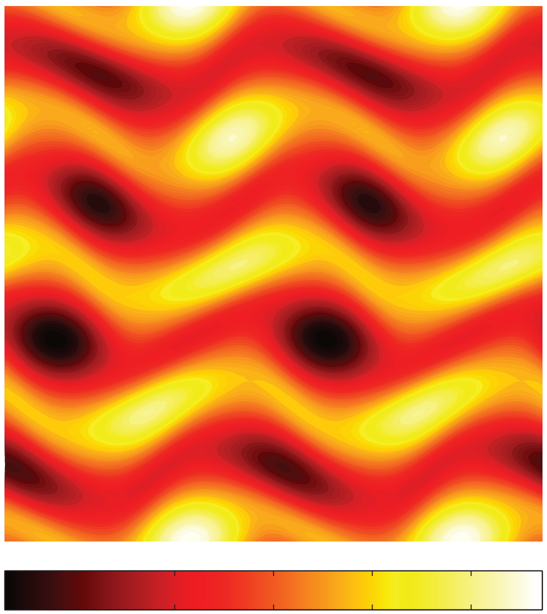
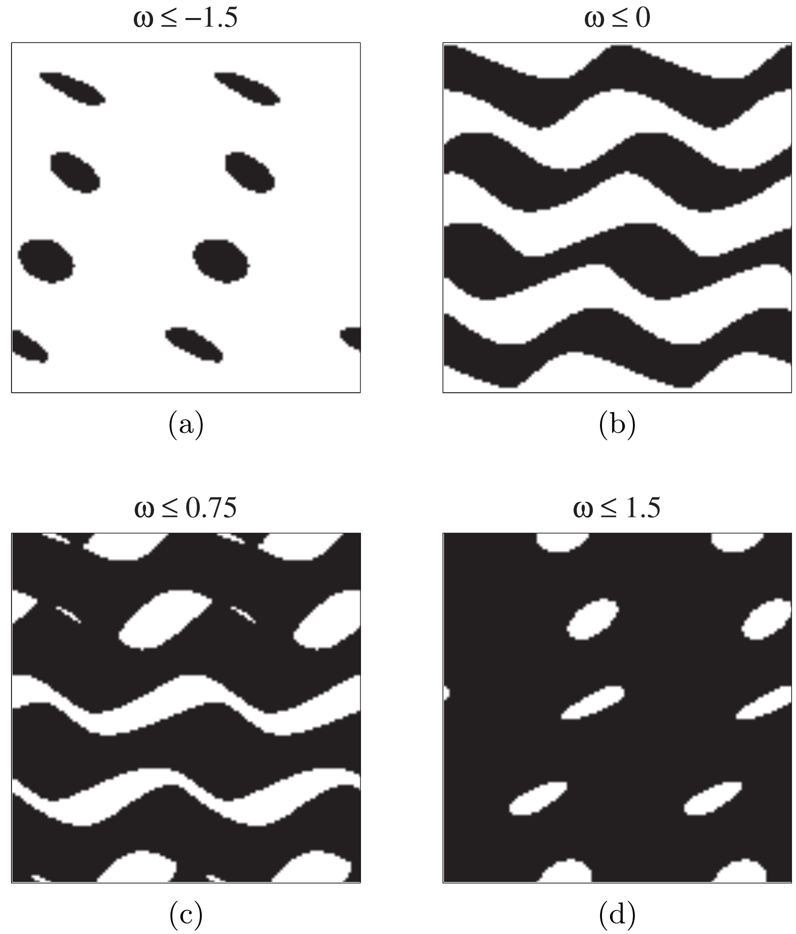
Persistent homology

Sublevelset persistence



Analysis of Kolmogorov flow and Rayleigh–Bénard convection using persistent homology by Miroslav Kramár, Rachel Levanger, Jeffrey Tithof, Balachandra Suri, Mu Xu, Mark Paul, Michael F Schatz, Konstantin Mischaikow



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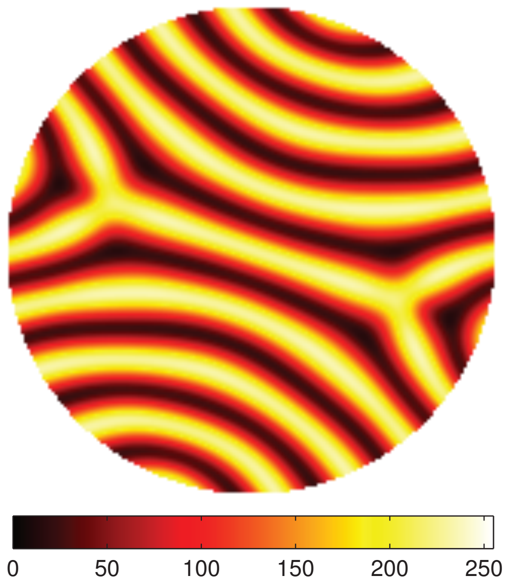

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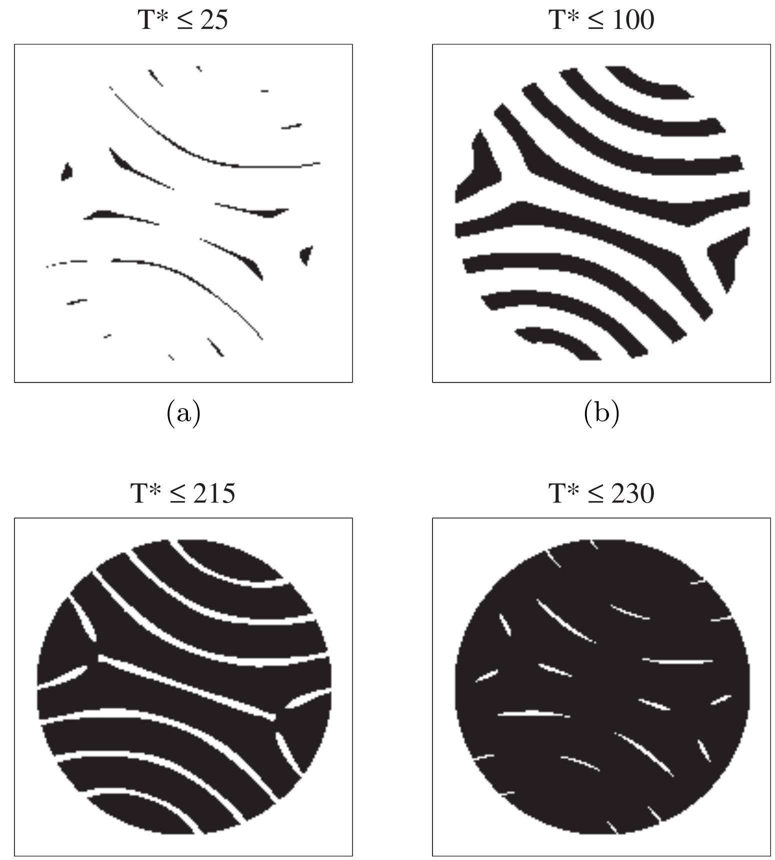

(a) (b) (c) (d)

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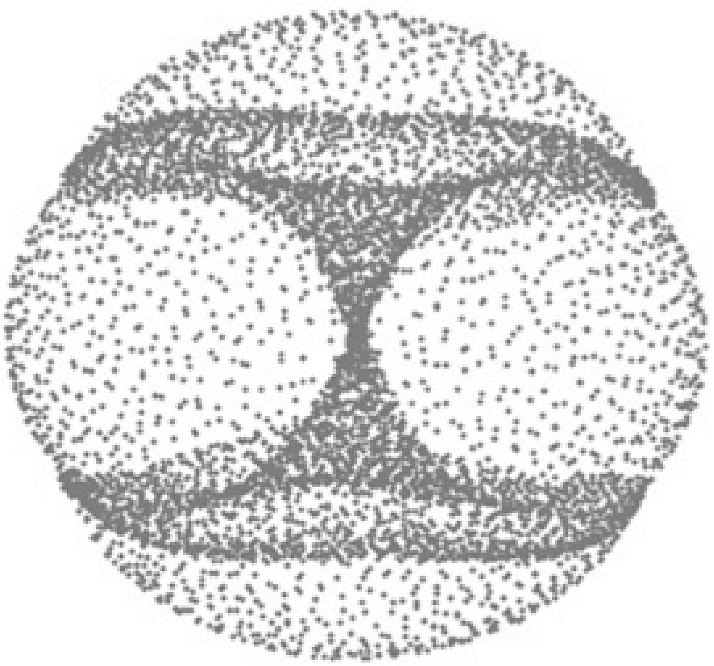
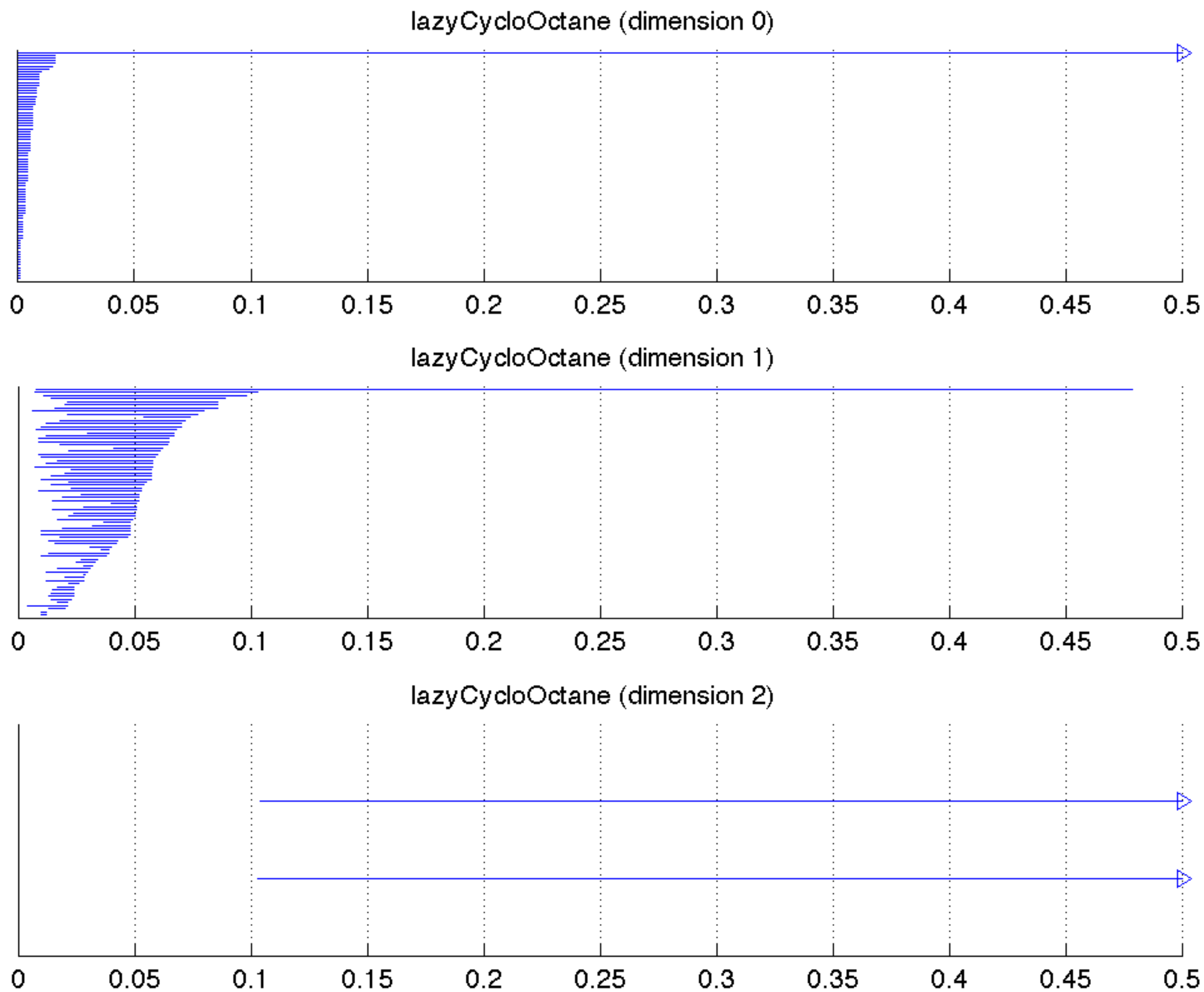
Persistent homology applied to data

Example: Cyclo-Octane (C_8H_{16}) data

1,000,000+ points in 72-dimensional space





Non-Manifold Surface Reconstruction from High Dimensional Point Cloud Data
by Shawn Martin and Jean-Paul Watson, 2010.

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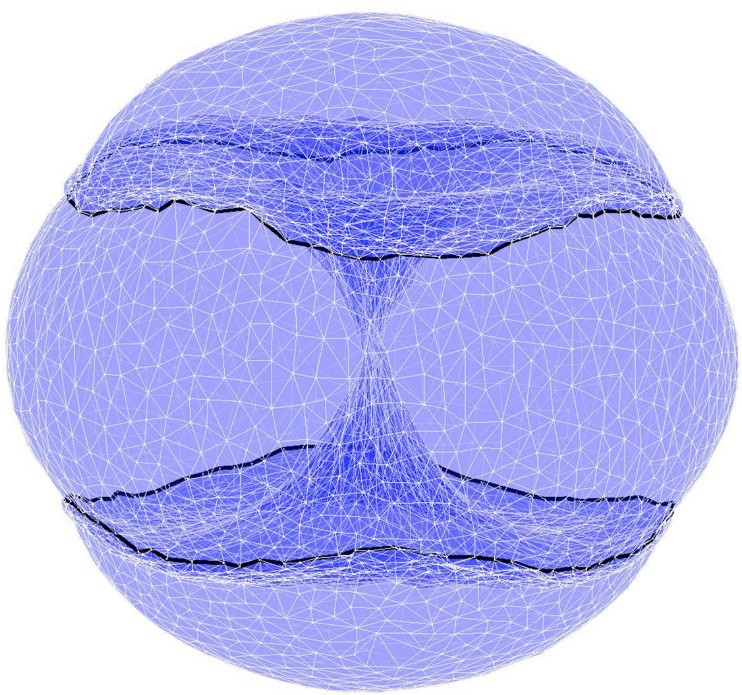

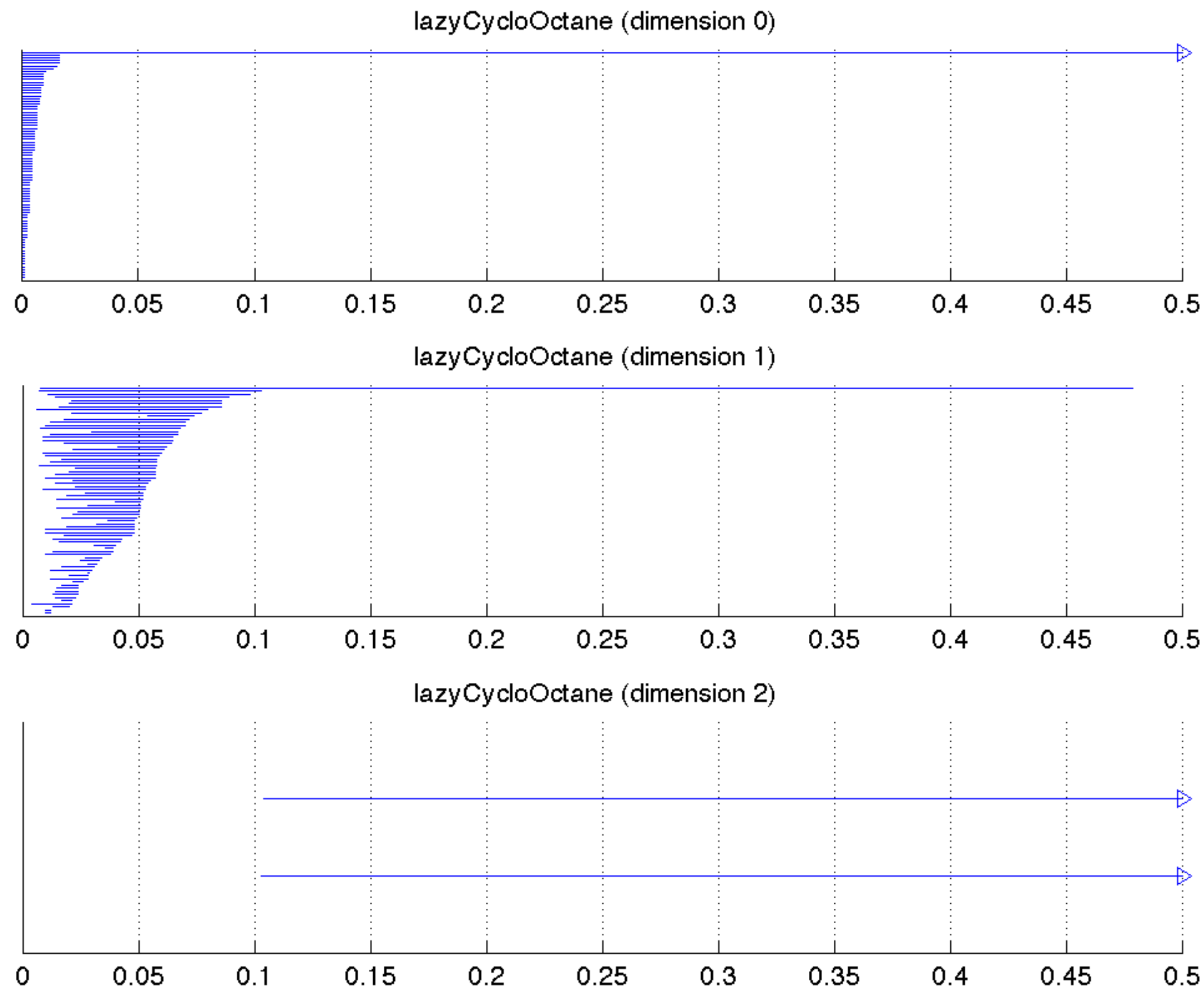

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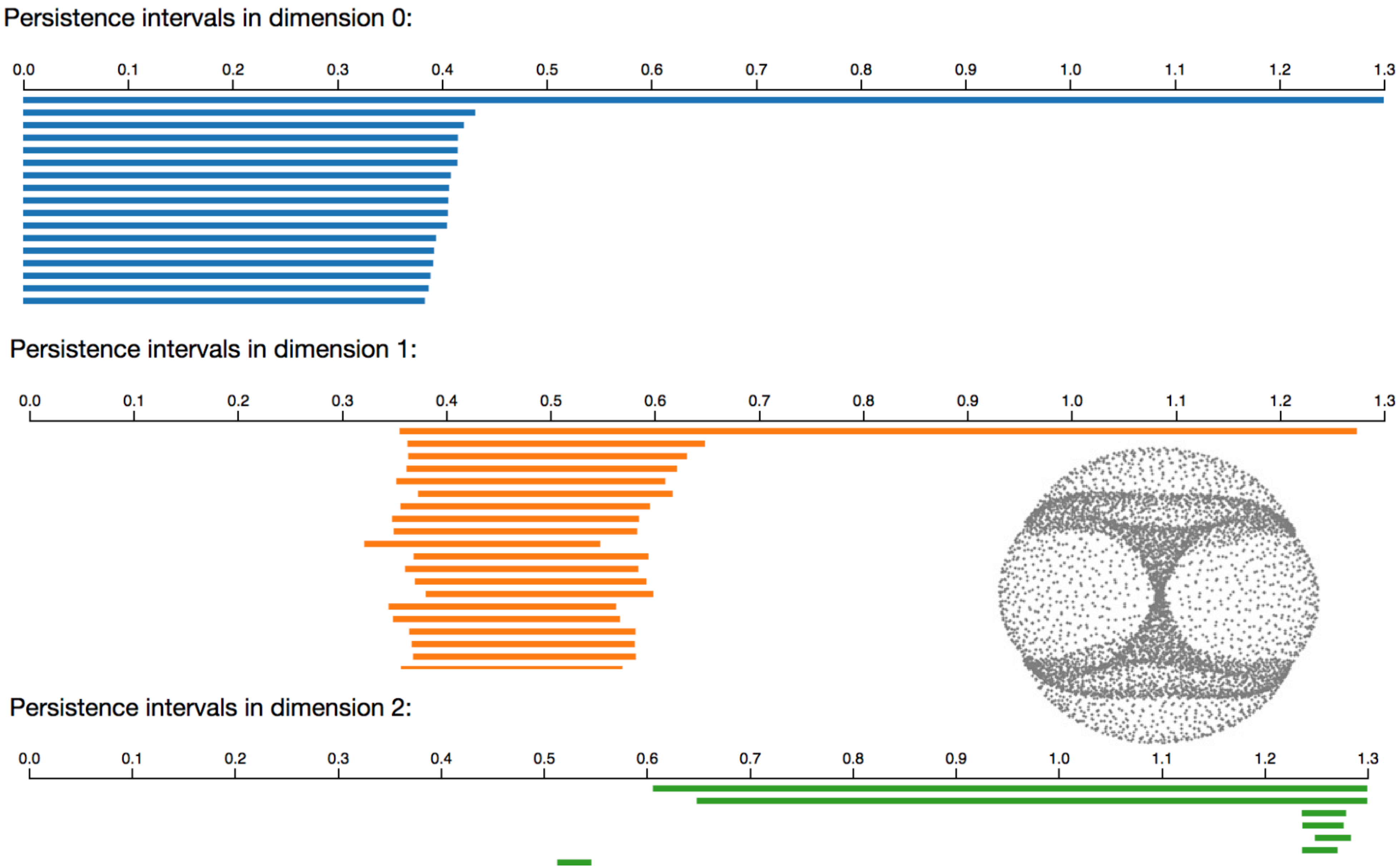

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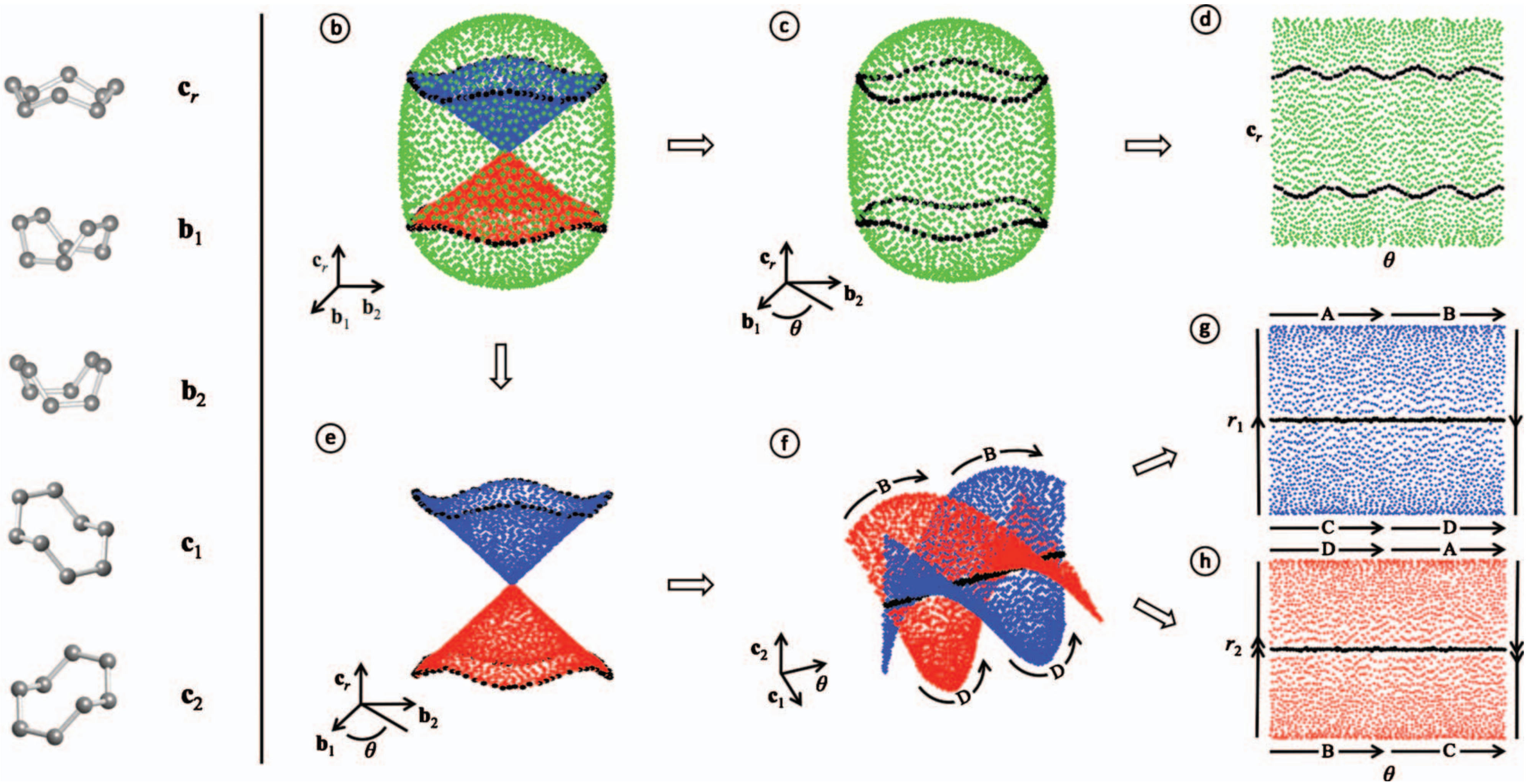

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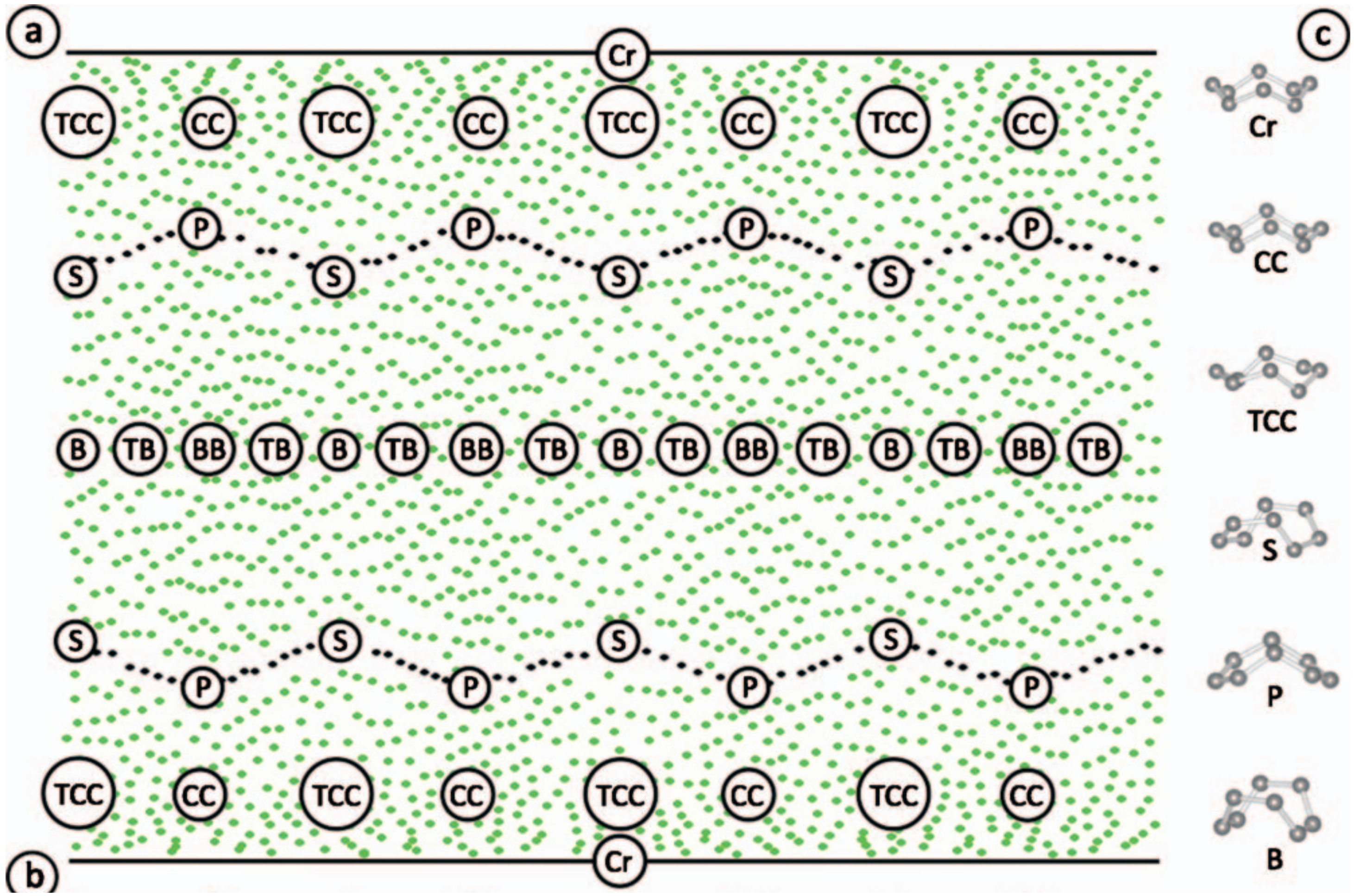

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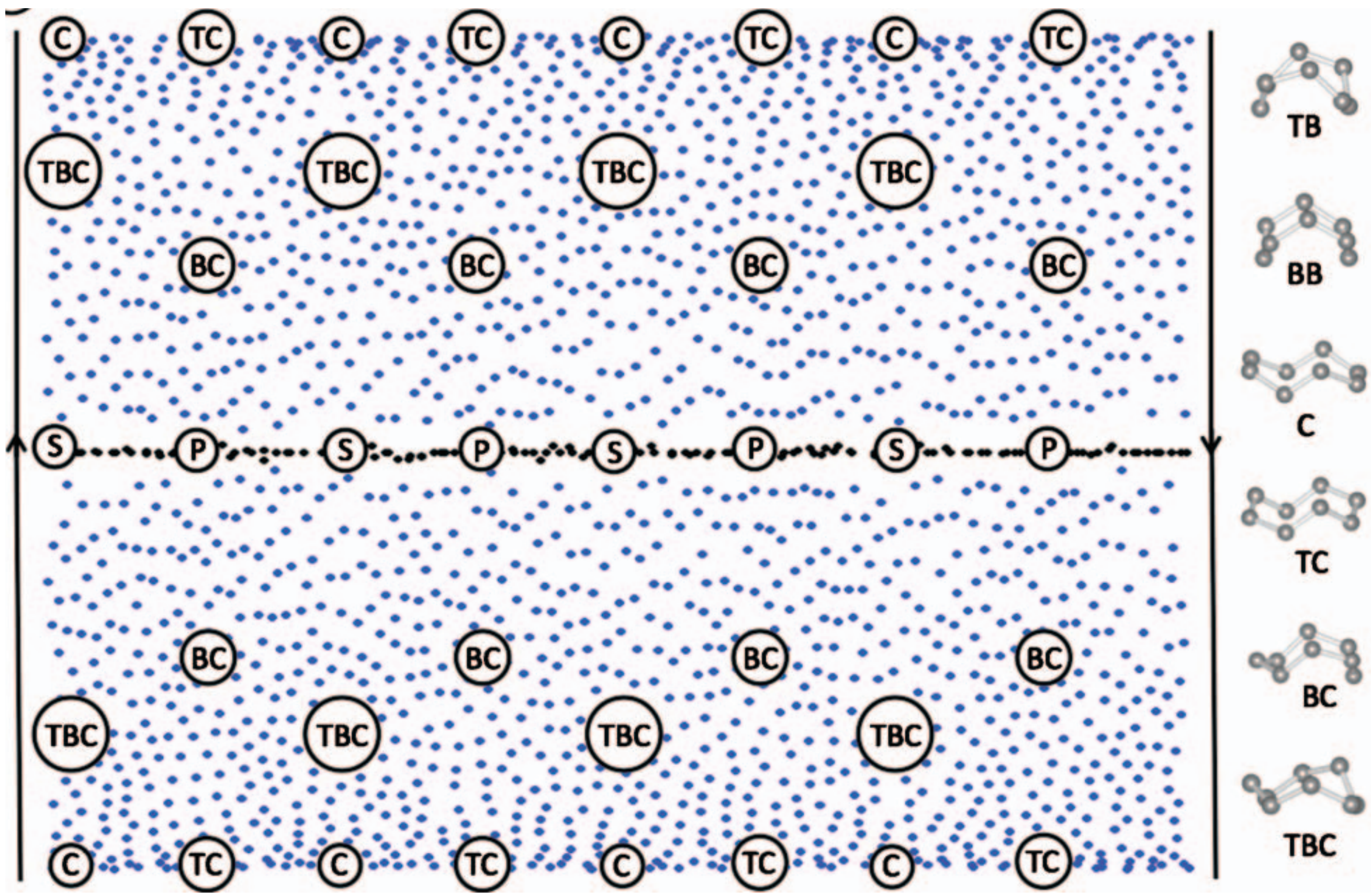

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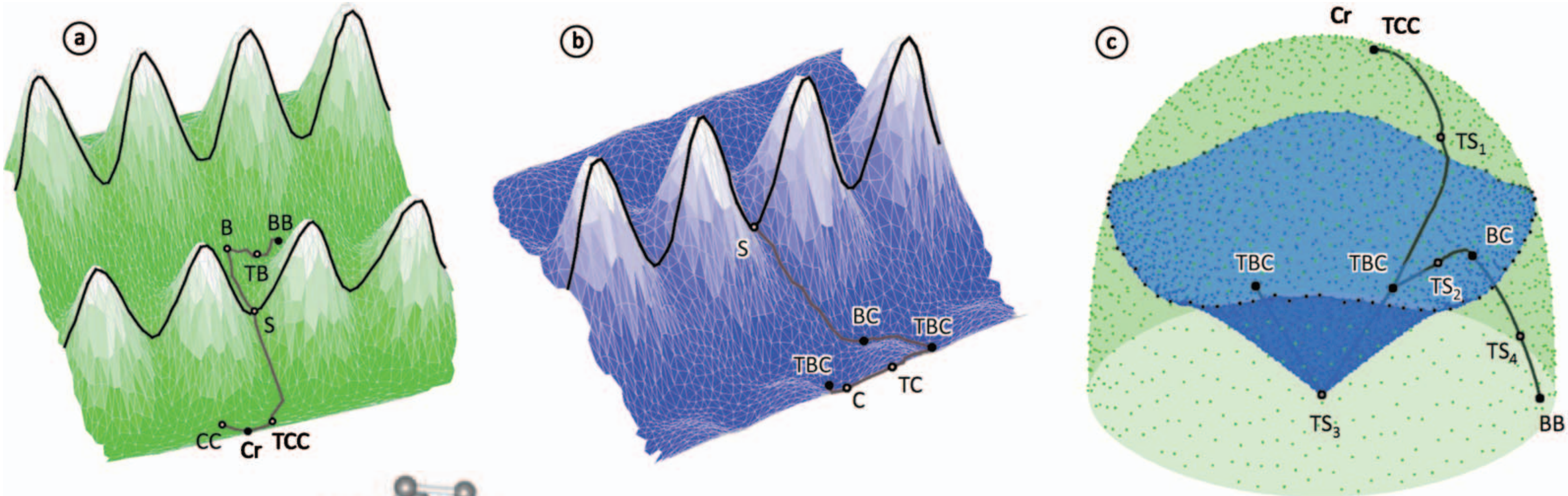

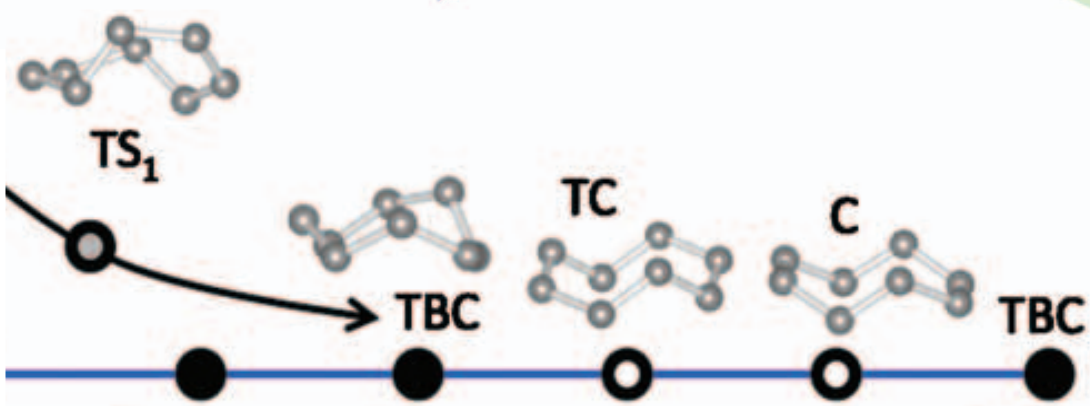

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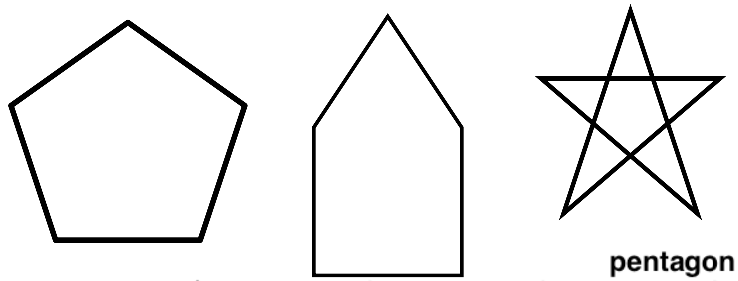
Persistent homology applied to data

Example: Equilateral pentagons in the plane



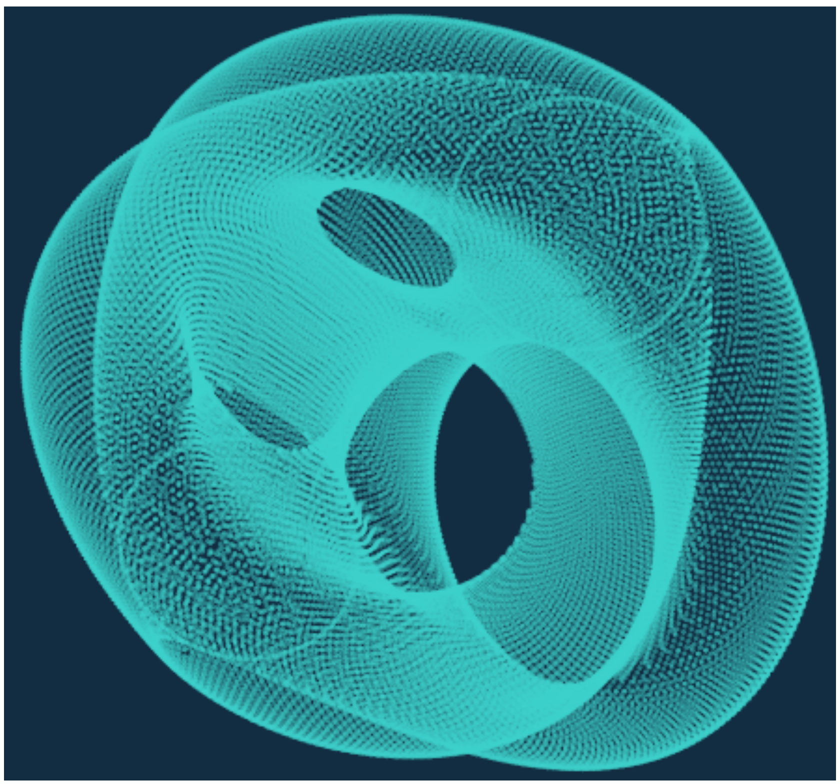
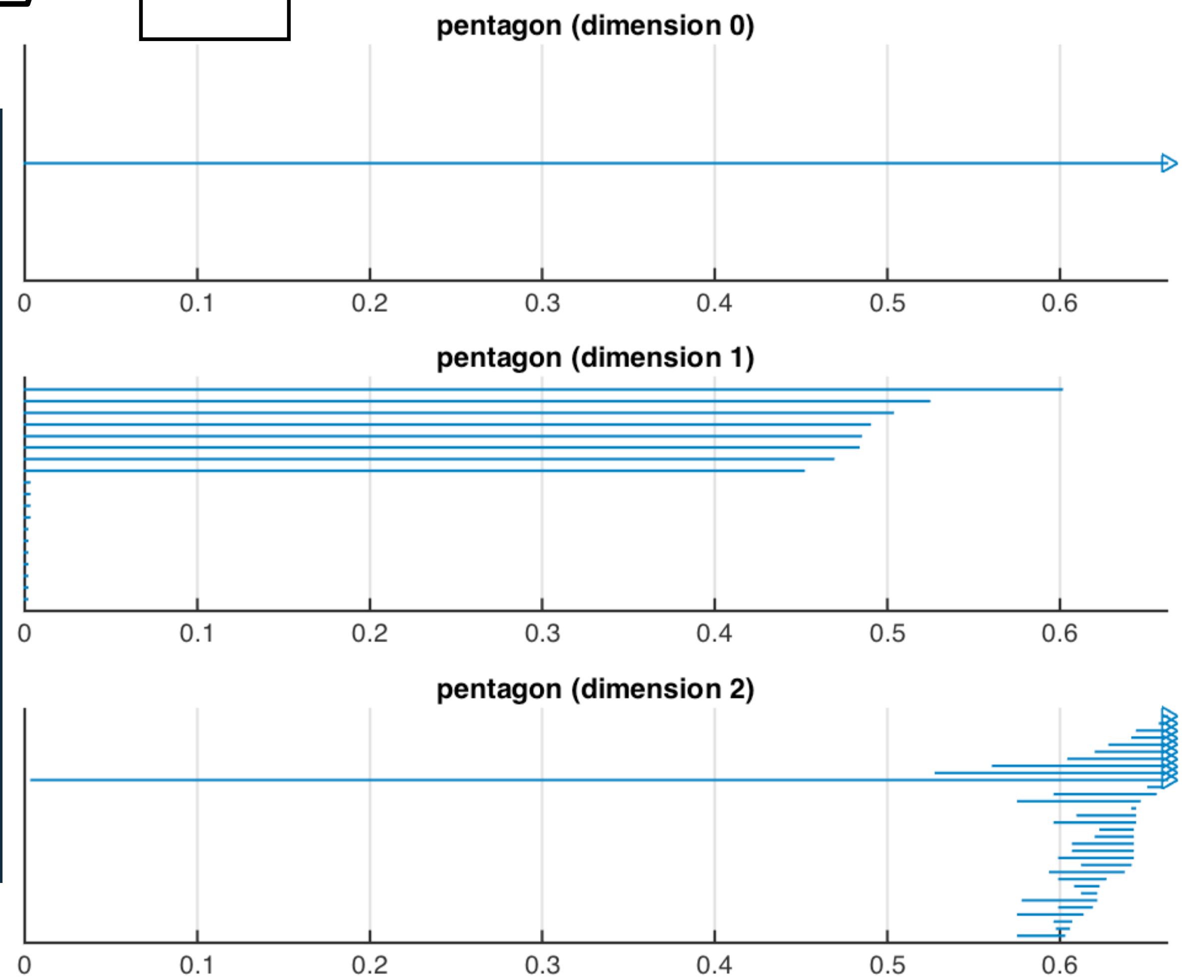


Image credit: Clayton Shonkwiler



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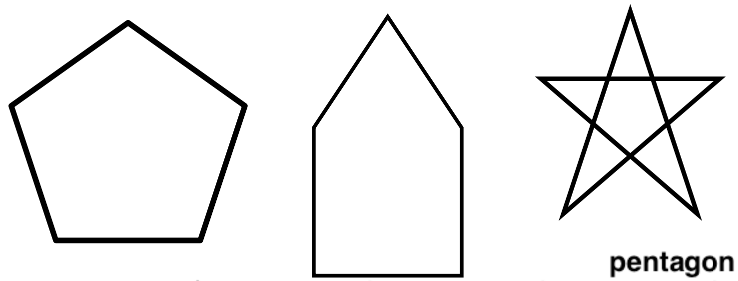

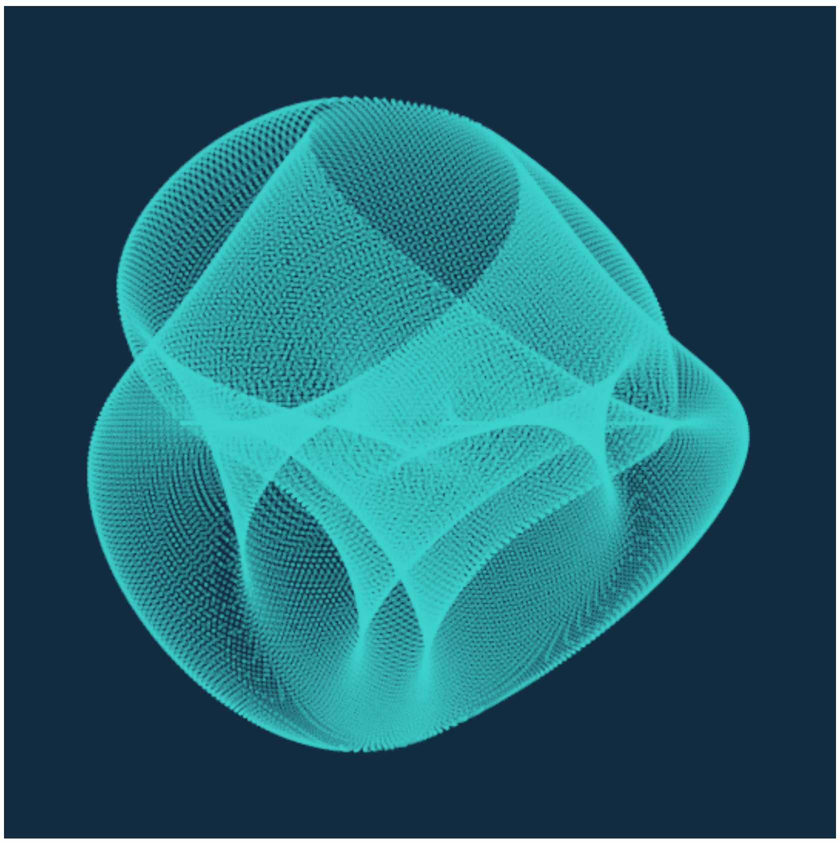

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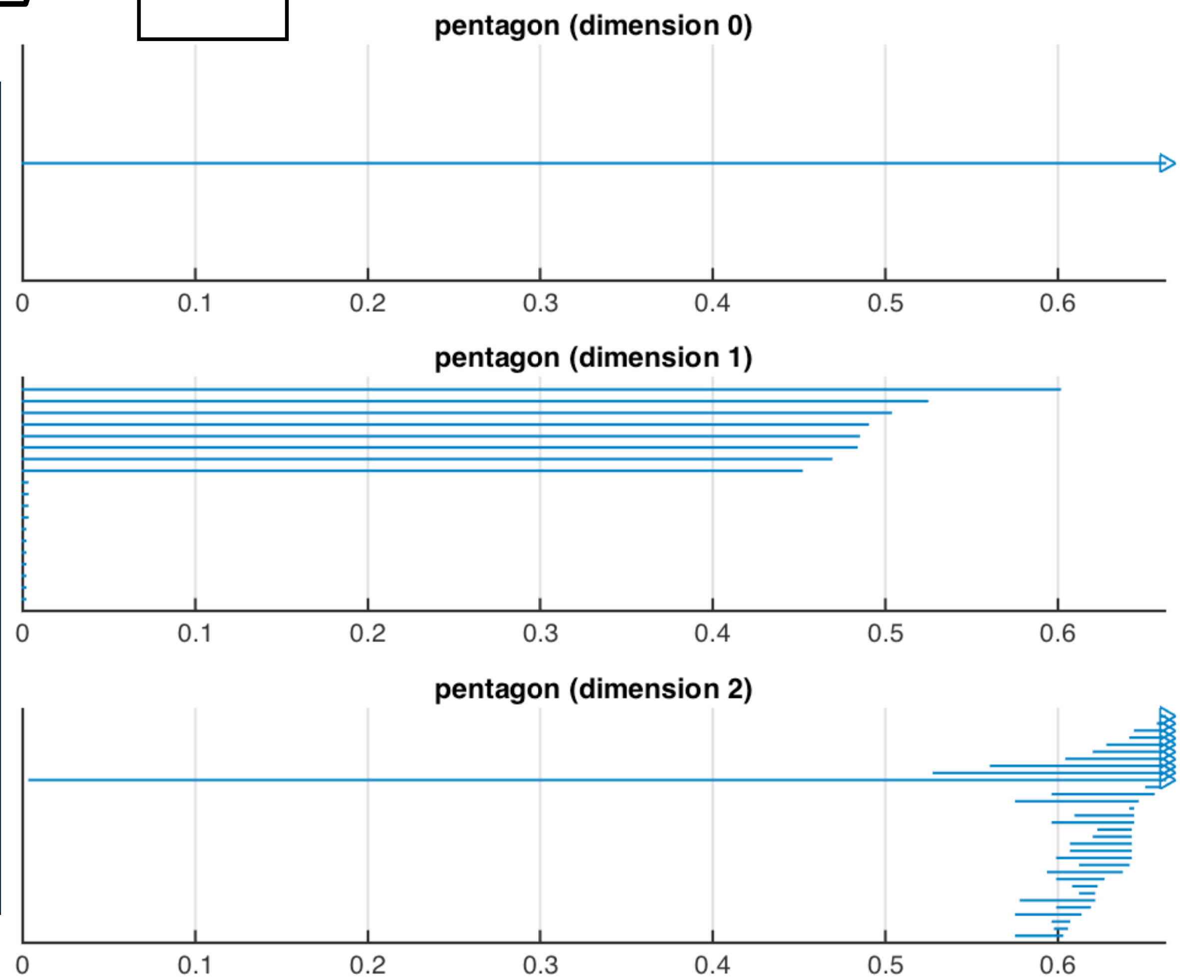

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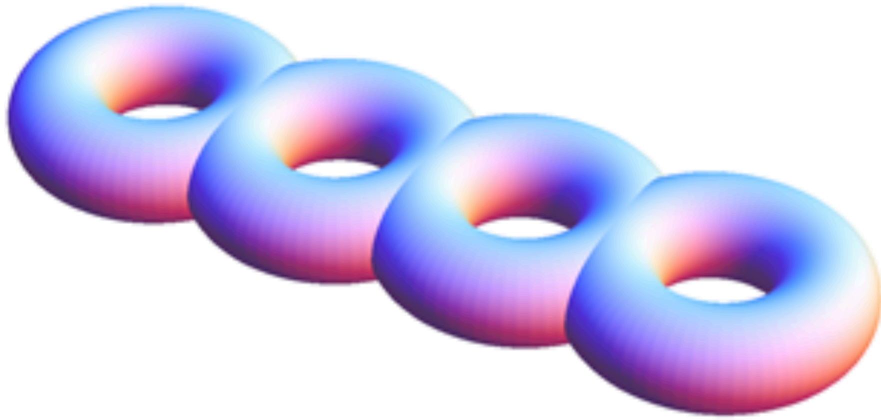

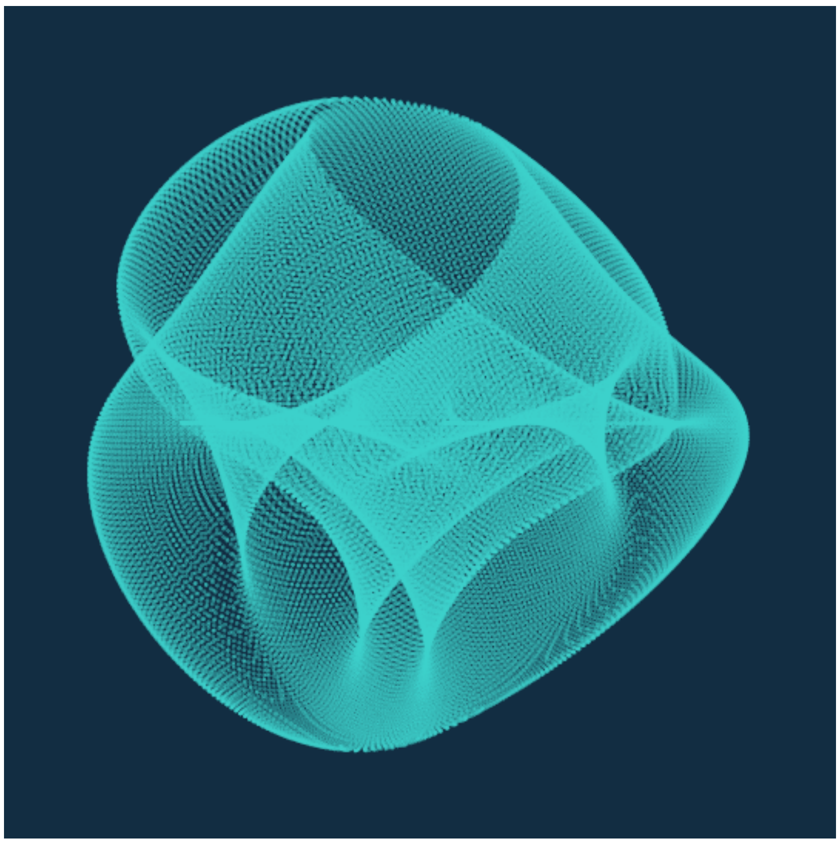

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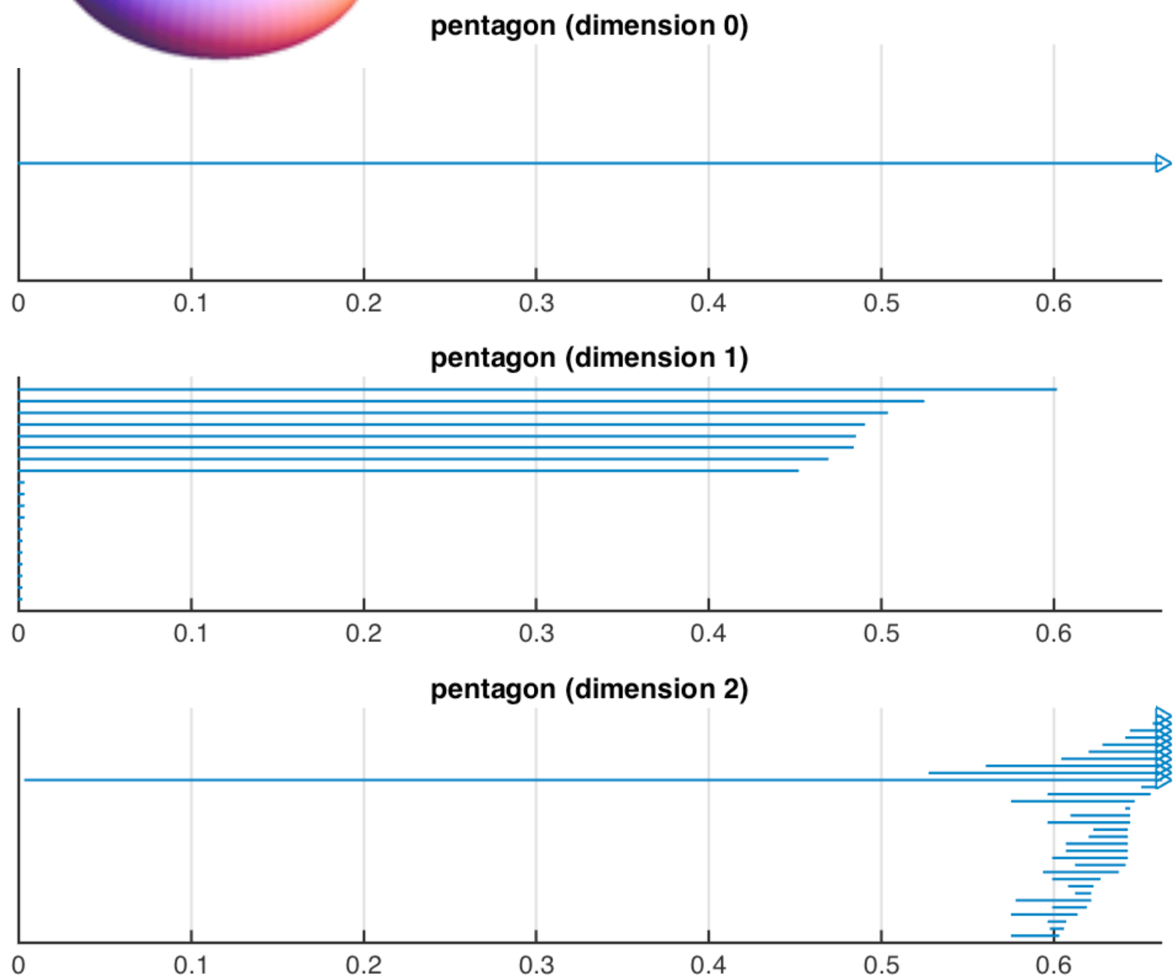
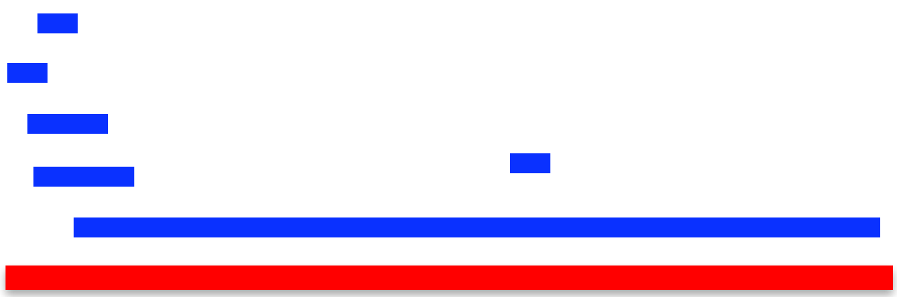
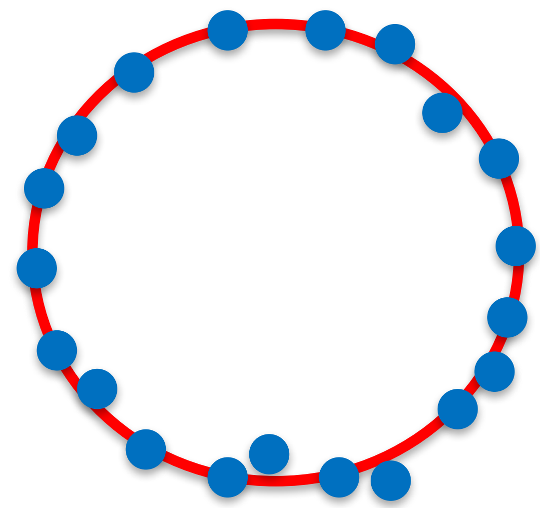


Persistent homology applied to data

- <u>Stability Theorem.</u>
  If X and Y are metric spaces, then

$$d_b(\mathrm{PH}(\check{\mathrm{C}}\mathrm{ech}(X)), \mathrm{PH}(\check{\mathrm{C}}\mathrm{ech}(Y))) \leq 2 d_{\mathrm{GH}}(X, Y)$$

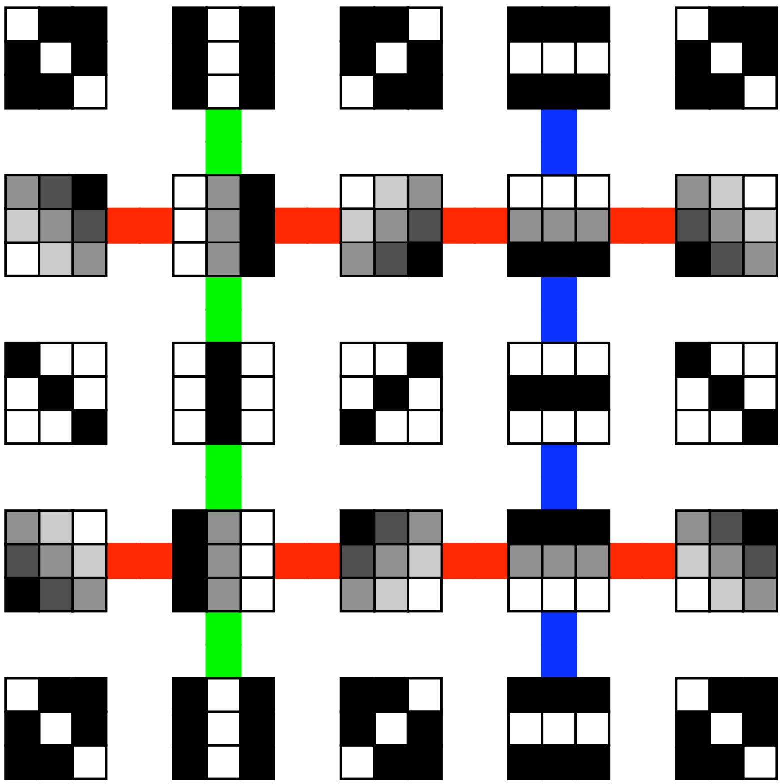




Conclusions for Part I

- Datasets have shape, which are reflective of patterns within.
- Persistent homology is a way to measure some of the local geometry and global topology of a dataset.

Topology applied to image data



Persistent homology applied to data

Groningen, The Netherlands

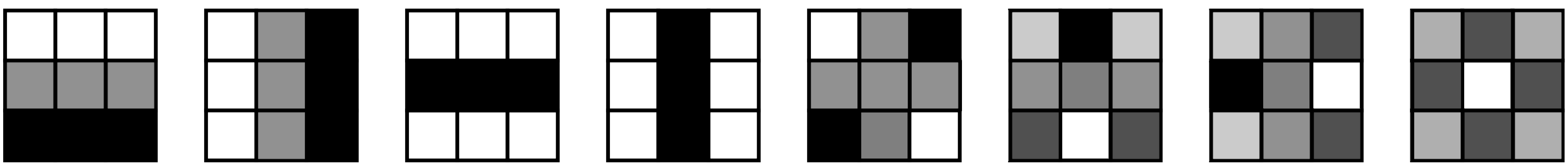
The receptive fields of cells in our primary visual cortex (V1) are related to the statistics natural images.

Independent component filters of natural images compared with simple cells in primary visual cortex by JH van Hateren and A van der Schaaf, 1997

Persistent homology applied to data

3x3 *high-contrast* patches from images

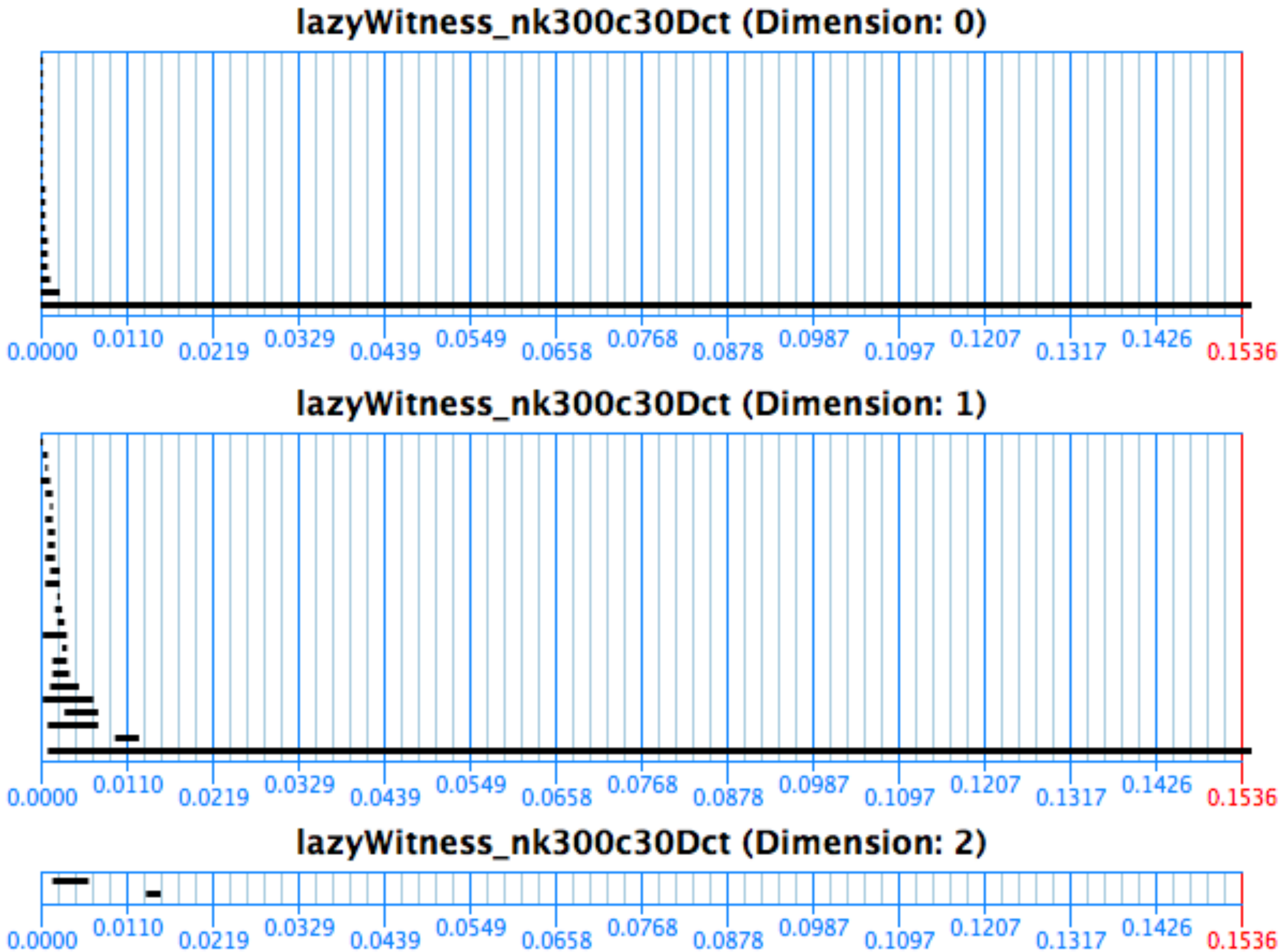
Points in 9-dimensional space, normalized to have average color gray and contrast norm one (on 7-sphere).



On the Local Behavior of Spaces of Natural Images by Gunnar Carlsson, Tigran Ishkhanov, Vin de Silva, and Afra Zomorodian, 2008.

Persistent homology applied to data

1. Densest patches according to a global estimate



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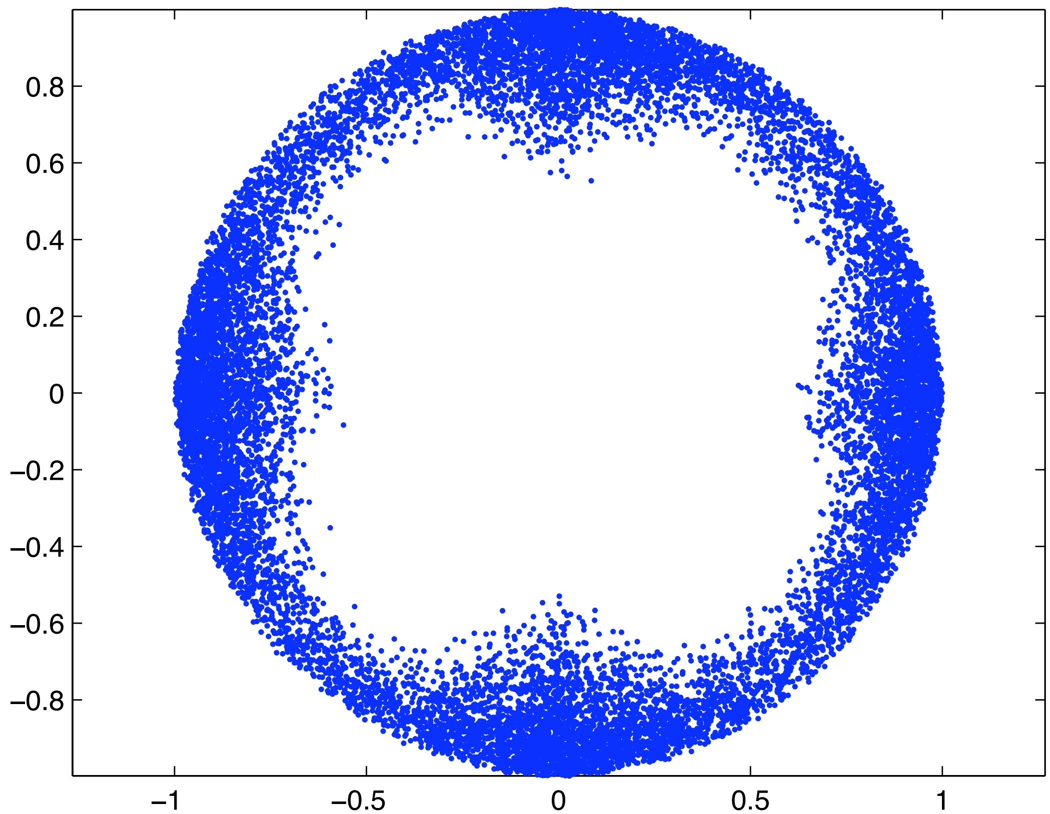
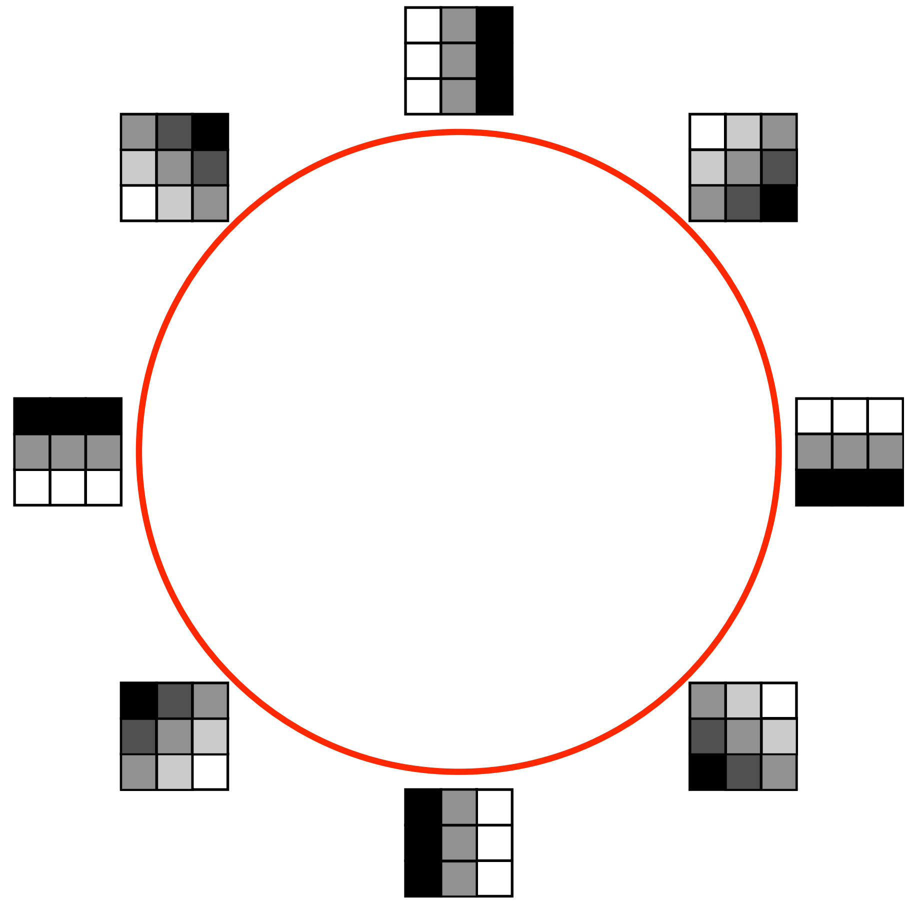
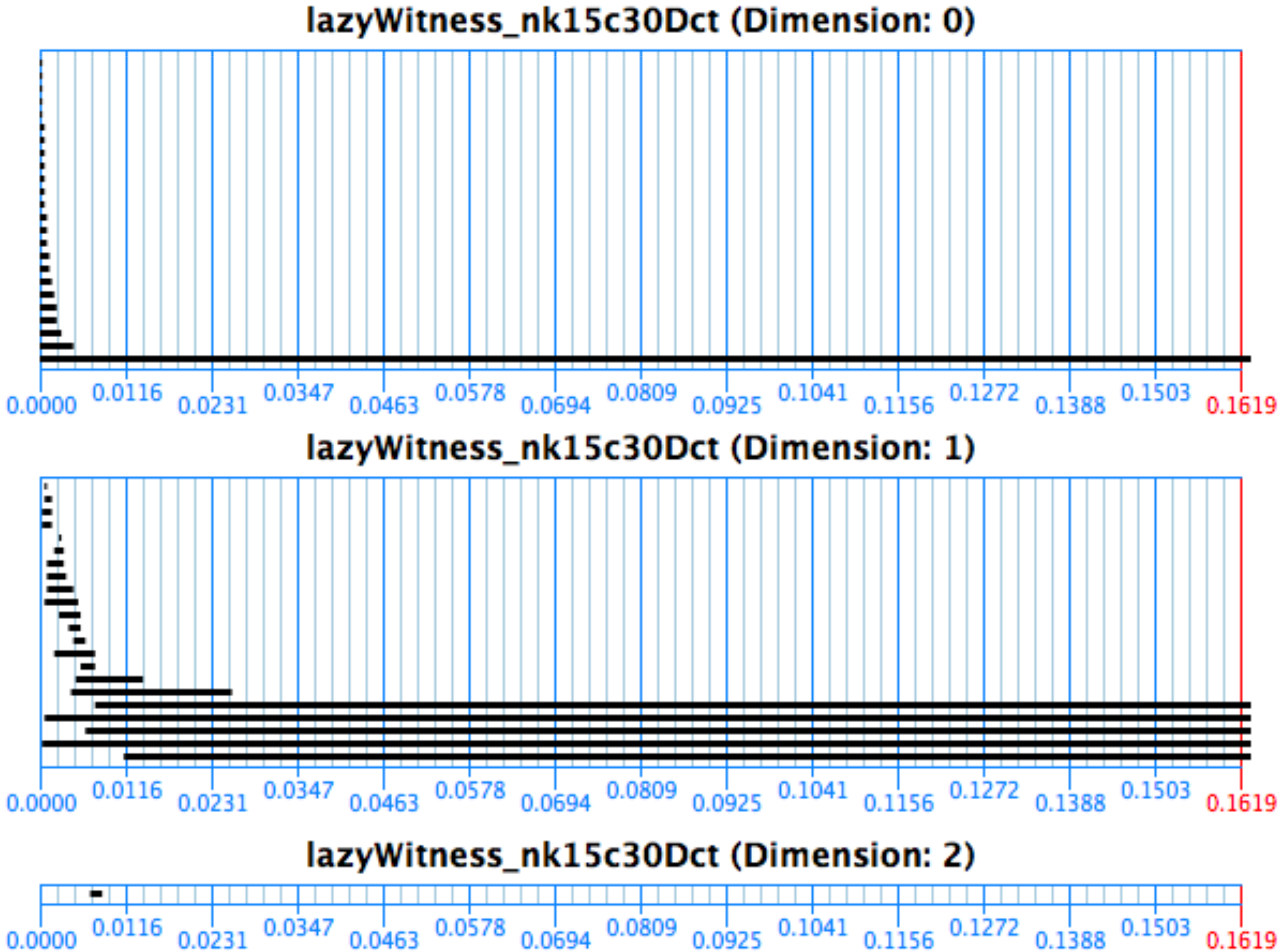




Interpretation: nature prefers linearity

Persistent homology applied to data

2. Densest patches according to an intermediate estimate



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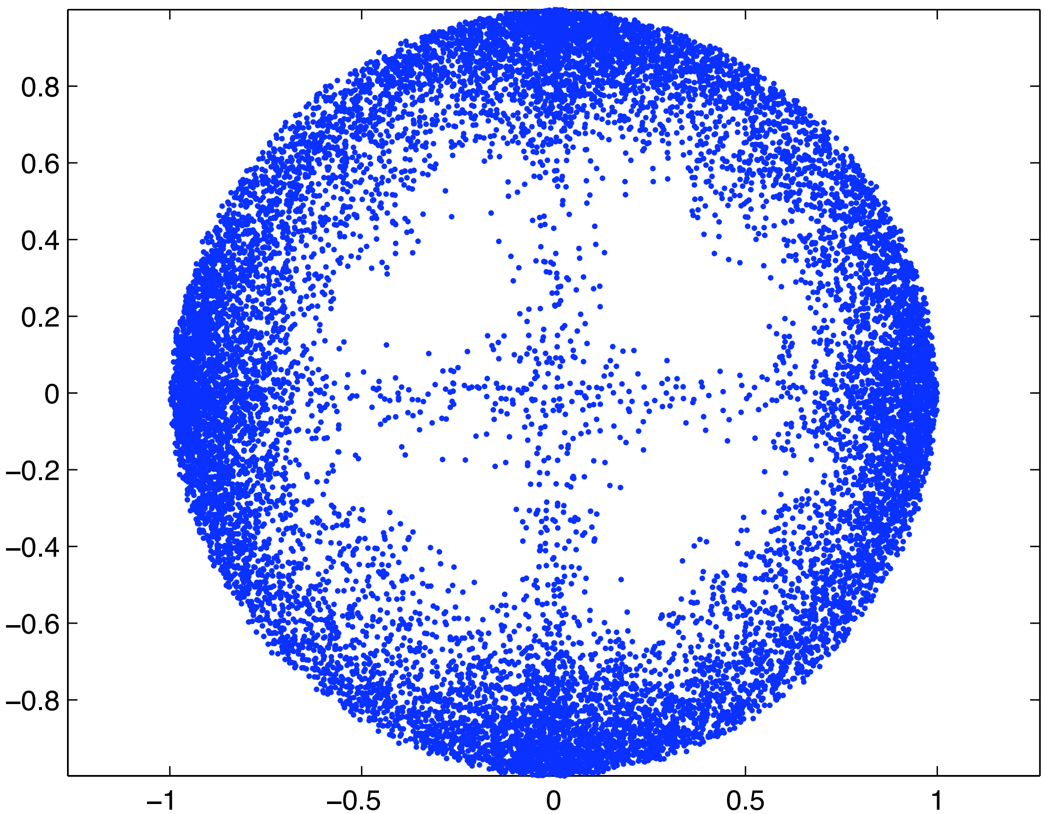
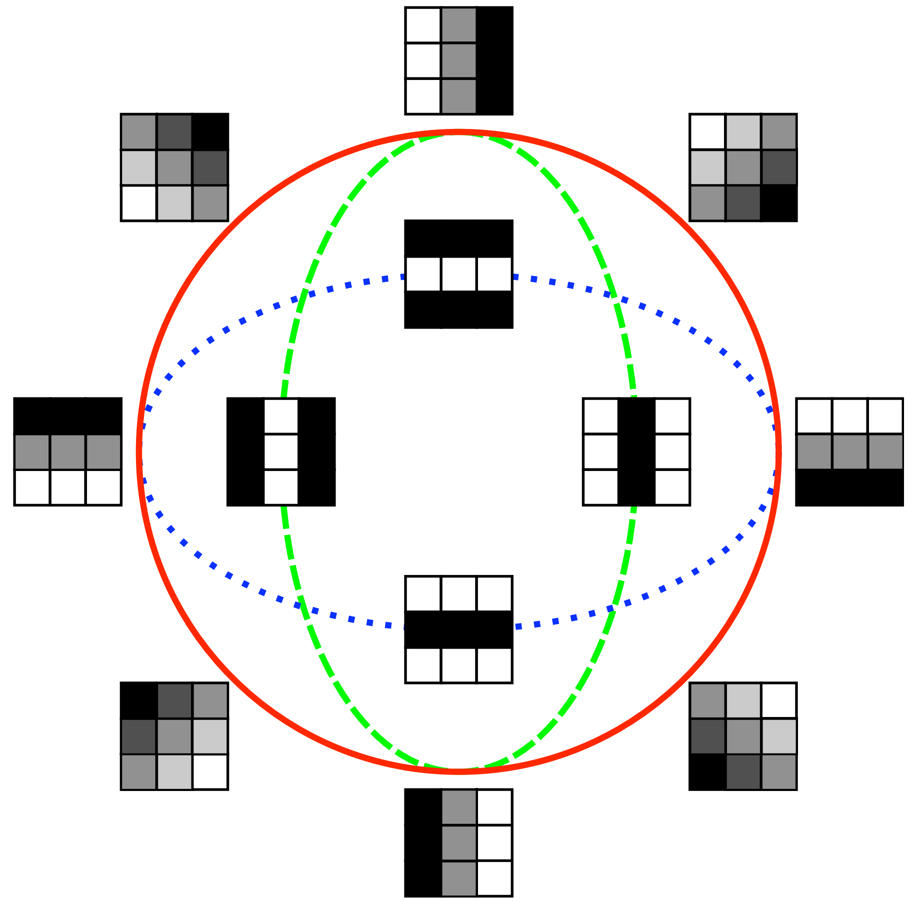
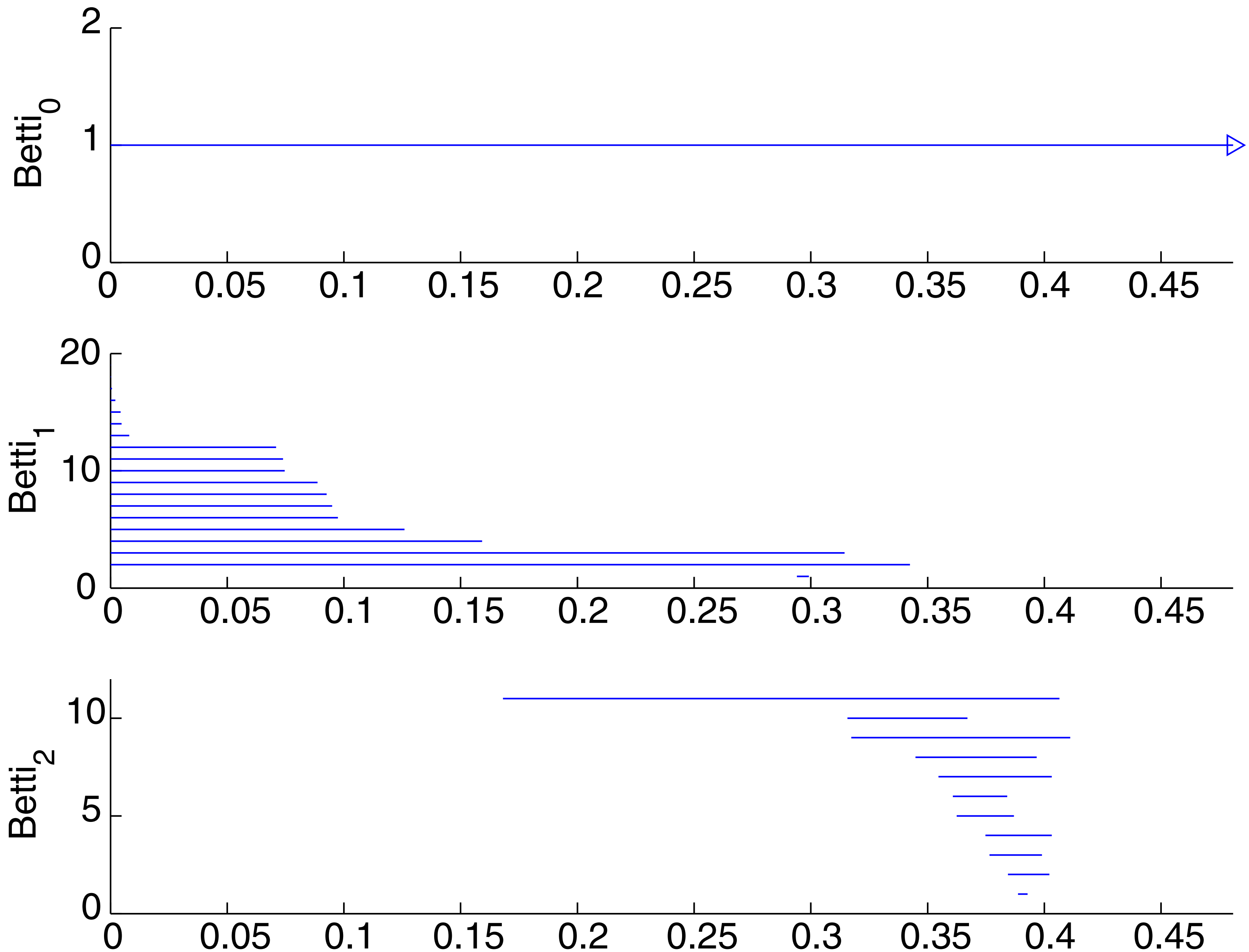




Interpretation: nature prefers horizontal and vertical directions

Persistent homology applied to data

3. Densest patches according to a local estimate



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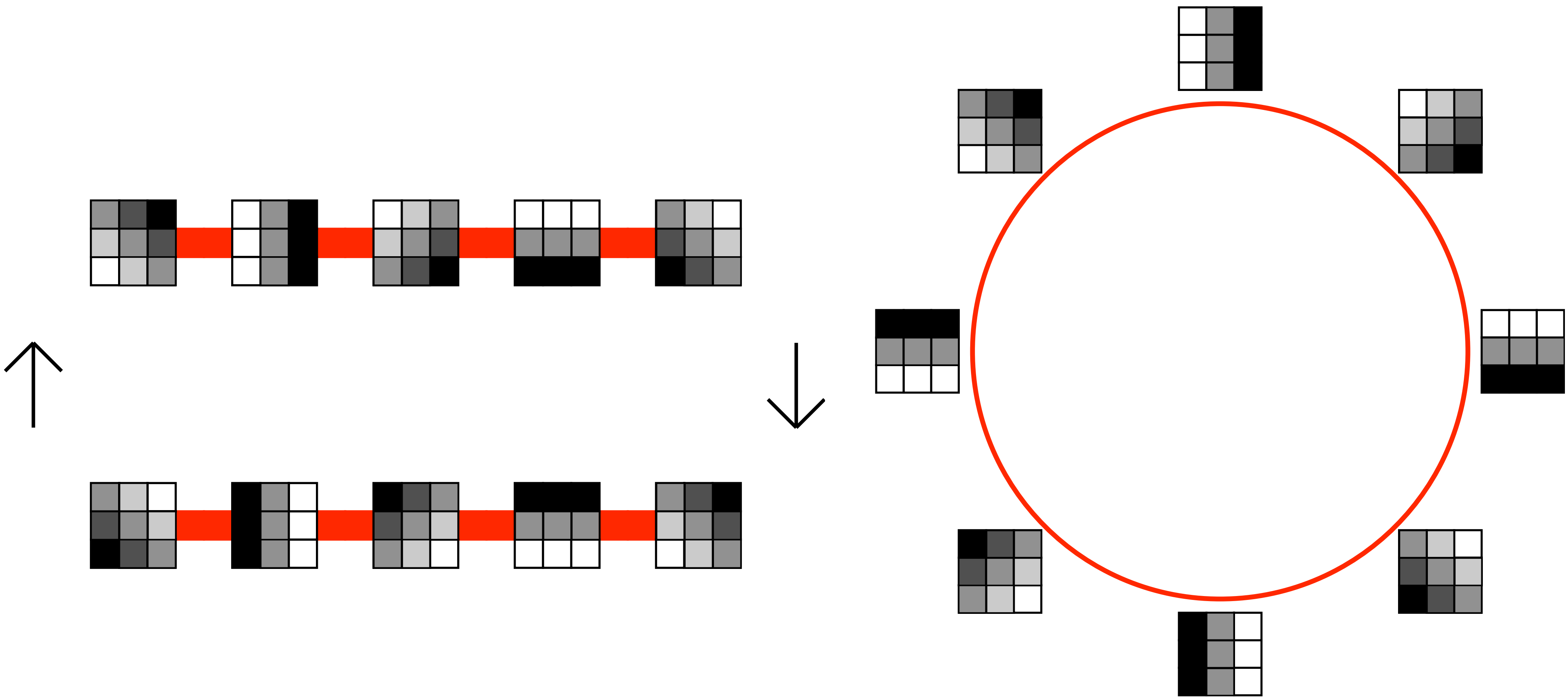

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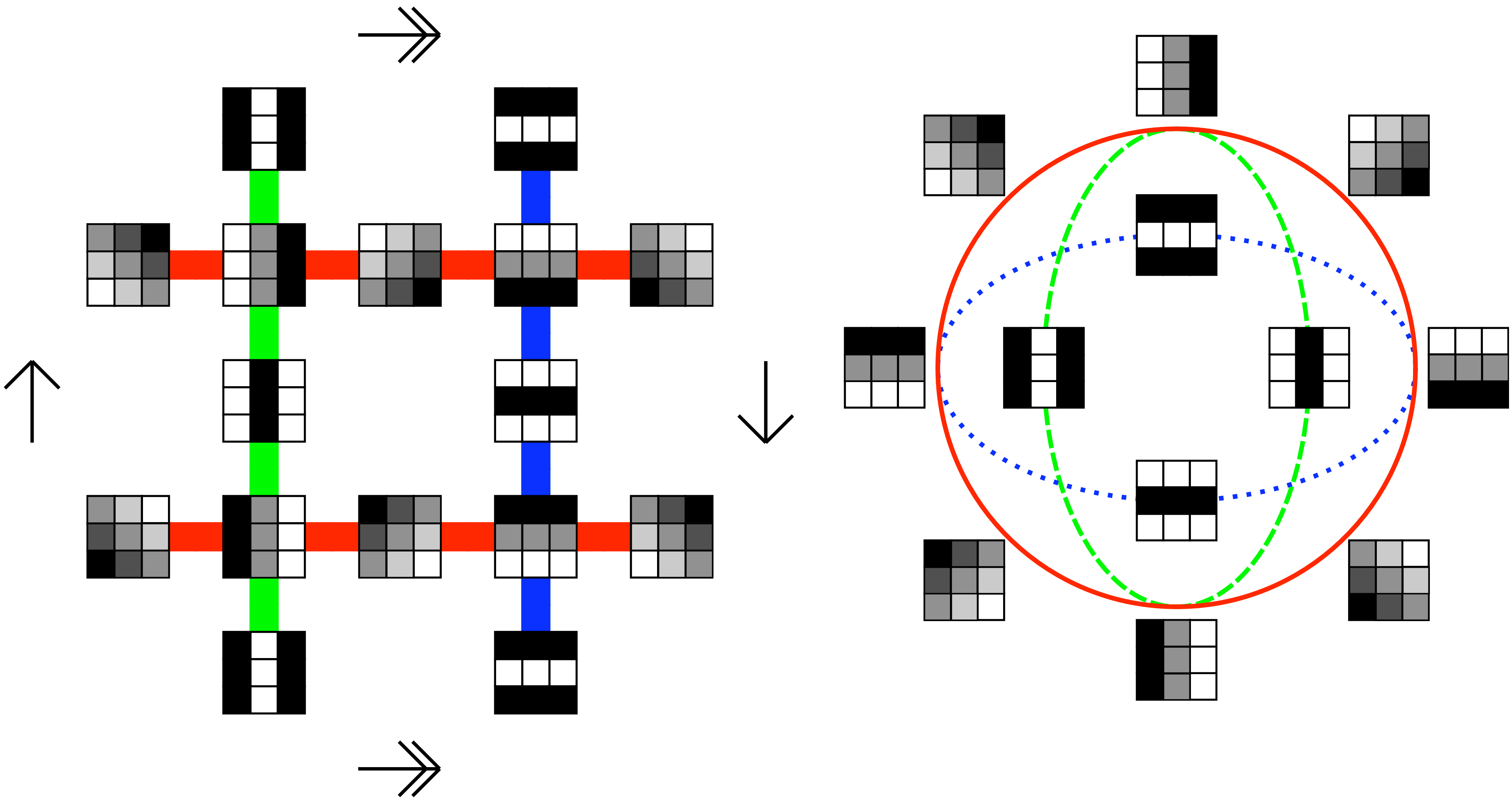

Persistent homology applied to data

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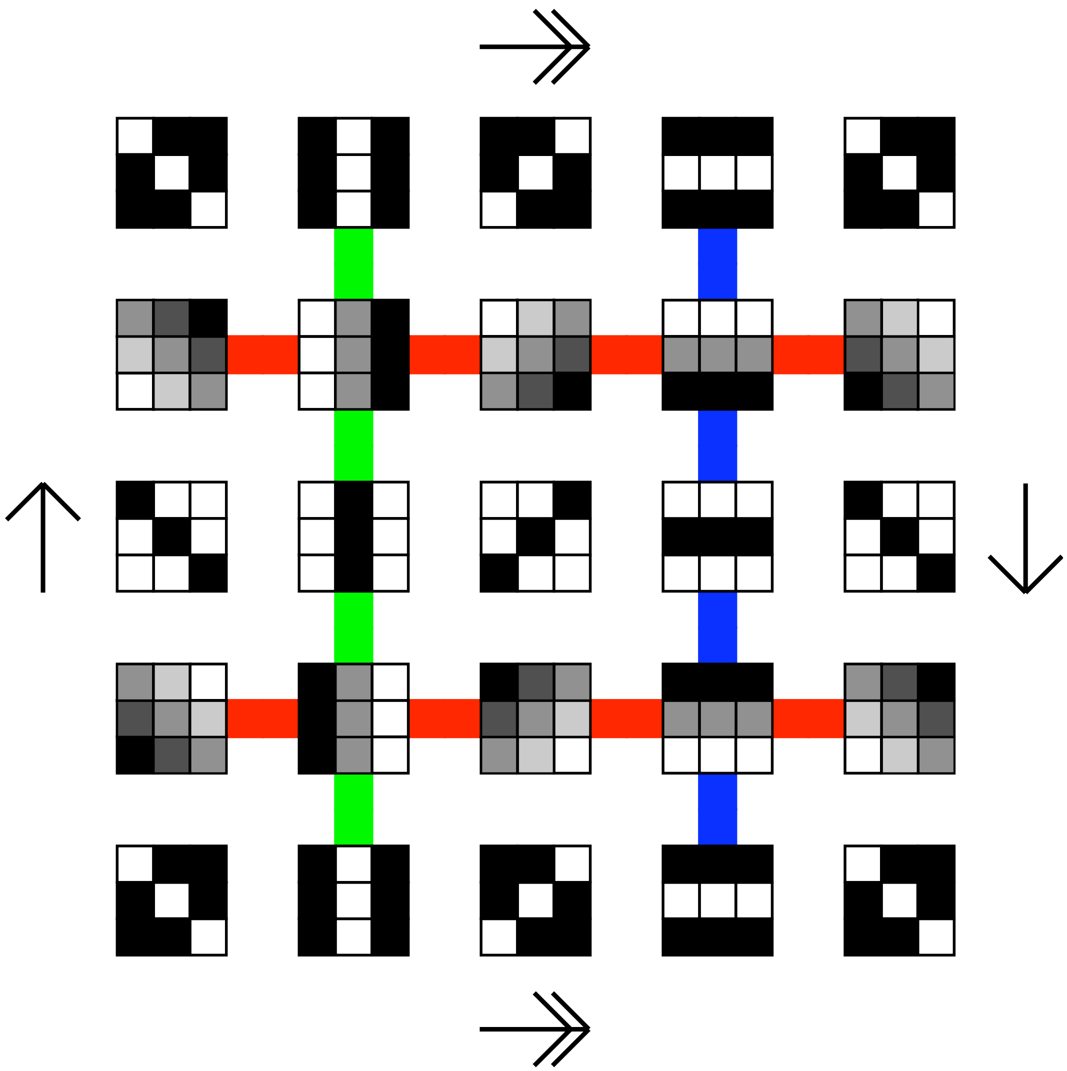

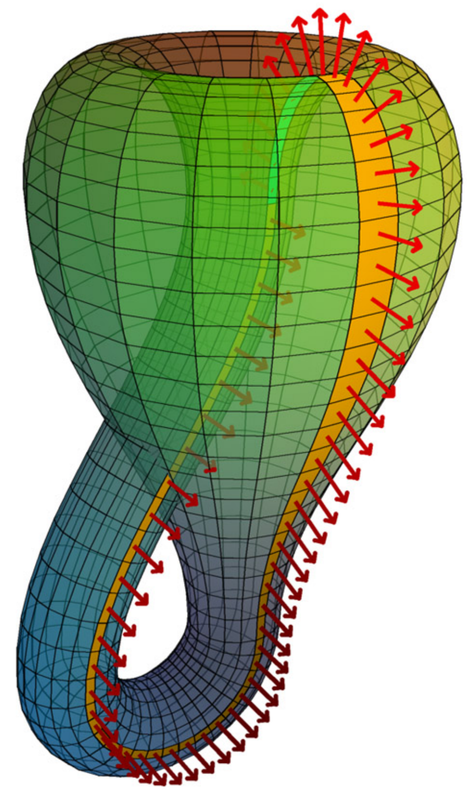

Image credit: https://plus.maths.org/content/imaging-maths-inside-klein-bottle

Interpretation: nature prefers linear and quadratic patches at all angles

Persistent homology applied to data

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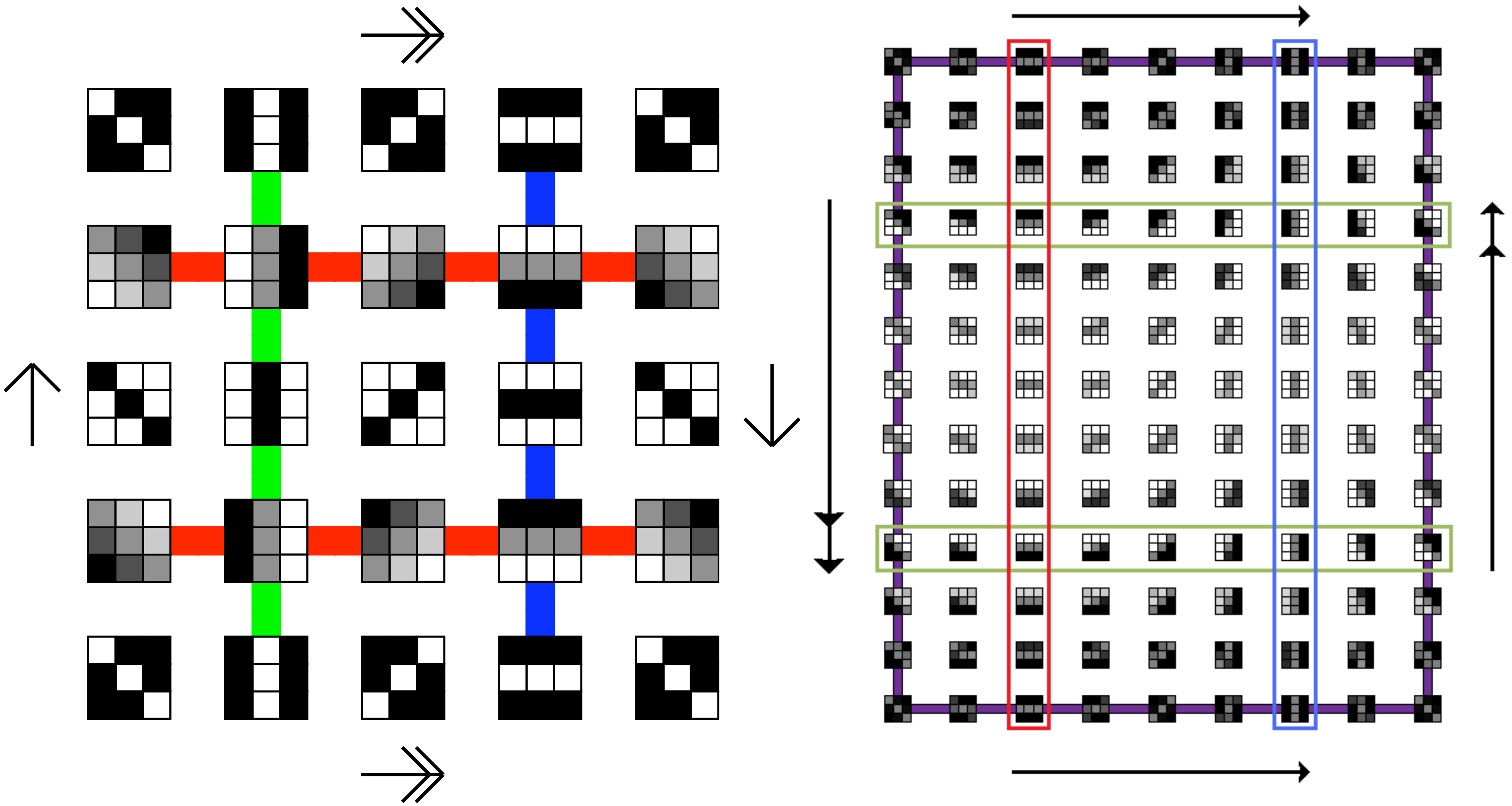

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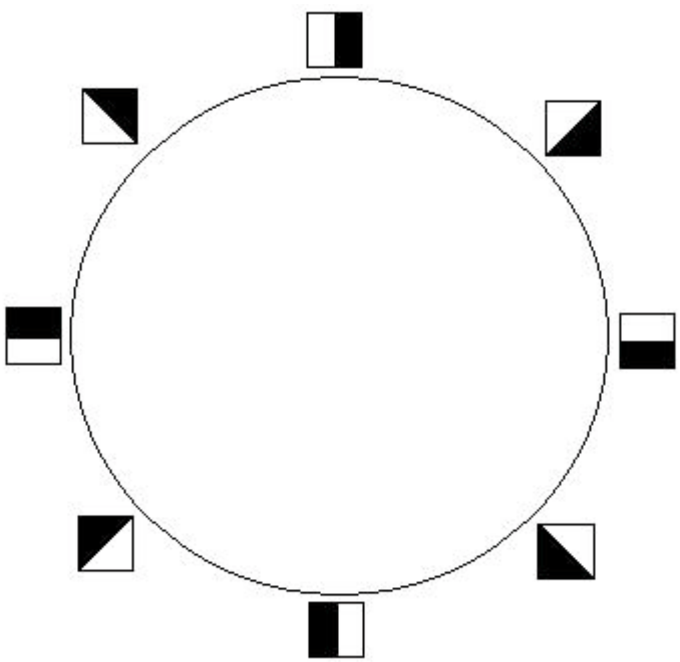
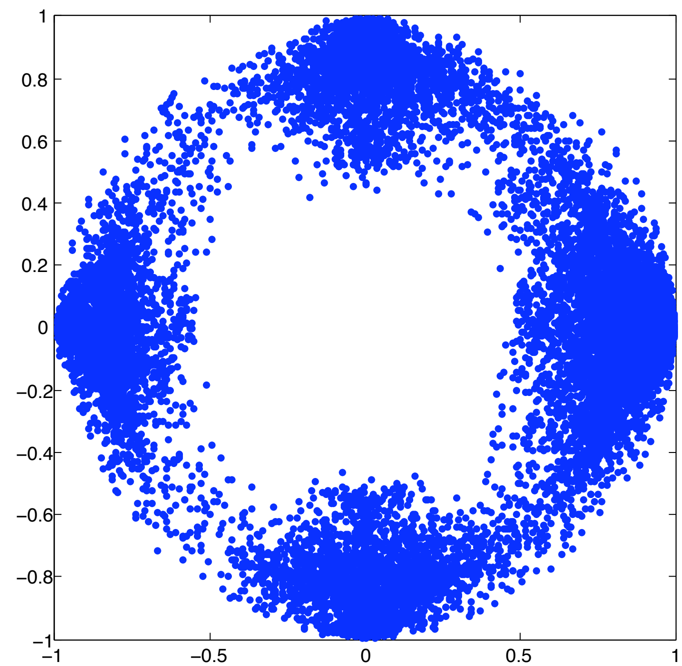
Persistent homology applied to data

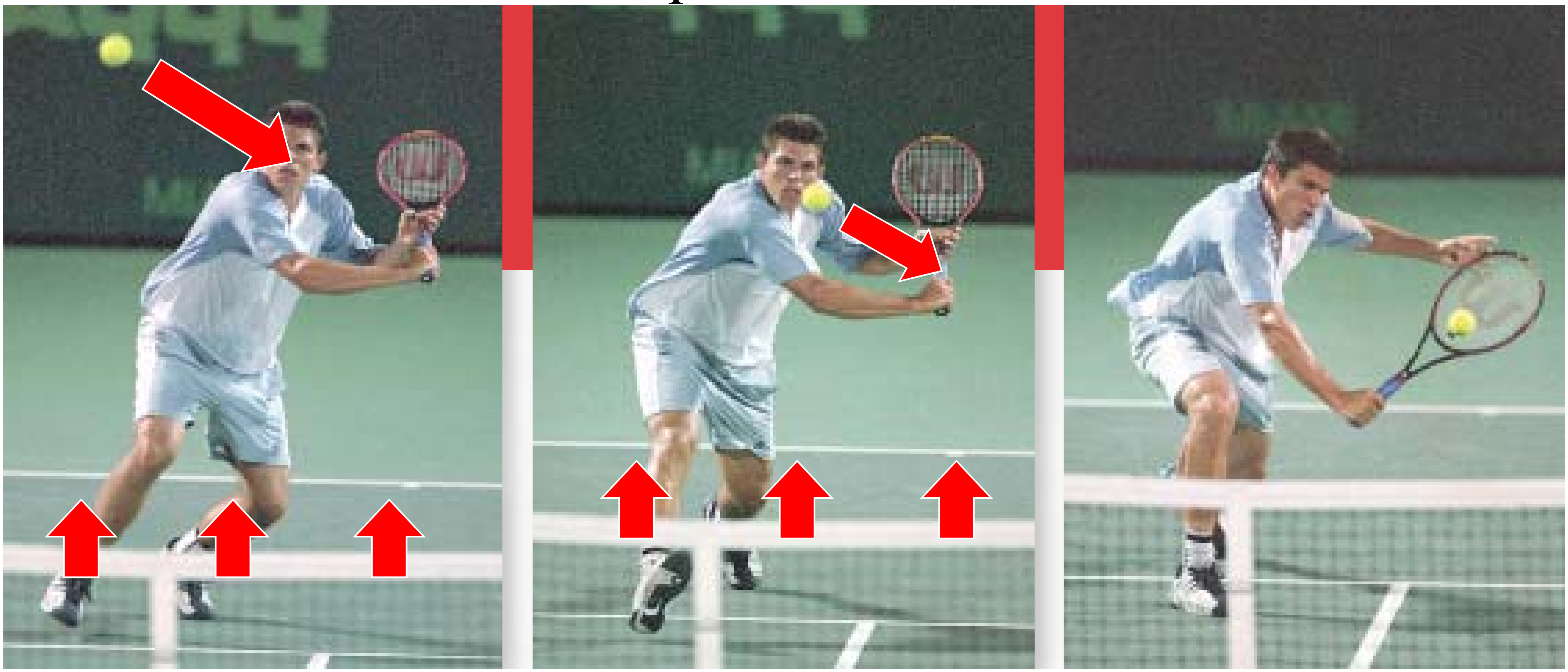
Range Images





Persistent homology applied to data

Optical Flow



Optical flow is a vector field representing the apparent motion (or projected motion) in a video.

On the nonlinear statistics of optical flow by HA, Johnathan Bush, Brittany Carr, Lara Kassab, and Joshua Mirth, 2018

Persistent homology applied to data

Optical Flow

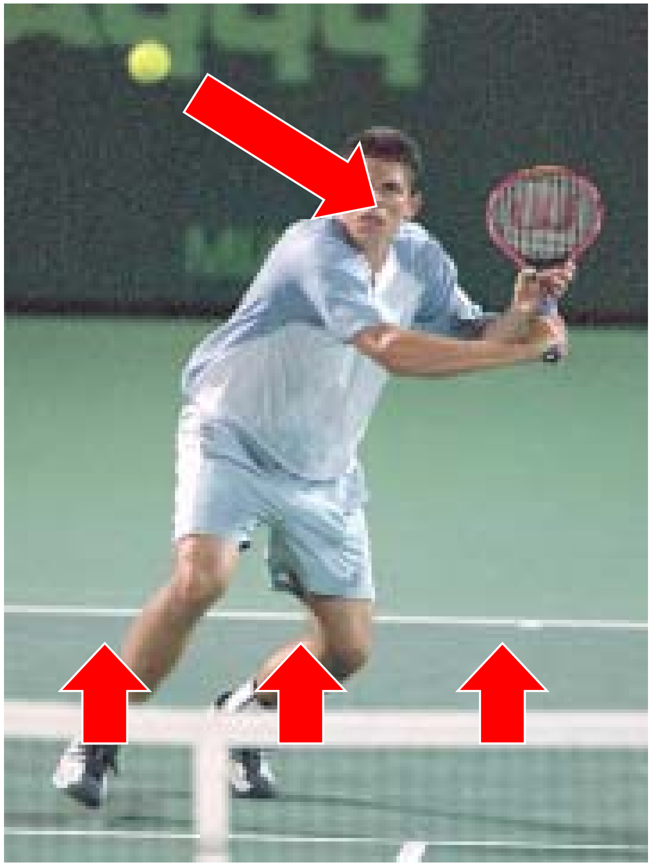
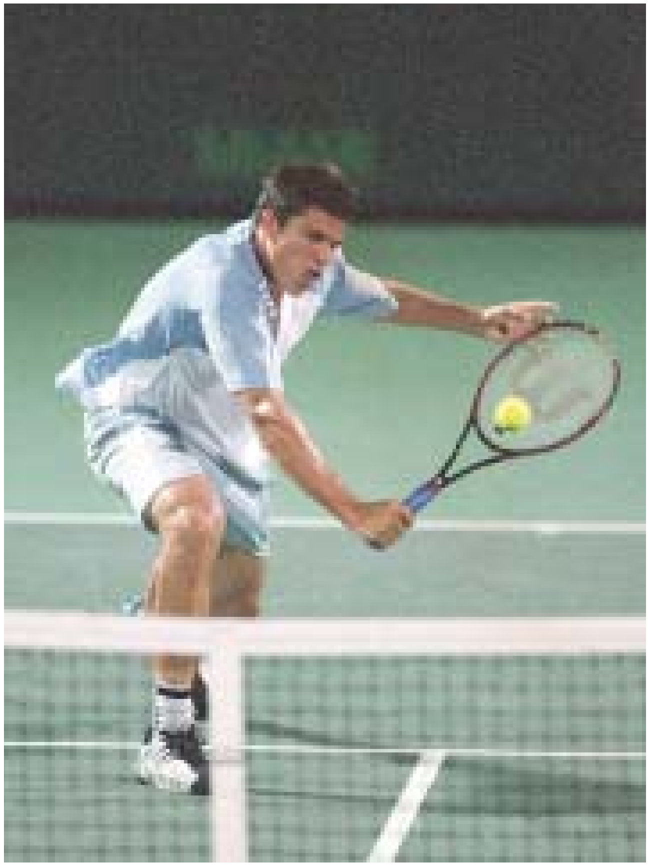


Image: Wikipedia



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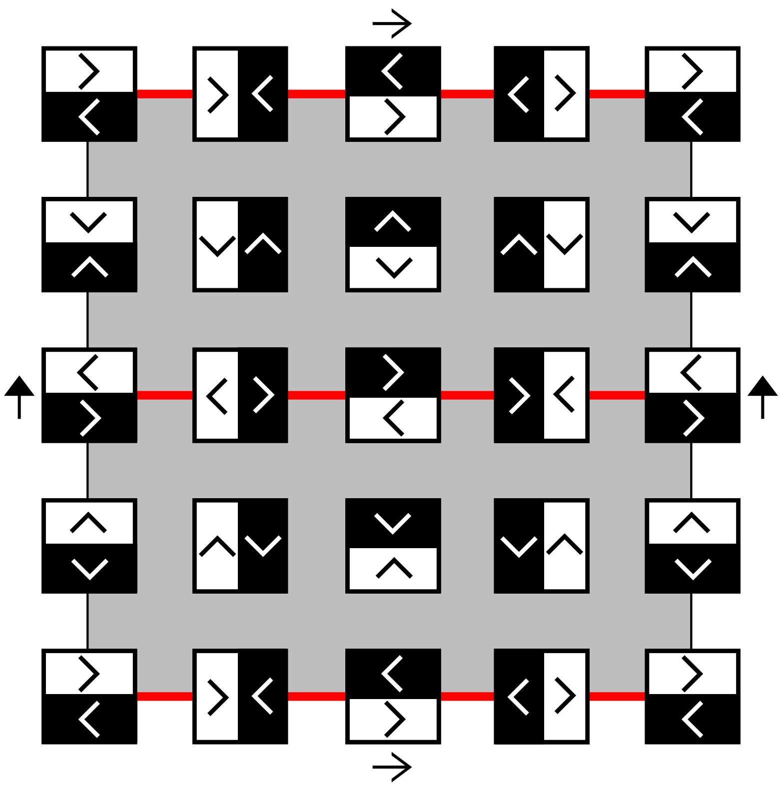

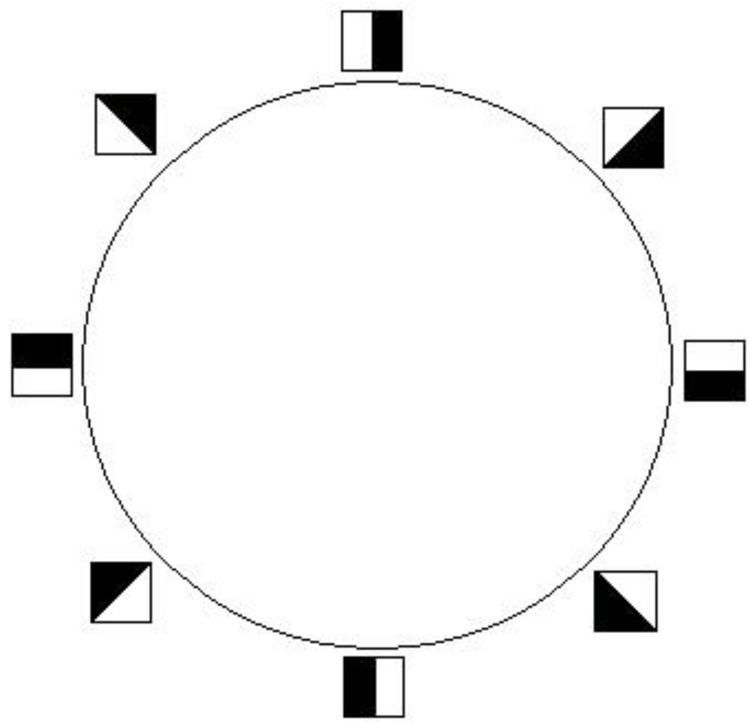

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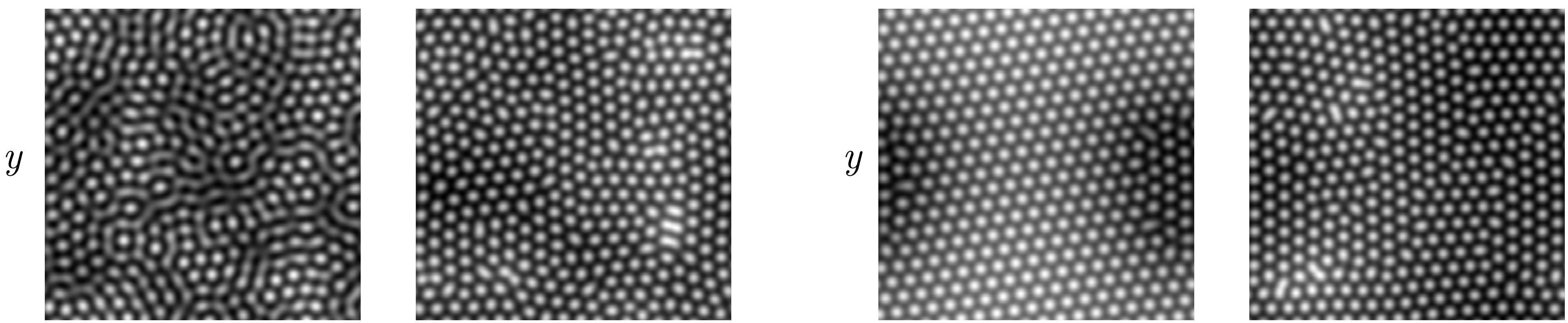
Why is applied topology popular when few datasets have Klein bottles?

- Many datasets have clusters & flares (as in the diabetes example)
- Motivates interesting questions in many pure disciplines: mathematics, computer science (computational geometry), statistics
- Interest from domain experts in biology, neuroscience, computer vision, dynamical systems, sensor networks, ...
- Materials science, pattern formation
- Machine learning: small features matter
- Agent-based modeling (swarming)



Measures of Order for nearly hexagonal lattices by Francis Motta, Rachel Neville, Patrick Shipman, Daniel Pearson, and Mark Bradley, 2018.

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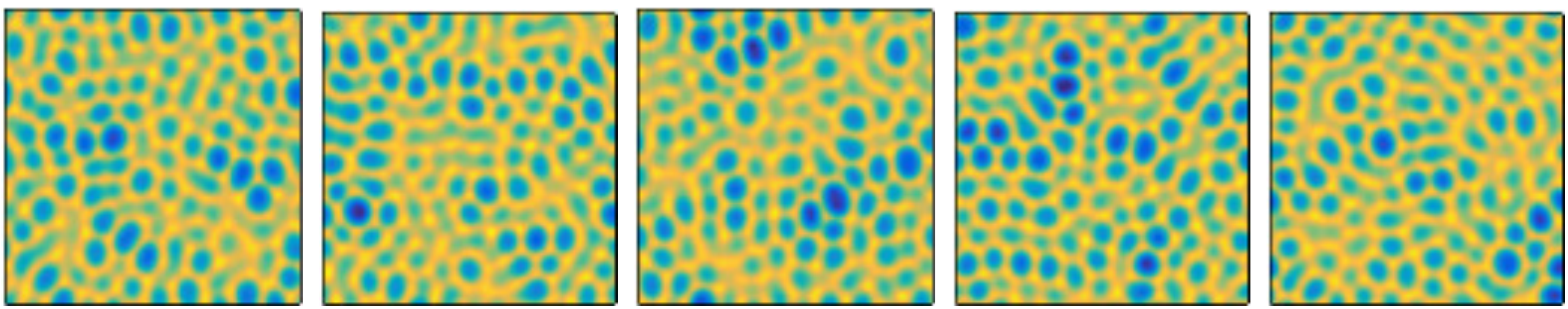

Answer: (from left) $r = 1.75, 2, 1.75, 2, 2.$

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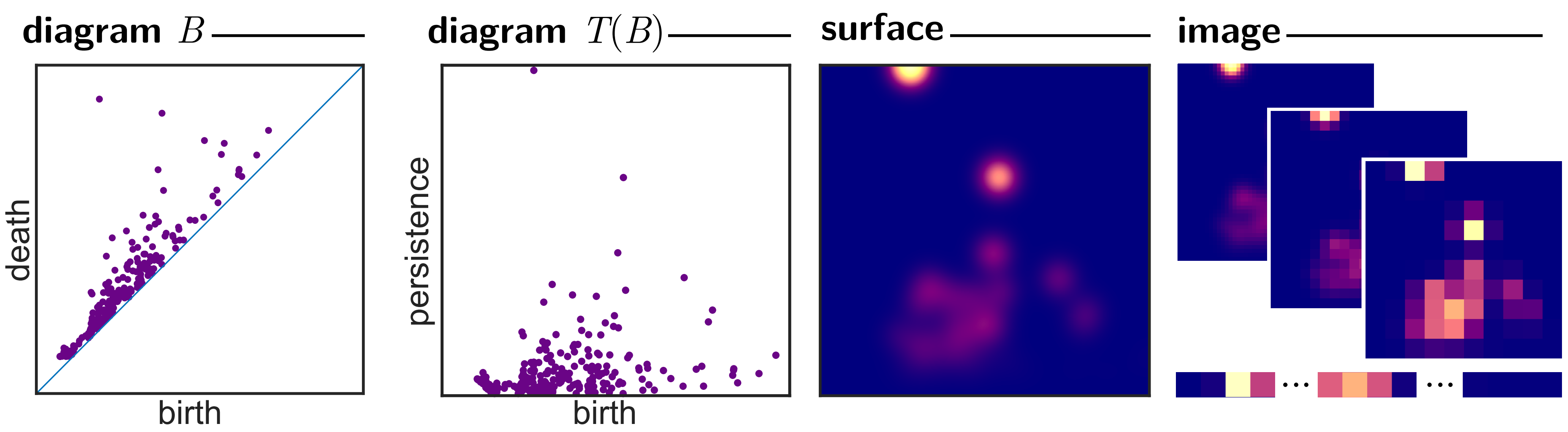

Agent-Based Modeling

Collective phenomenon, self-organization

Dutch Starling murmuration
filmed by Roald van Stijn

https://www.youtube.com/watch?v=YjDYE5CUb7Q

The following slides and images are largely from: Chad Topaz, Lu Xuan, Lori Ziegelmeier

Biological Aggregations

- A system in which agents interact with each other
- Formed through social interaction and coordinated behaviors like attraction, repulsion, and/or alignment
- Self-organization

Video retrieved from [1], images from [4] [3] [5]

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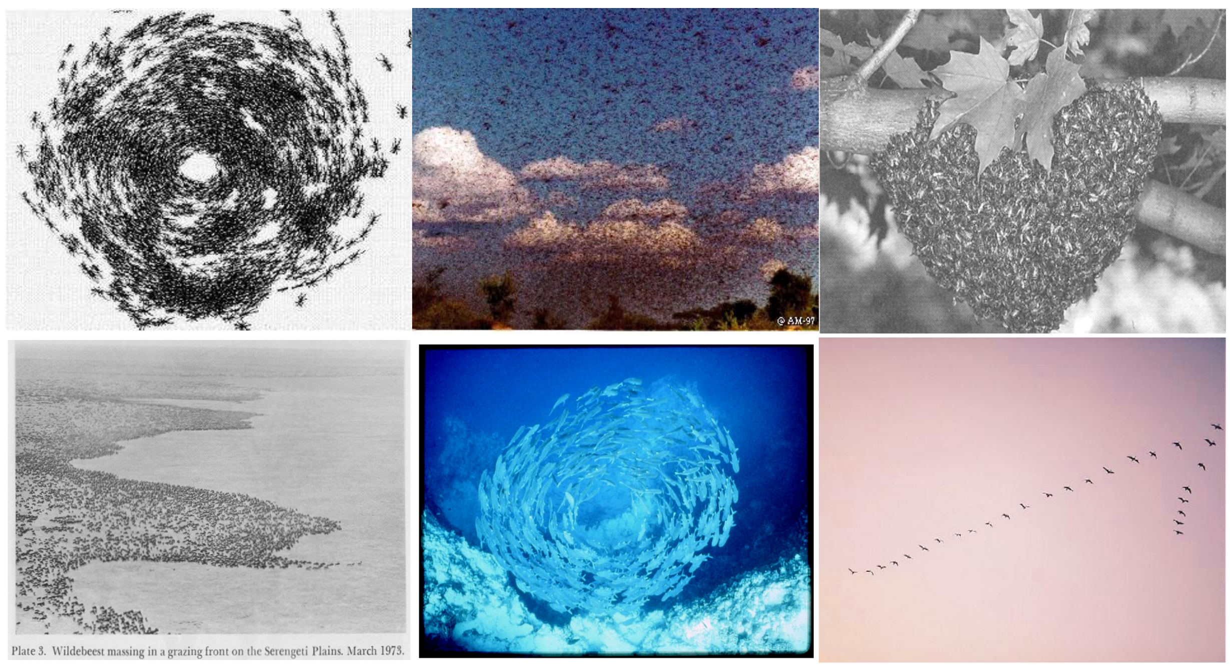
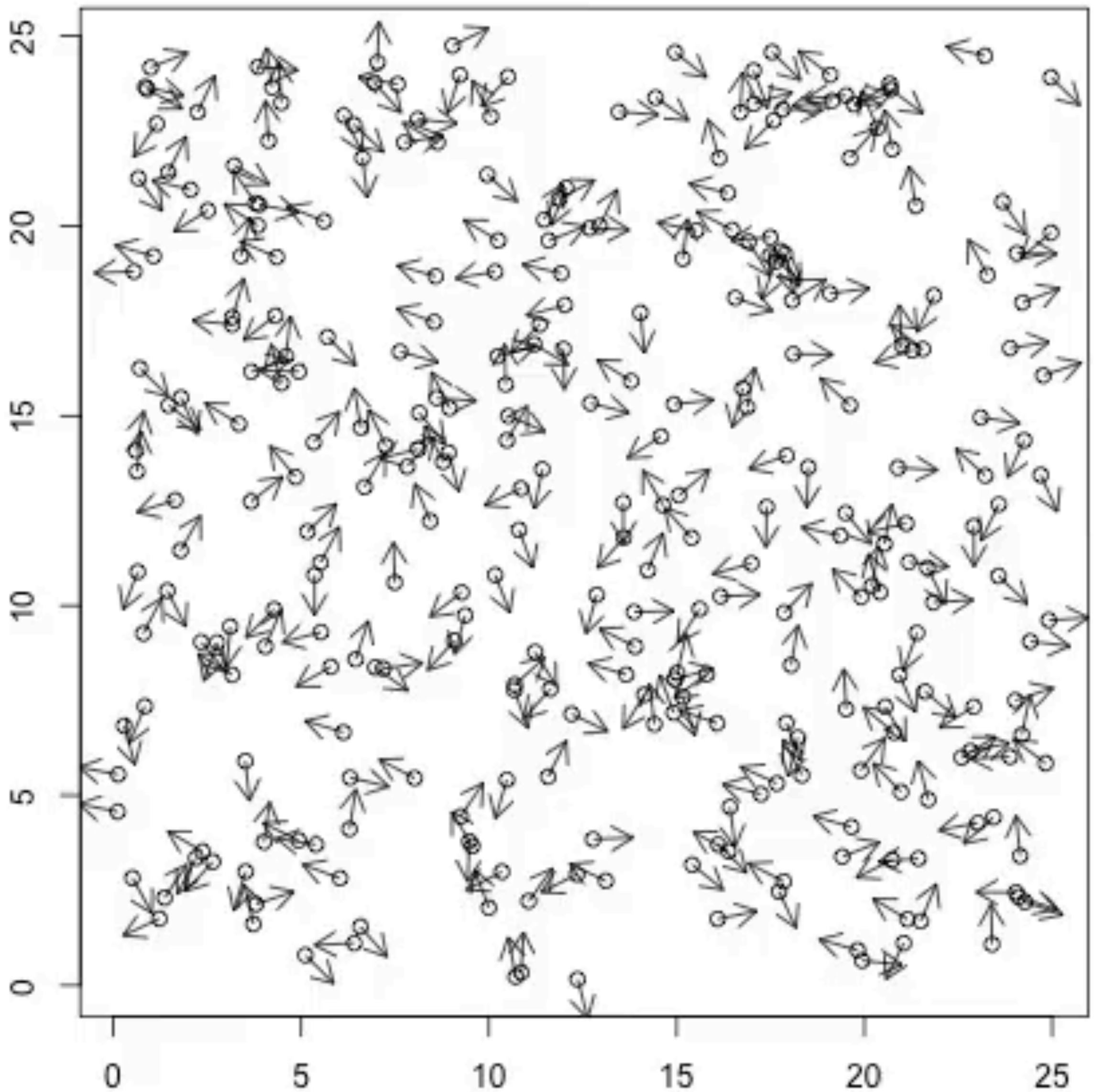
BIOLOGICAL AGGREGATIONS



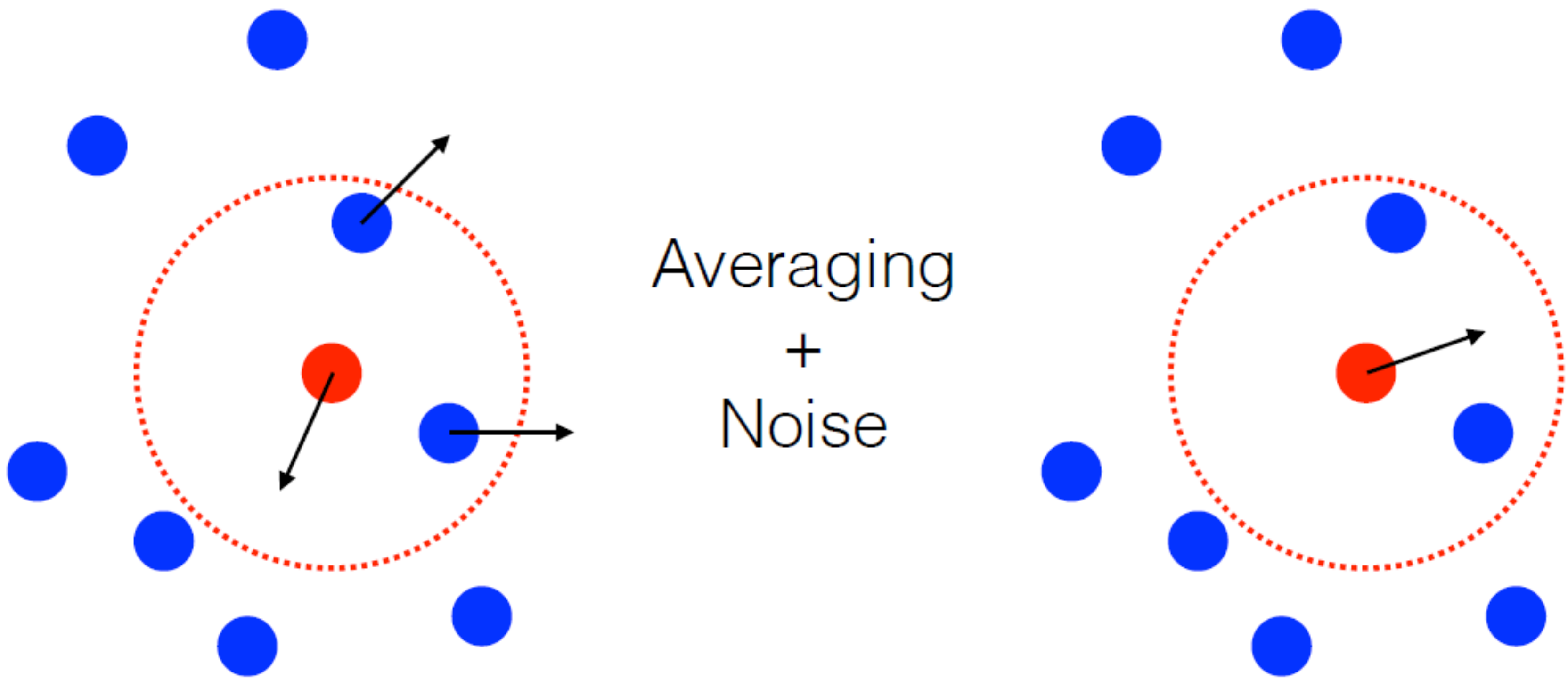
In many natural systems, particles, organisms, or agents interact locally according to rules that produce aggregate behavior.

Vicsek Model



Topological data analysis of biological aggregation models
by Chad M Topaz, Lori Ziegelmeier, Tom Halverson, 2015.

Vicsek Model



$$\theta_i(t+\Delta t) = \frac{1}{N}\Big(\sum_{|\mathbf{x}_i-\mathbf{x}_j|\leq R} \theta_j(t)\Big) + U\left(-\frac{\eta}{2}, \frac{\eta}{2}\right)$$

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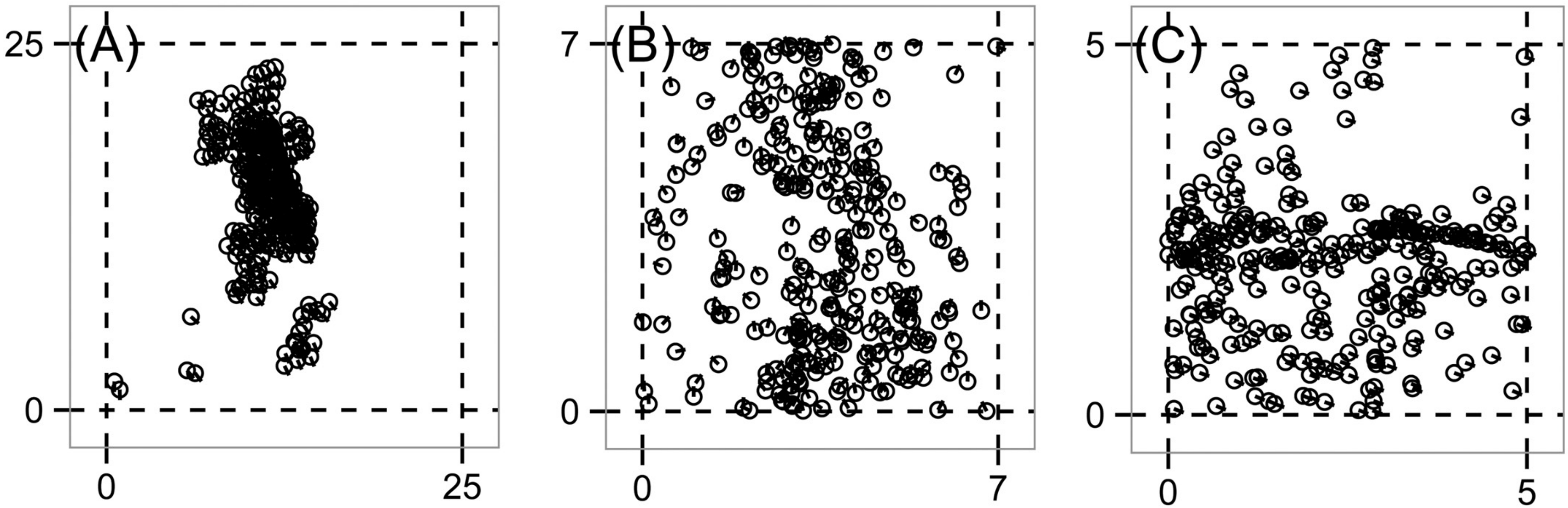

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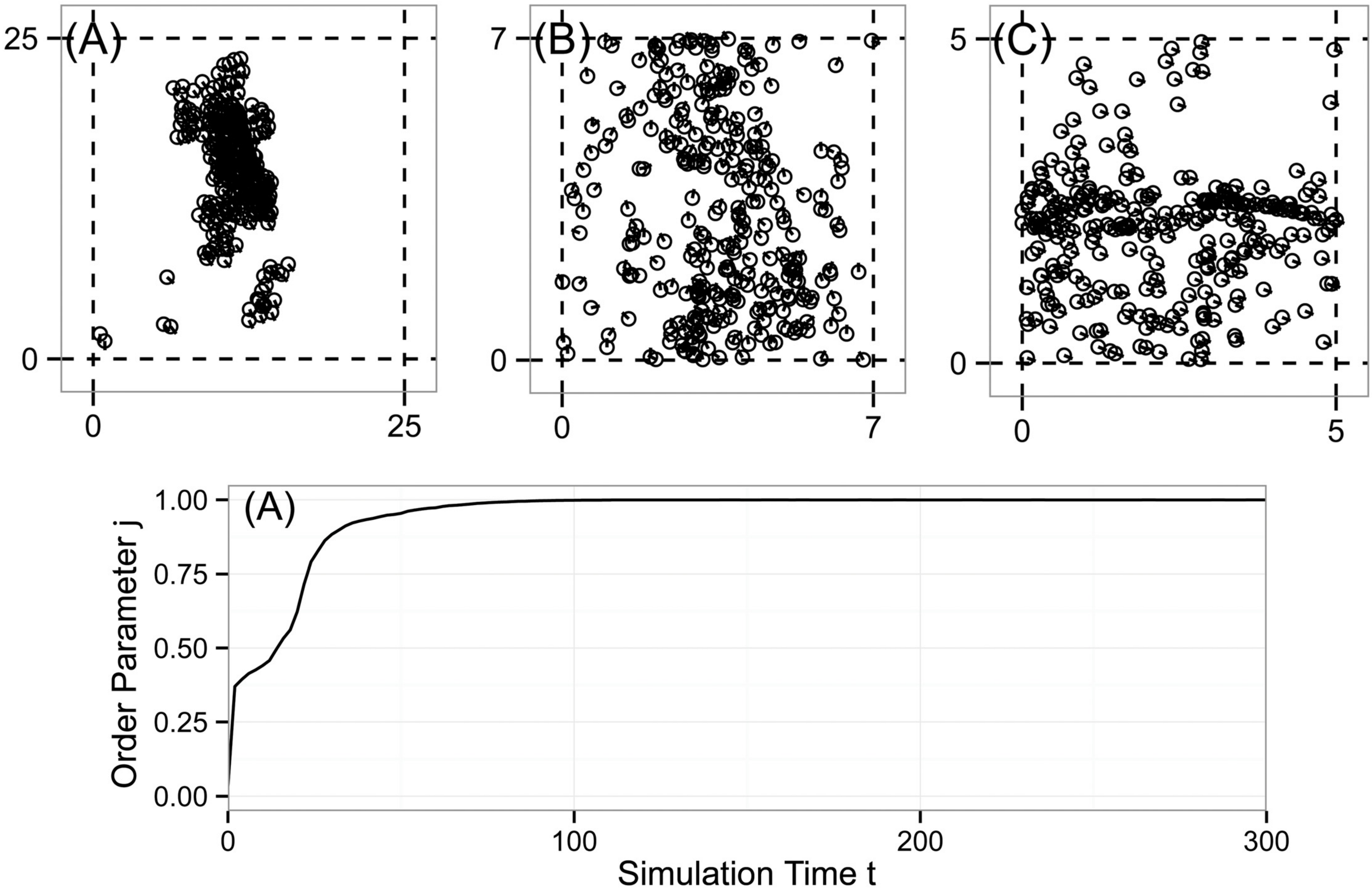

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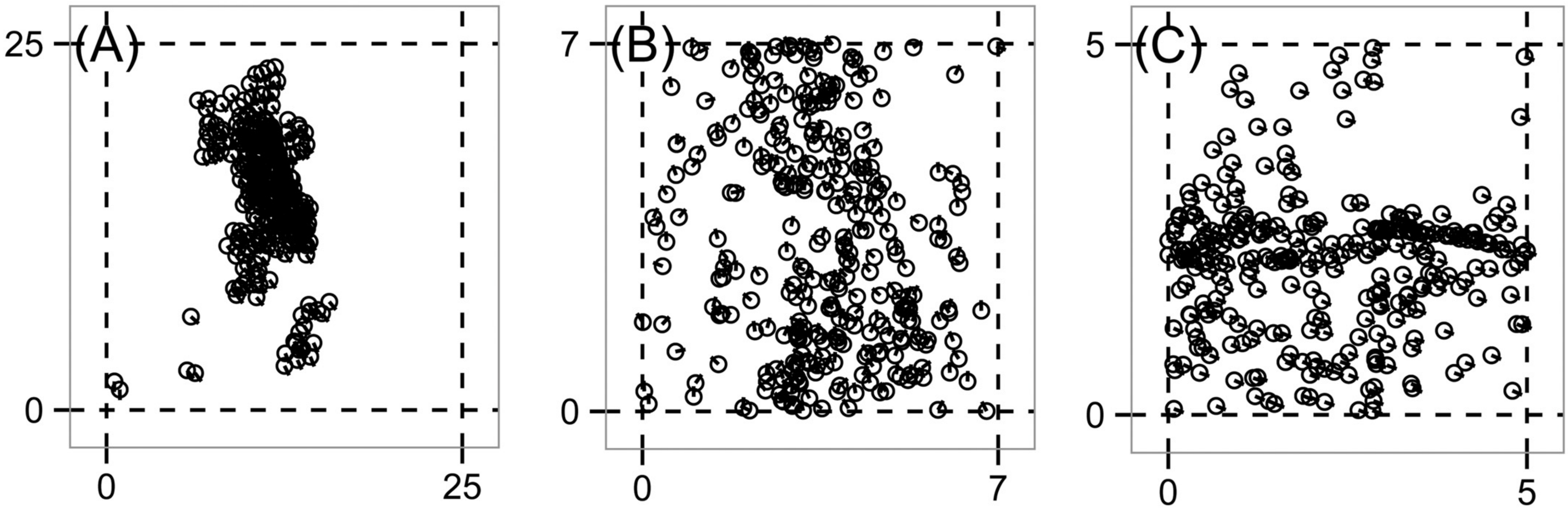
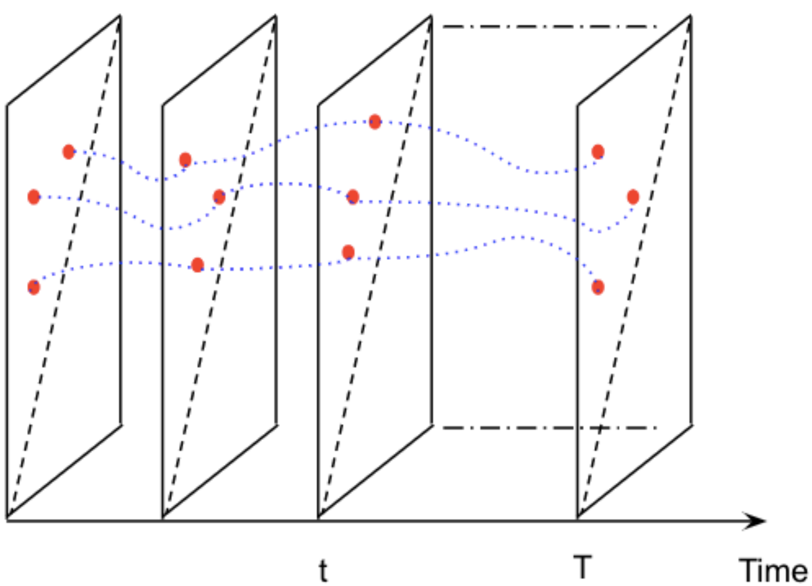
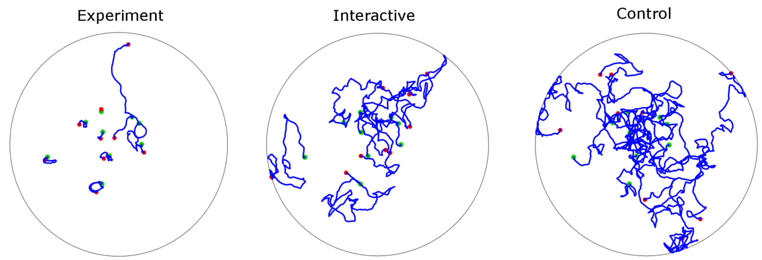




- Persistent homology is a reasonable summary of local geometry and global topology
- Can you reject the null hypothesis that your model fits the data?

Aphid Motion

ASSESSING MODEL VALIDITY



Summaries of statistical tests comparing models of aphid motion using order parameters.

Exp	P D	P $R_{95\%}$	M_{ang} D	M_{ang} $R_{95\%}$	M_{abs} D	M_{abs} $R_{95\%}$	d_a D	d_a $R_{95\%}$	$Mov_{\%}$ D	$Mov_{\%}$ $R_{95\%}$
1	0.17	0.13	1.50	0.16	-2.00	0.15	0.26	0.05	20.2	0.40
2	-0.25	0.10	-0.97	0.11	-1.66	0.09	0.25	0.02	0.78	0.24
3	1.09	0.10	0.40	0.11	-1.71	0.10	0.54	0.04	25.1	0.26
4	1.70	0.08	-0.78	0.09	-1.24	0.07	0.16	0.02	1.68	0.21
5	-2.22	0.09	-0.63	0.09	-1.10	0.07	0.31	0.03	7.68	0.24
6	1.04	0.09	-0.49	0.10	-1.68	0.09	0.53	0.04	24.0	0.26
7	-0.97	0.11	-1.28	0.13	-1.99	0.11	0.57	0.04	23.9	0.29
8	-3.30	0.25	-2.11	0.22	-2.87	0.26	0.57	0.25	13.2	0.63
9	-4.69	0.25	-3.14	0.22	-2.81	0.20	0.48	0.15	11.7	0.34

Summaries of statistical tests comparing models of aphid motion using topology.

Exp	$b_{0(pos)}$ D^*	$b_{0(pos)}$ $R^*_{95\%}$	$b_{1(pos)}$ D^*	$b_{1(pos)}$ $R^*_{95\%}$	$b_{0(posvel)}$ D^*	$b_{0(posvel)}$ $R^*_{95\%}$	$b_{1(posvel)}$ D^*	$b_{1(posvel)}$ $R^*_{95\%}$
1	1147	38.73	135.1	5.149	831.9	42.81	124.2	5.640
2	1703	23.98	297.0	6.500	858.9	16.88	261.1	7.258
3	1893	37.36	290.4	6.836	1347	35.60	271.3	7.218
4	1825	18.36	359.0	5.741	552.3	19.04	332.5	6.301
5	1370	15.81	249.2	6.039	270.8	15.25	225.1	6.401
6	2086	40.53	322.9	7.061	1484	37.35	307.7	7.116
7	1747	34.32	264.8	6.235	1289	33.69	247.5	6.69
8	298.5	20.81	10.28	1.920	147.9	19.81	3.573	1.641
9	467.5	19.66	22.34	2.148	245.3	21.10	10.93	2.067

(Ulmer, Z., Topaz 2018) Assessing Biological Models Using Topological Data Analysis

Aphid Motion

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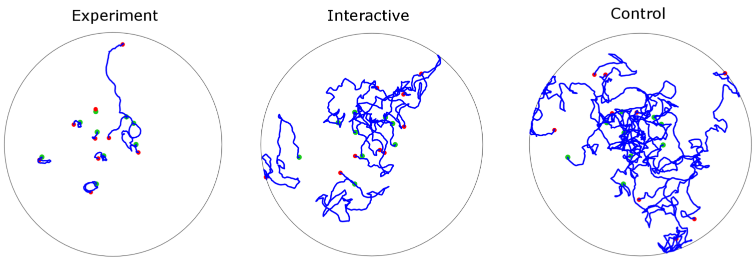

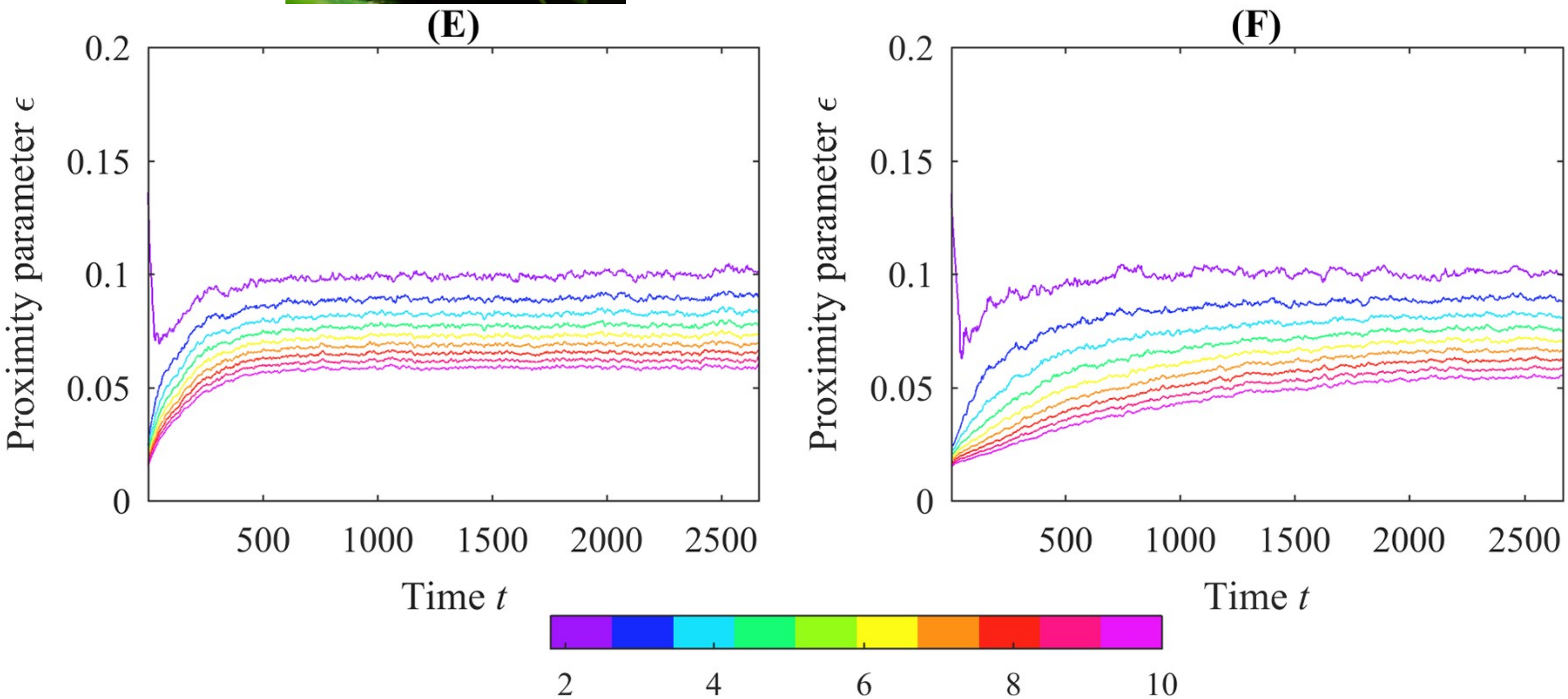

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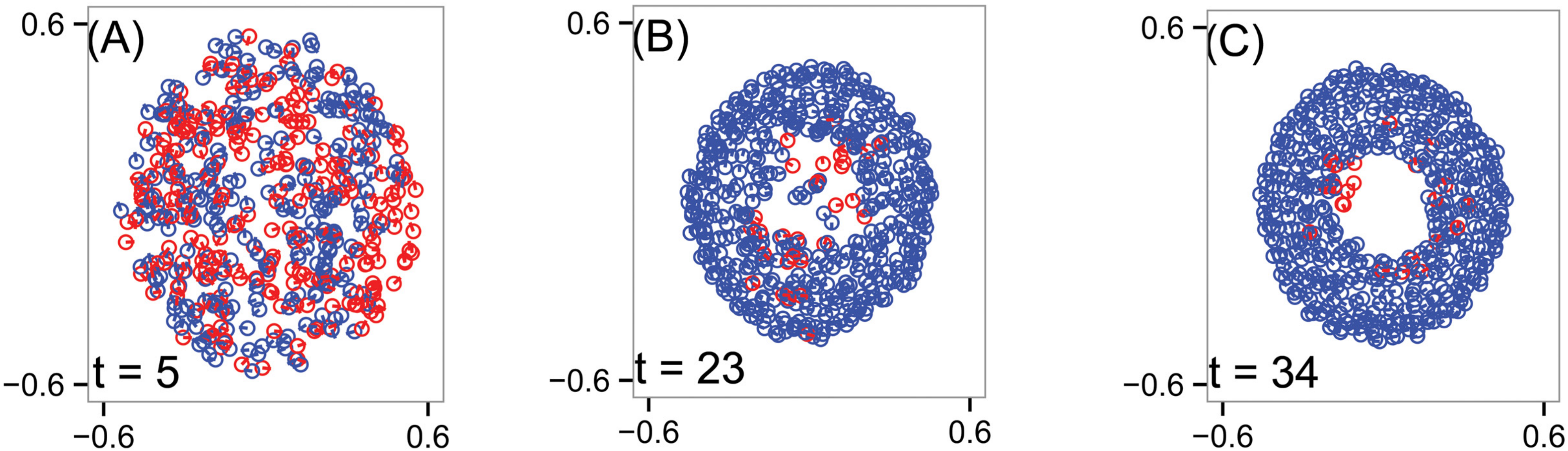
D'Orsogna Model



Repulsion on short length scales L_r, attraction on large length scales L_a

$$\begin{aligned}
\dot{\mathbf{x}}_i &= \mathbf{v}_i, \\
m\dot{\mathbf{v}}_i &= (\alpha - \beta|\mathbf{v}_i|^2)\mathbf{v}_i - \nabla_i Q_i, \\
Q_i &= \sum_{j \neq i} C_r e^{-|\mathbf{x}_i - \mathbf{x}_j|/L_r} - C_a e^{-|\mathbf{x}_i - \mathbf{x}_j|/L_a}
\end{aligned}$$

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Applied Mathematical Modeling with Topological Techniques

Aug 5 - 9, 2019

Organizing Committee

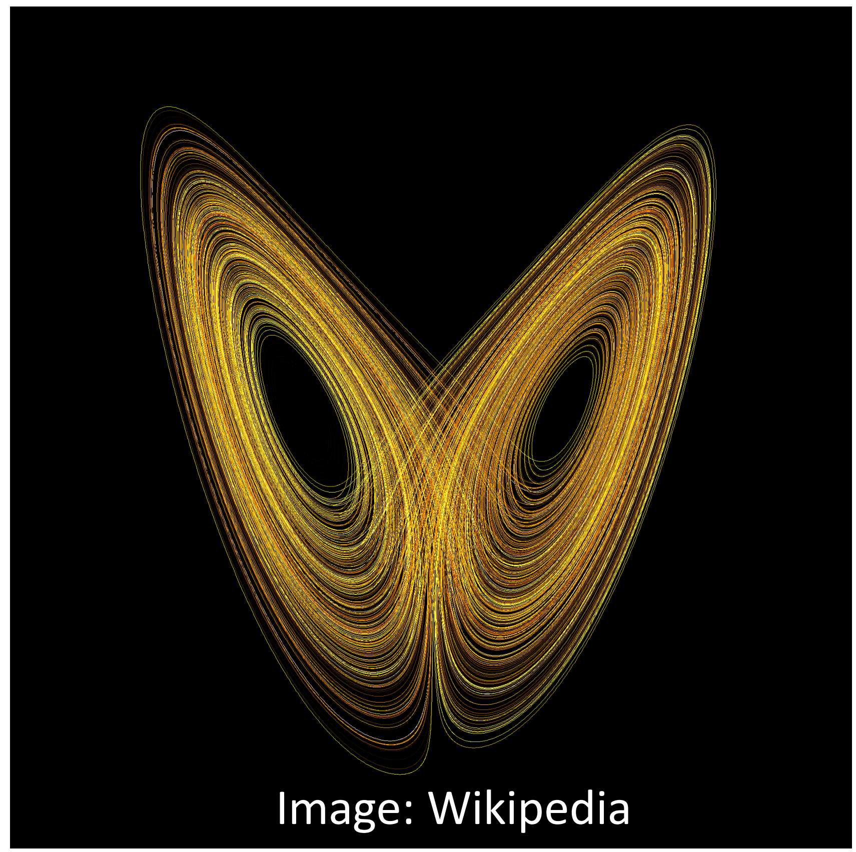
- Henry Adams
  Colorado State University
- Jose Perea
  Michigan State University
- Maria D'Orsogna
  California State University, Northridge
- Chad Topaz
  Williams College
- Rachel Neville
  University of Arizona

Takens' Theorem

Roughly speaking: Let M be a d-dimensional compact manifold, let $\phi\colon M \to M$ be a flow, and let $f: M \to \mathbb{R}$ be a measurement. Then *generically*,

$$m \mapsto \big(f(m), f(\phi(m), f(\phi^2(m), \ldots, f(\phi^{2d}(m))\big)$$

is an embedding $M \hookrightarrow \mathbb{R}^{2d+1}$.



Detecting strange attractors in turbulence by Floris Takens, 1982

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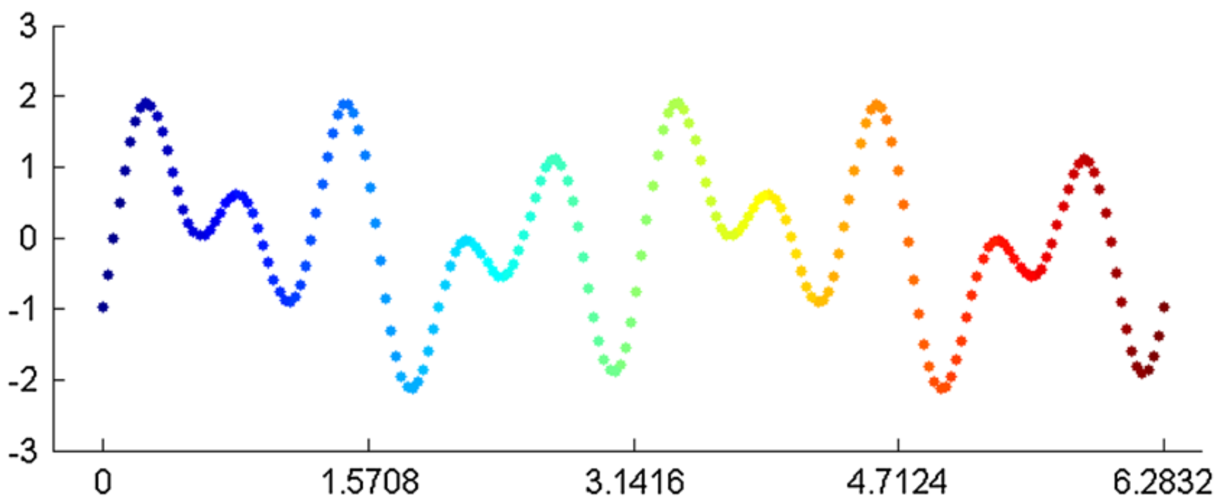

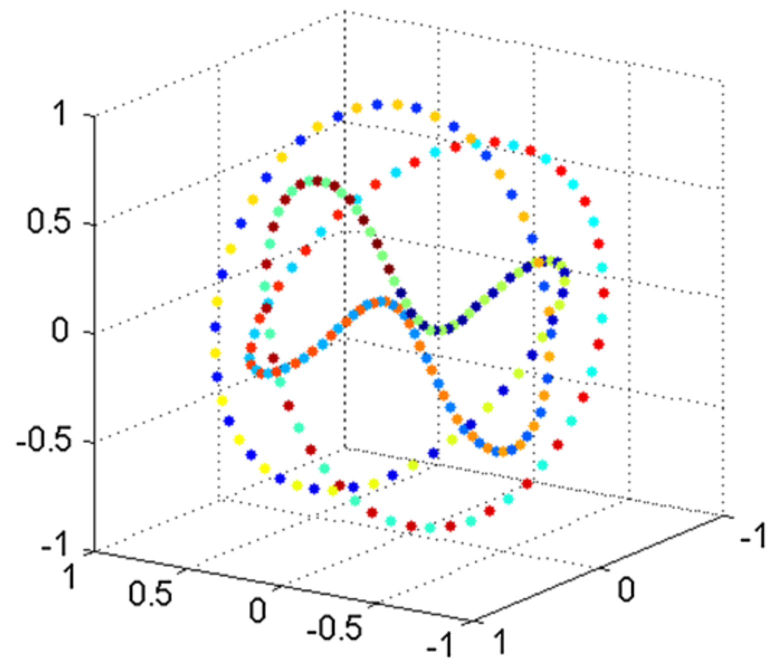

Sliding windows and persistence: An application of topological methods to signal analysis by Jose Perea and John Harer, 2014

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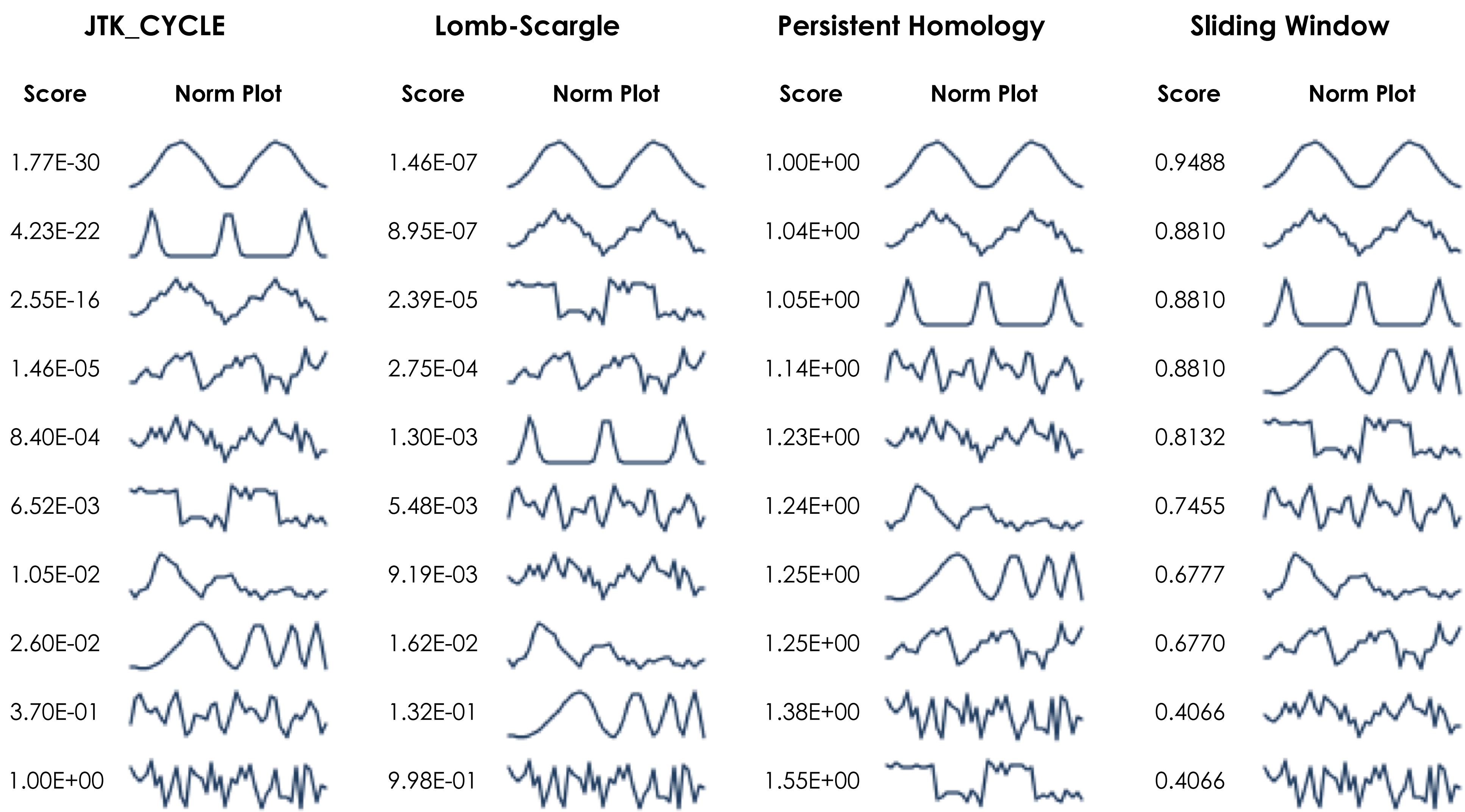

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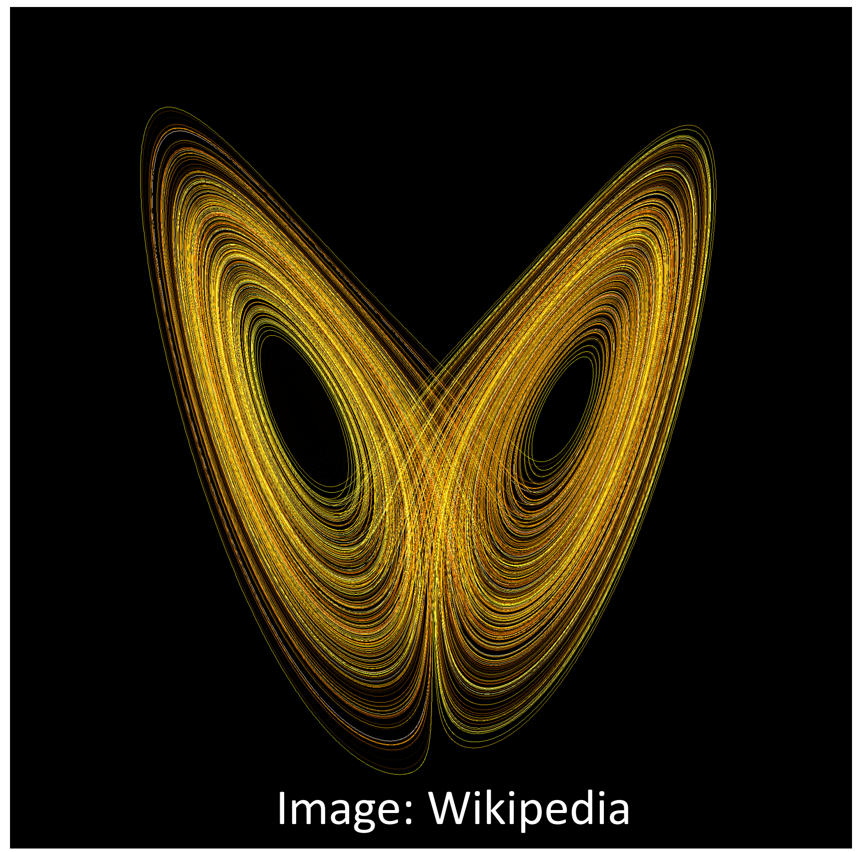

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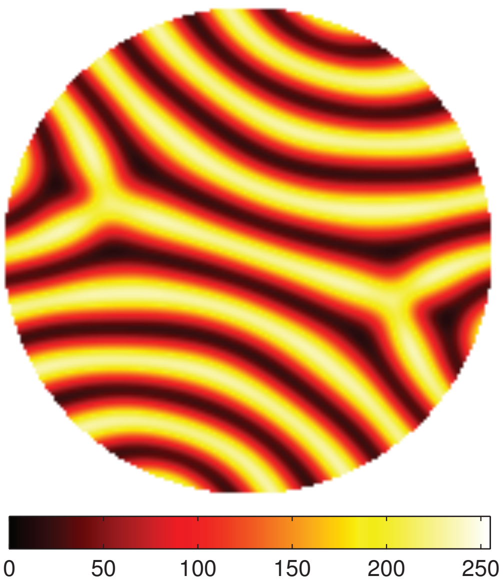
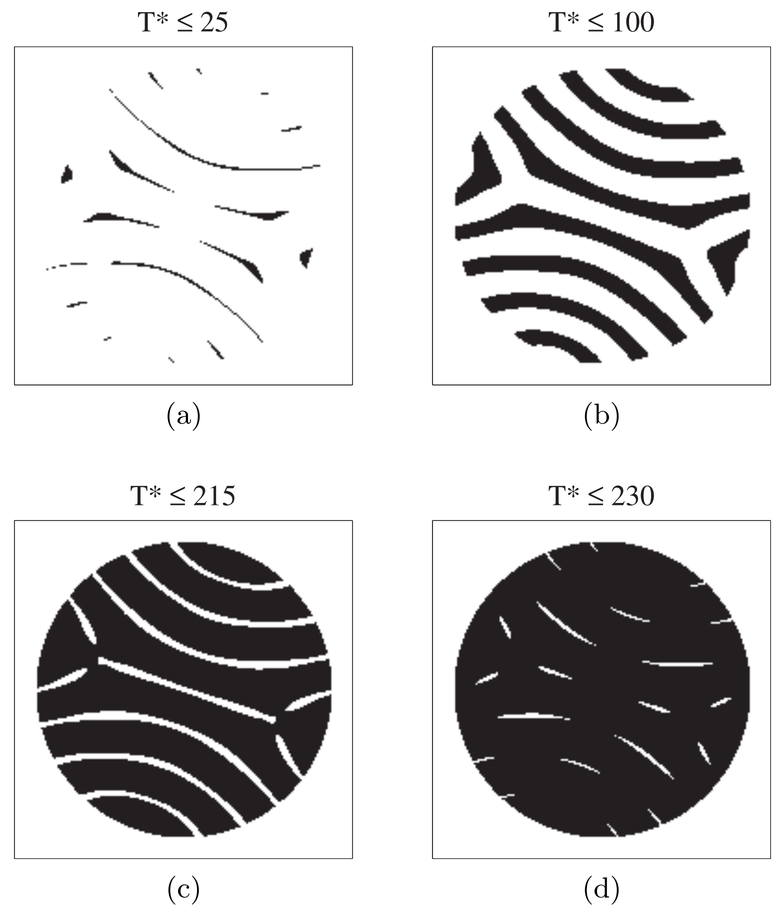
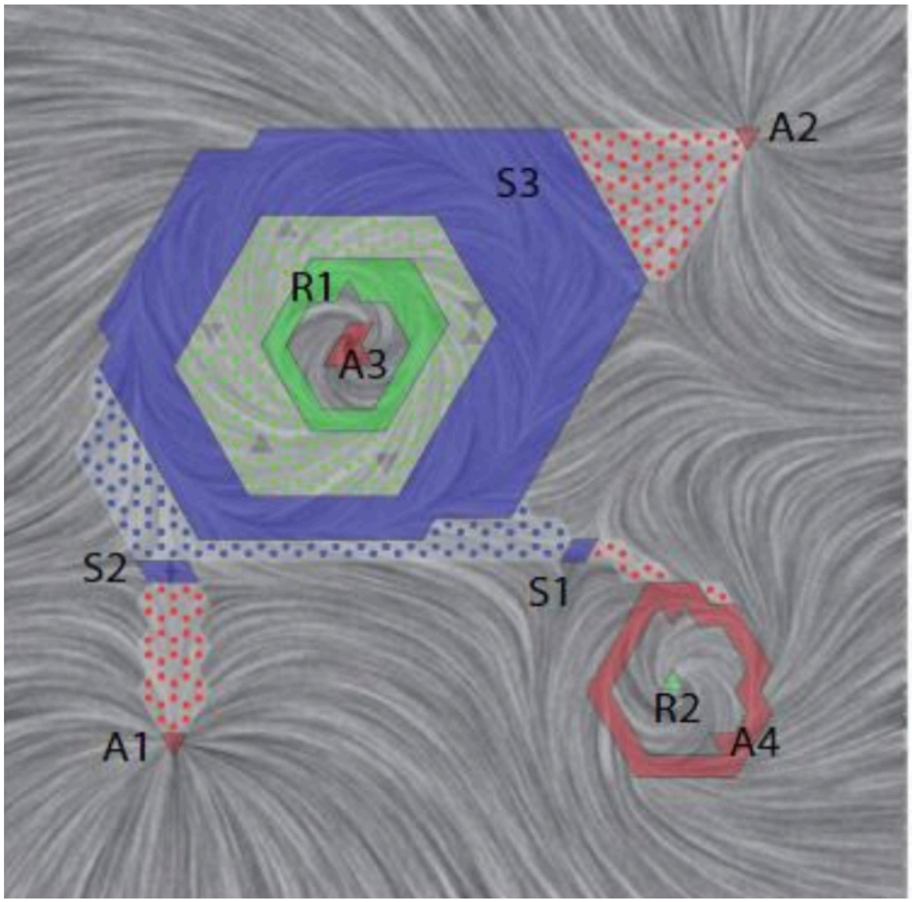
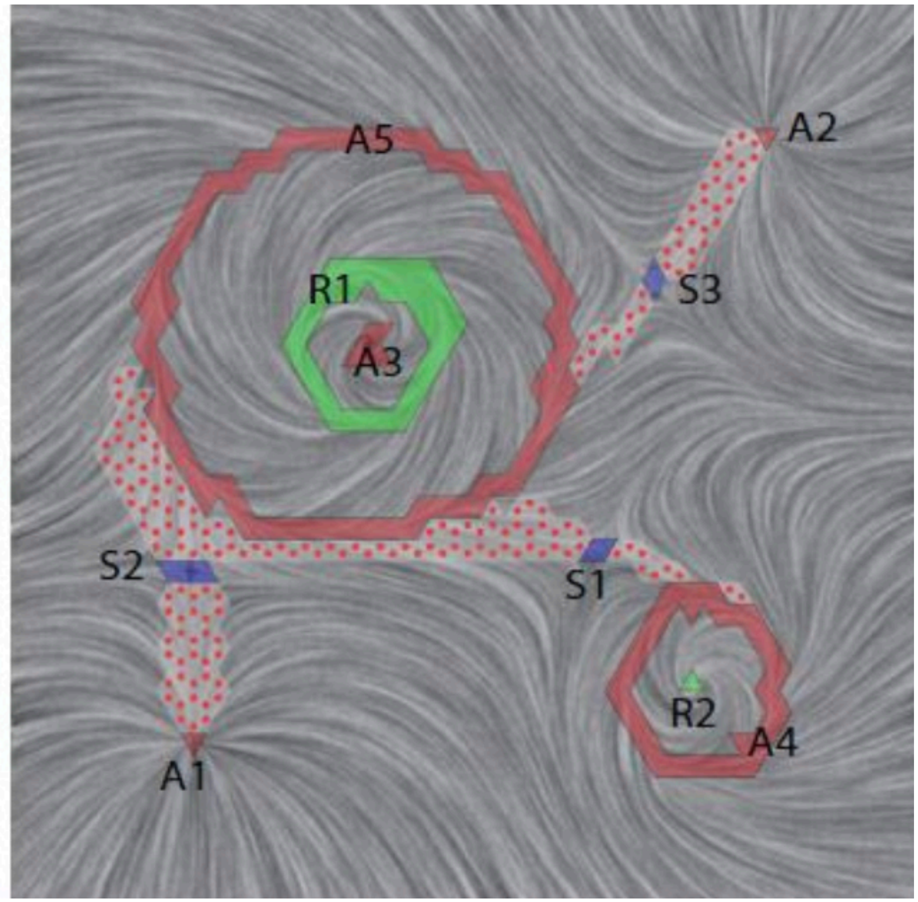
Sublevelset filtrations of dynamical system data





Conley index theory



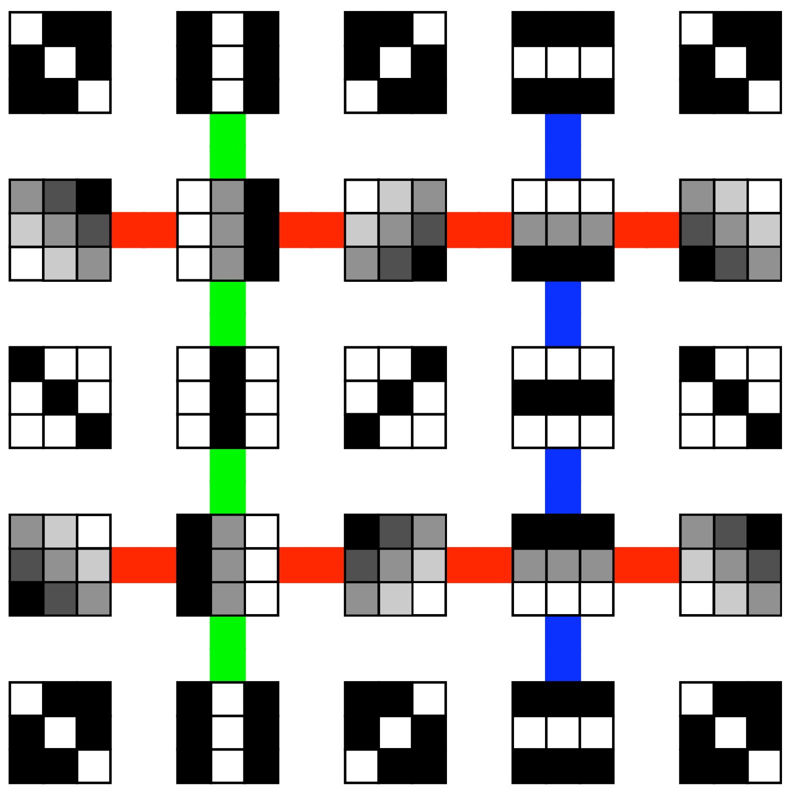


“Topology! The stratosphere of human thought! In the twenty-fourth century it might possibly be of use to someone …”

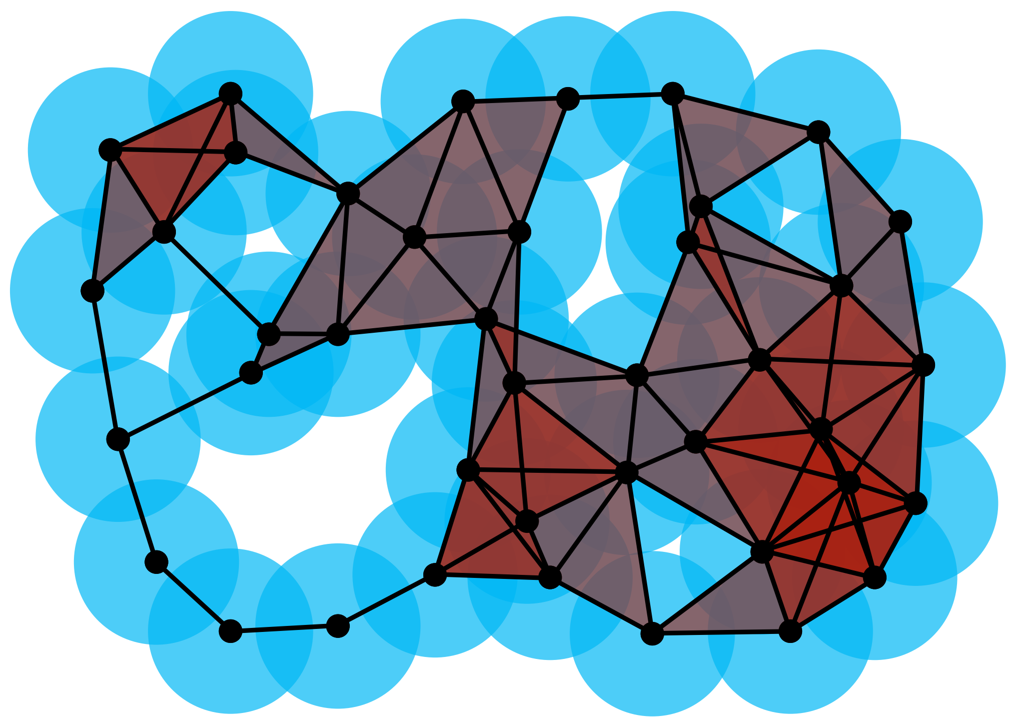
-Aleksandr Solzhenitsyn, *The First Circle*

Conclusions for Part II

- Data density matters
- Persistent homology is now being used as a summary statistic of local geometry and global topology, for use both in machine learning and in dynamical systems.

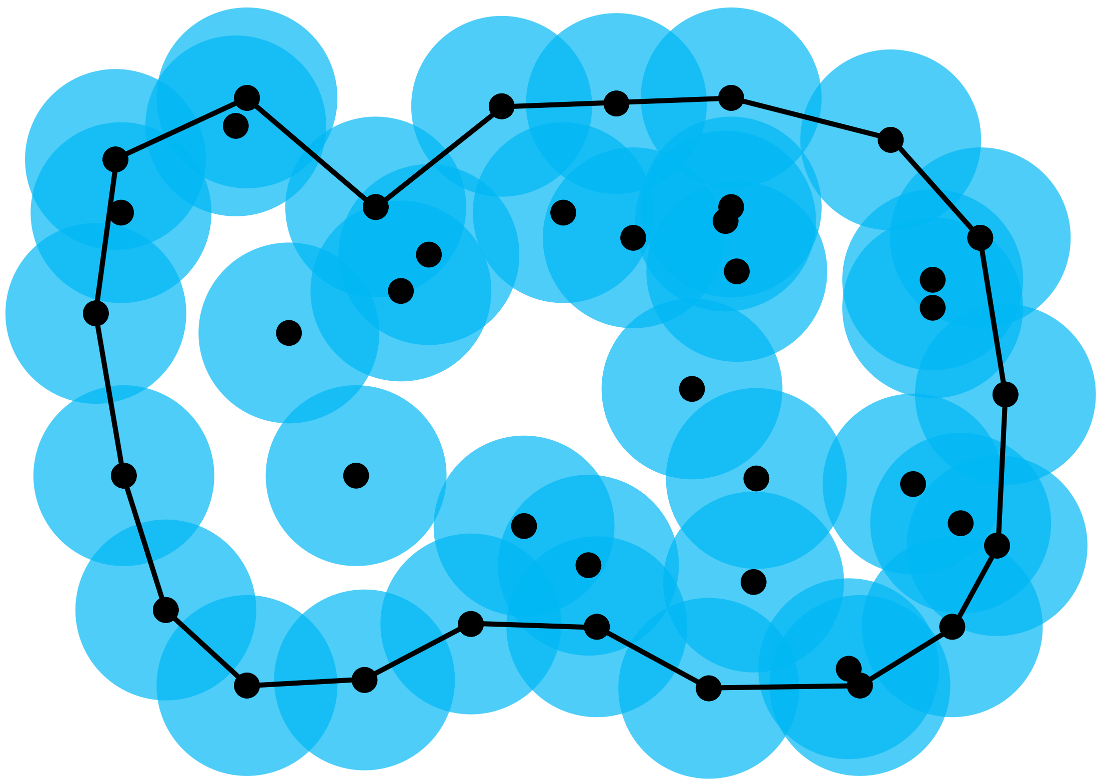


Evasion Paths in Mobile Sensor Networks



Topology applied to sensor networks

- Sensors move in a ball-shaped domain $B \subset \mathbb{R}^d$ over time interval $I = [0, 1]$. Fixed sensors cover the boundary.
- Measure only the Čech complex.
- Is there an evasion path?



Coordinate-free Coverage in Sensor Networks with Controlled Boundaries via Homology by V. de Silva and R. Ghrist

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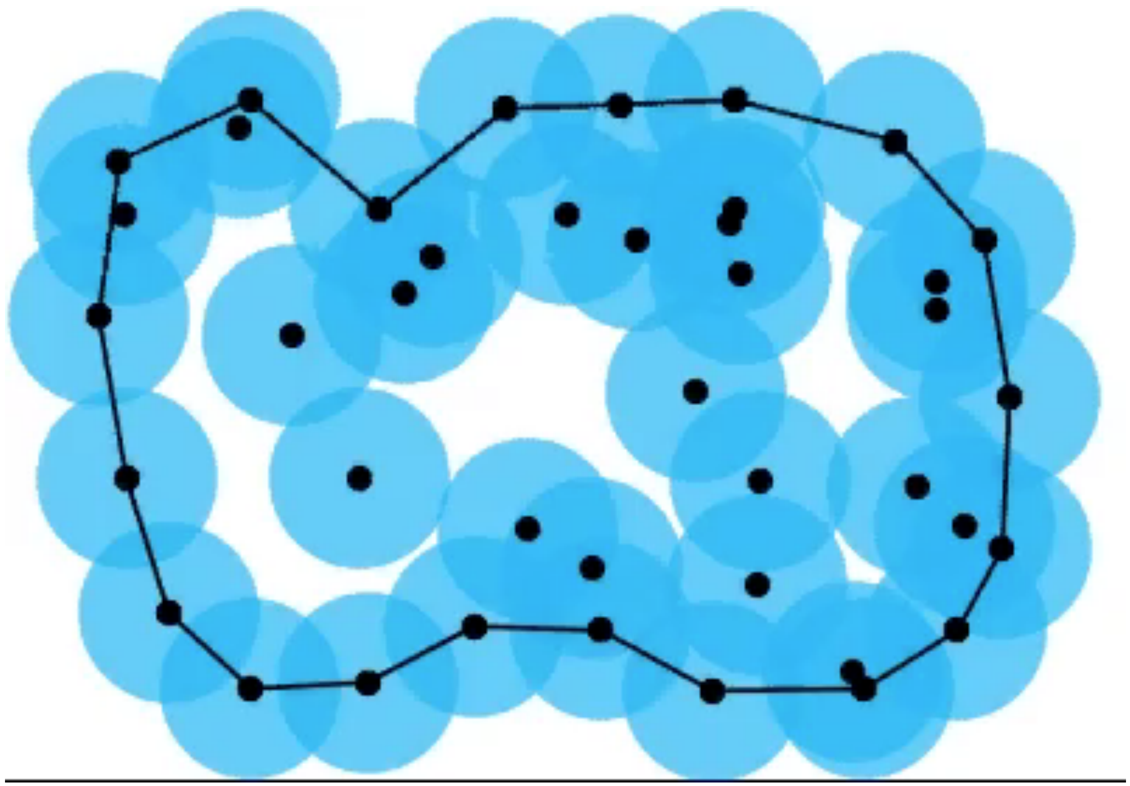

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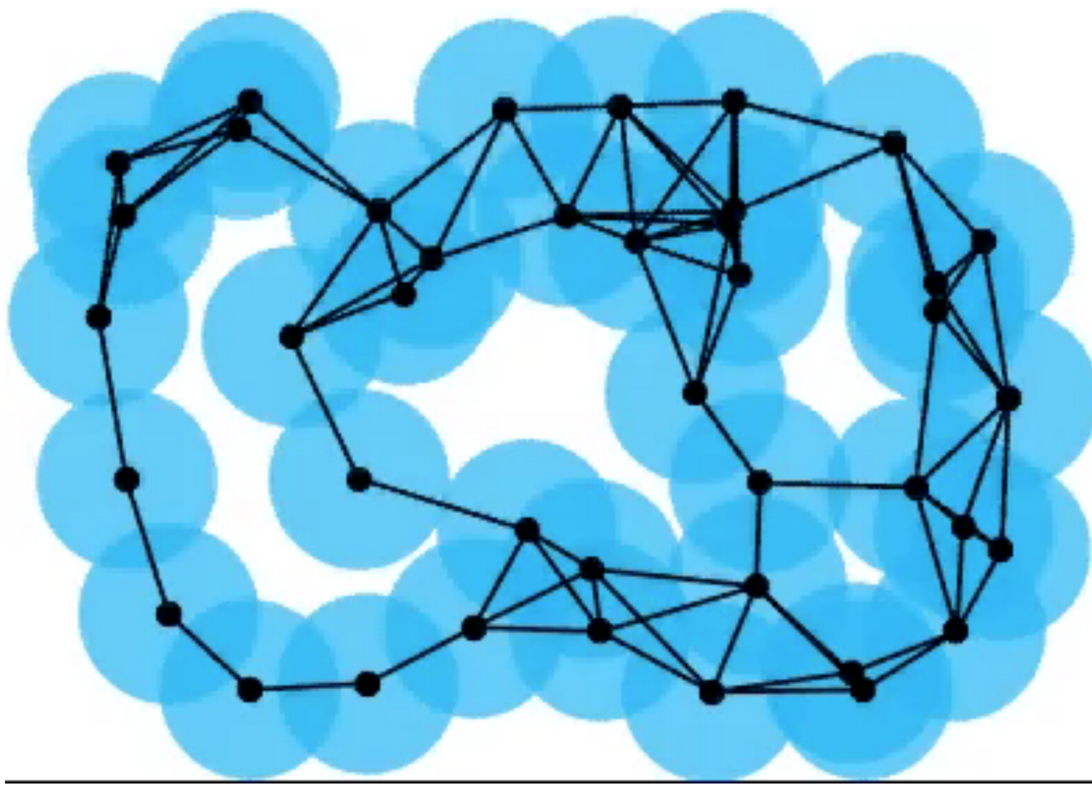

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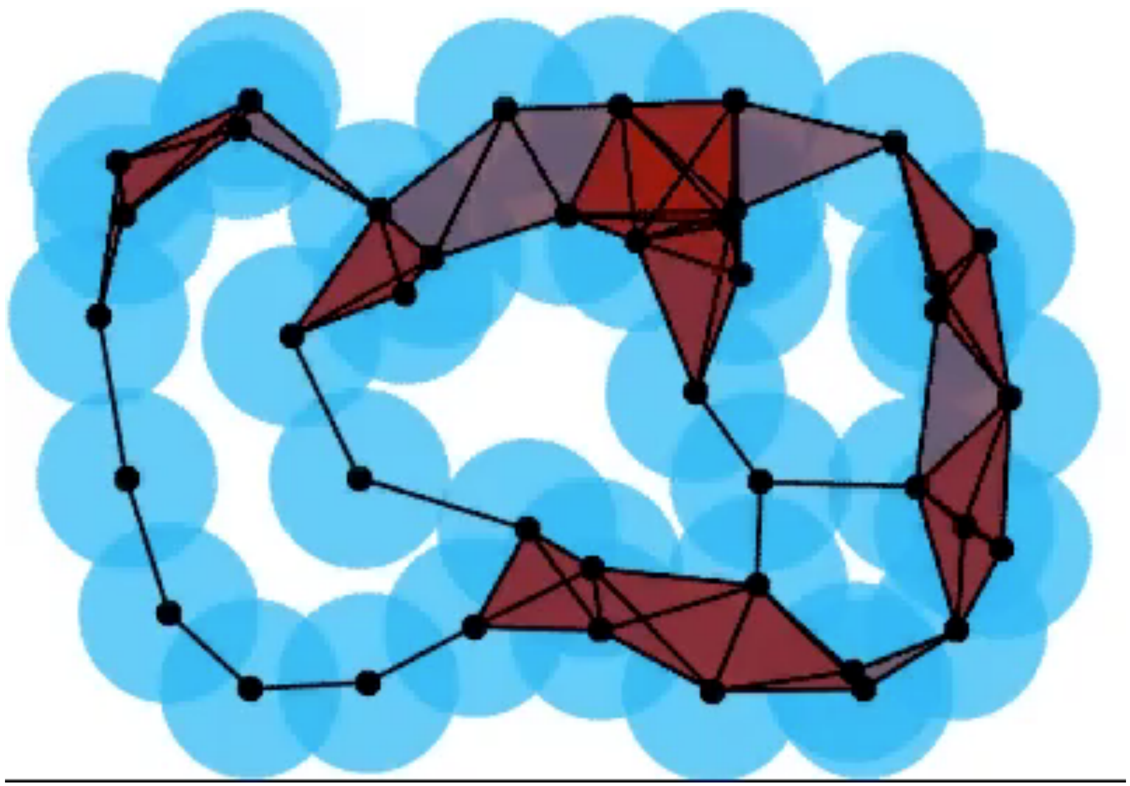

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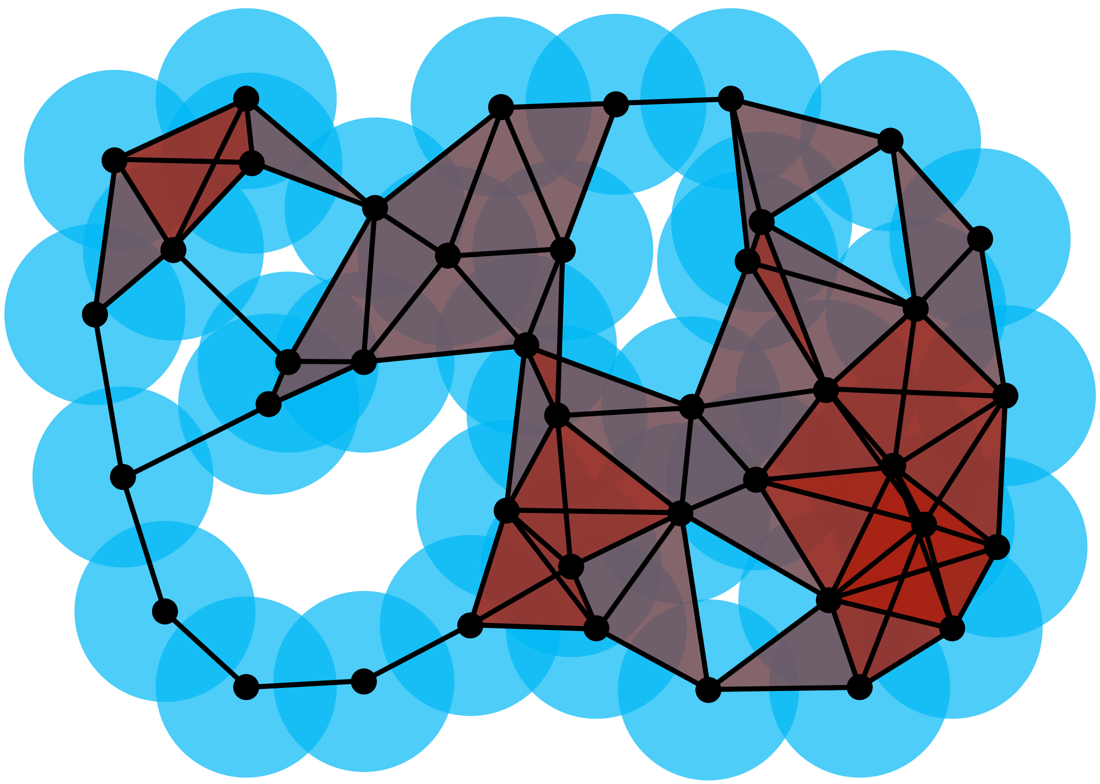

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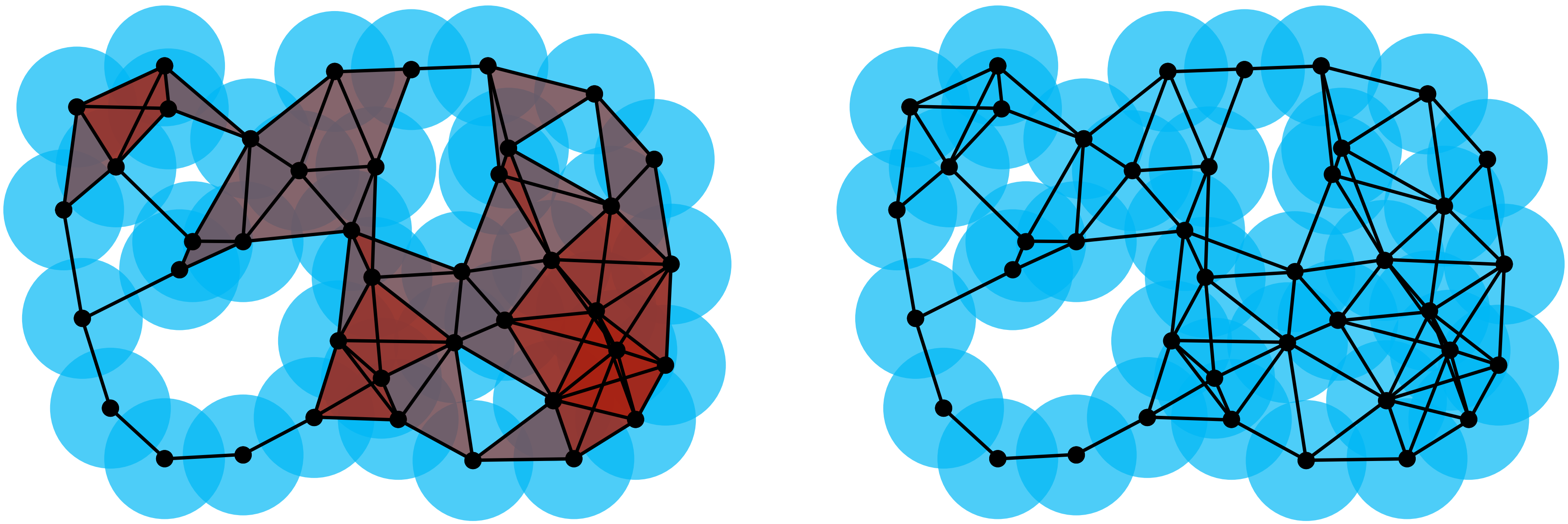
Topology applied to sensor networks

Čech complex



- One vertex for each ball
- Edges when 2 balls overlap
- Triangles when 3 balls overlap

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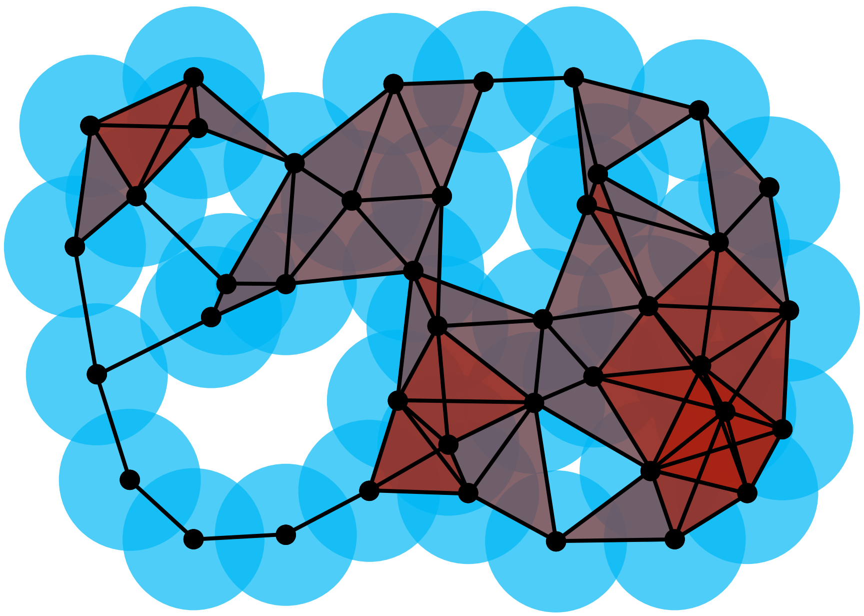
Topology applied to sensor networks

Čech complex



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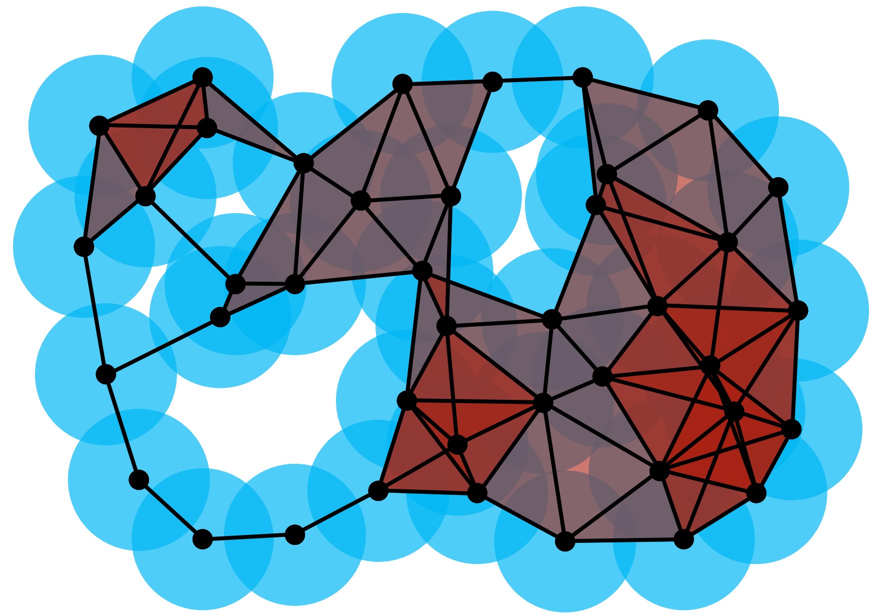
Vietoris-Rips complex



- One vertex for each ball
- Edges when 2 balls overlap
- All possible triangles

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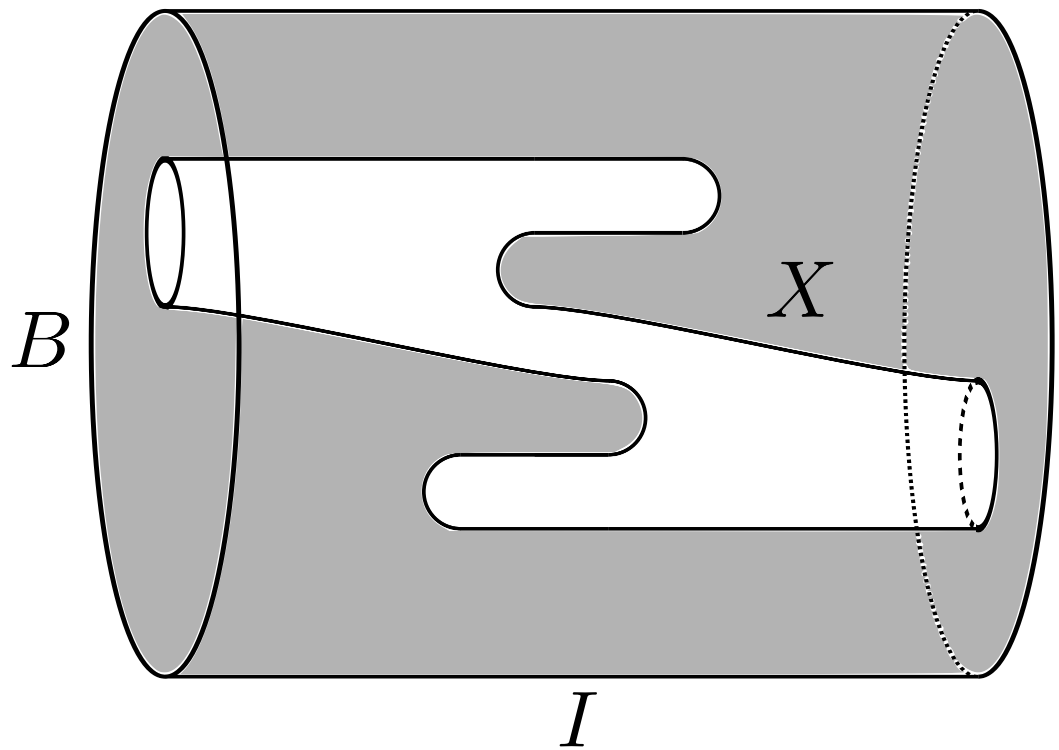

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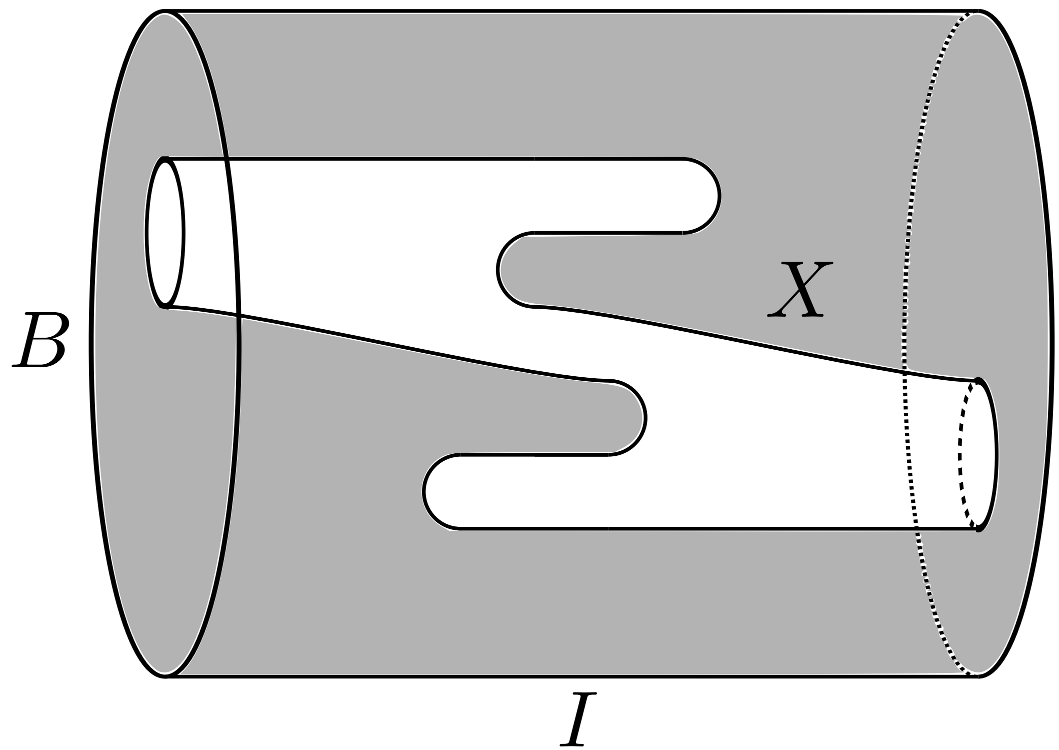

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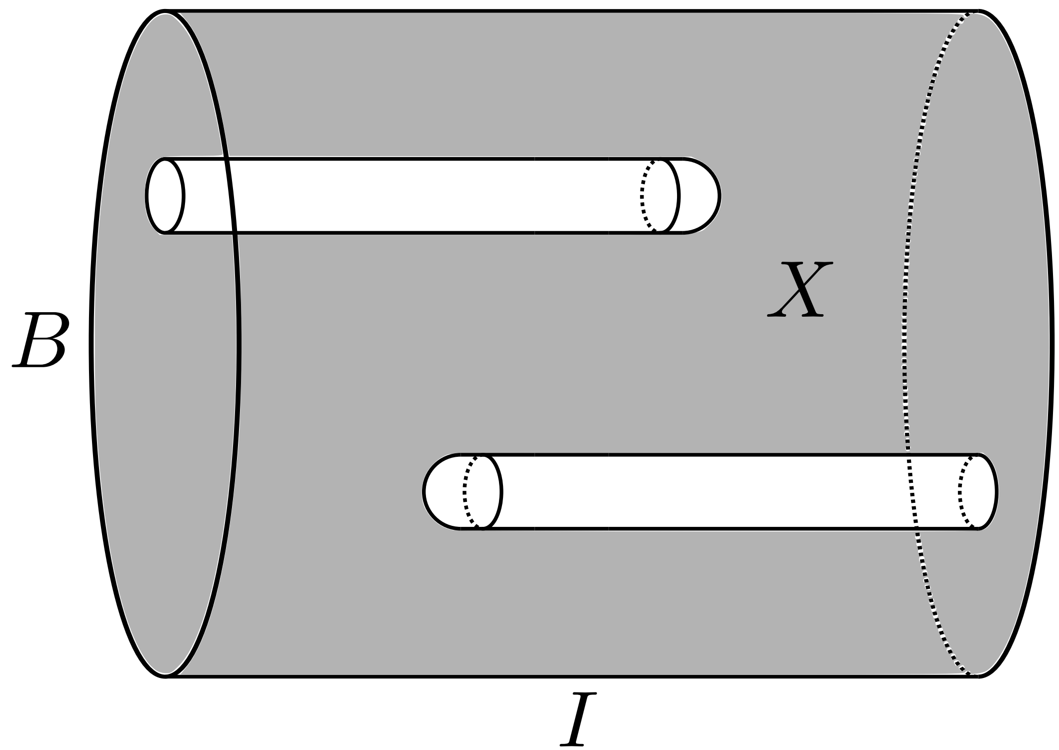

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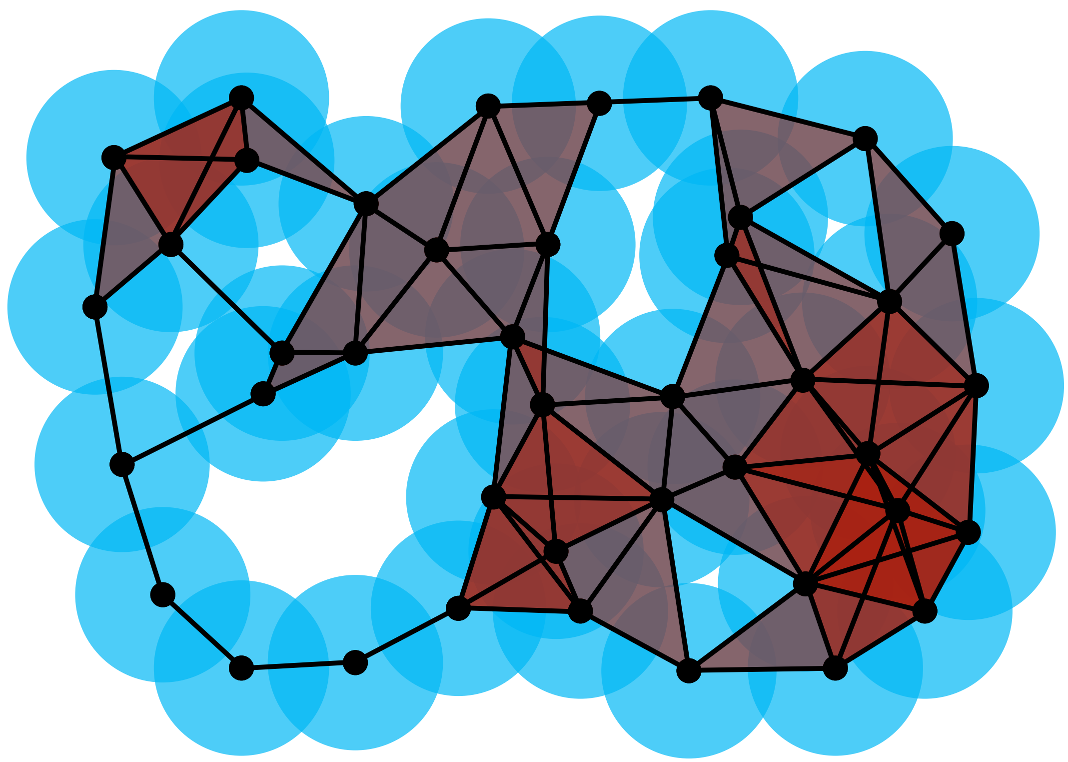

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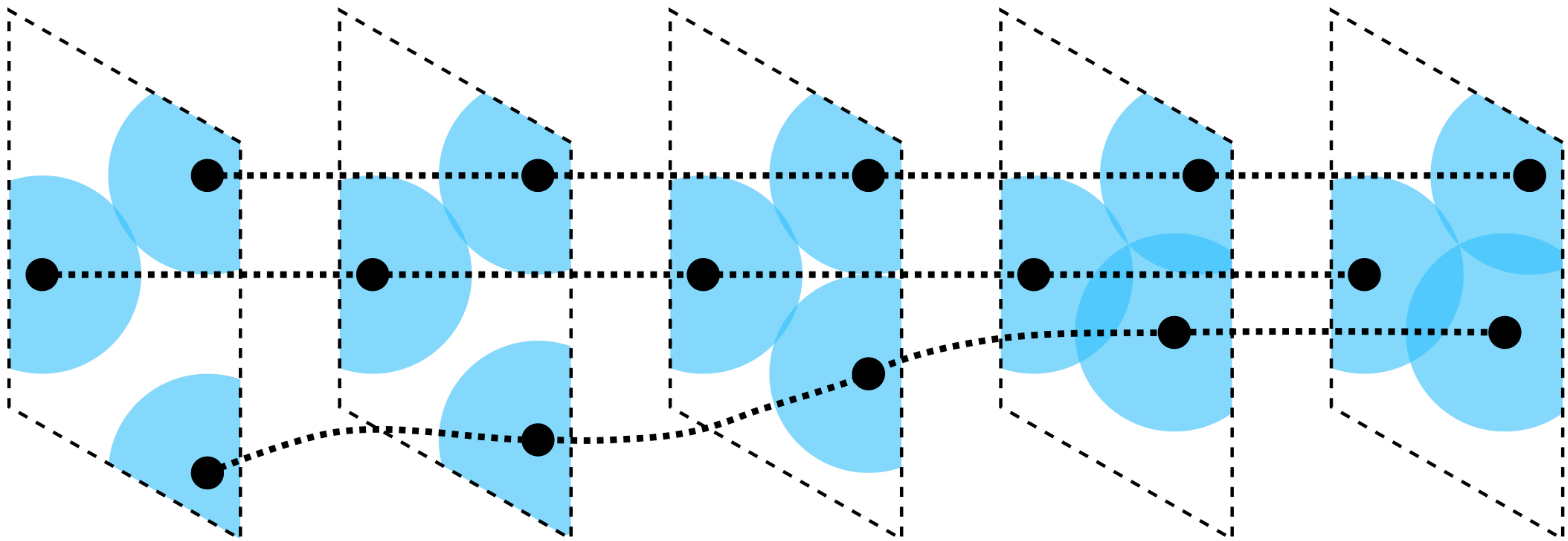

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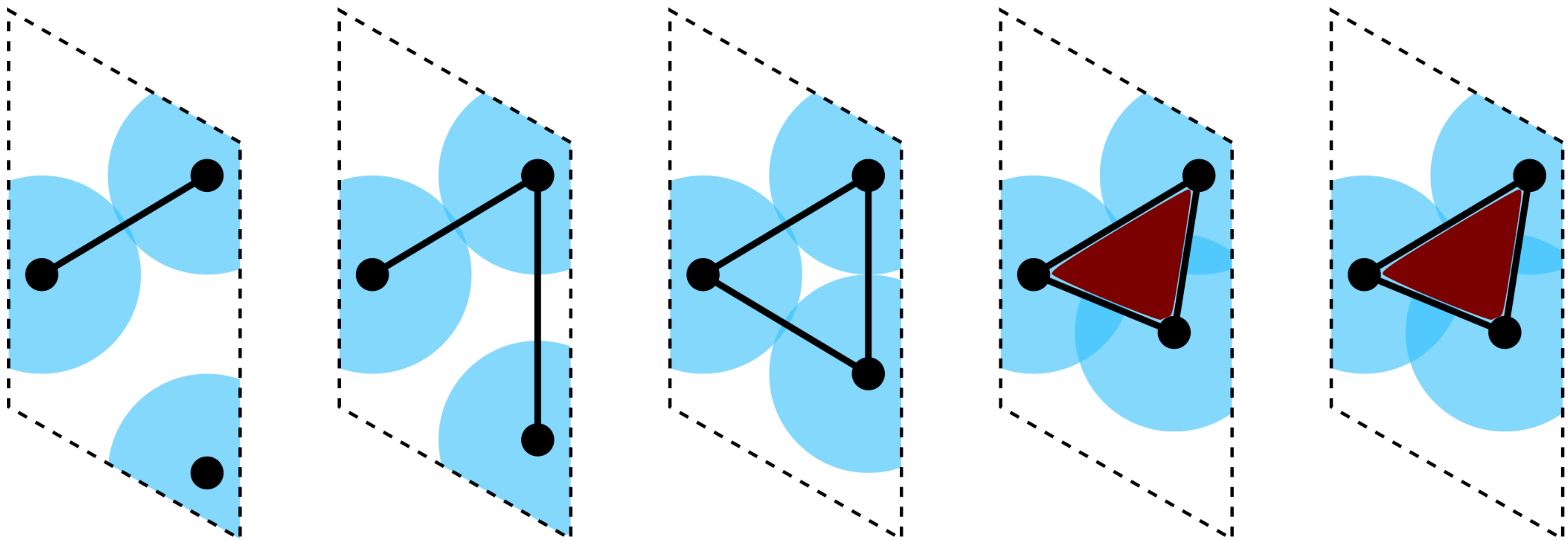

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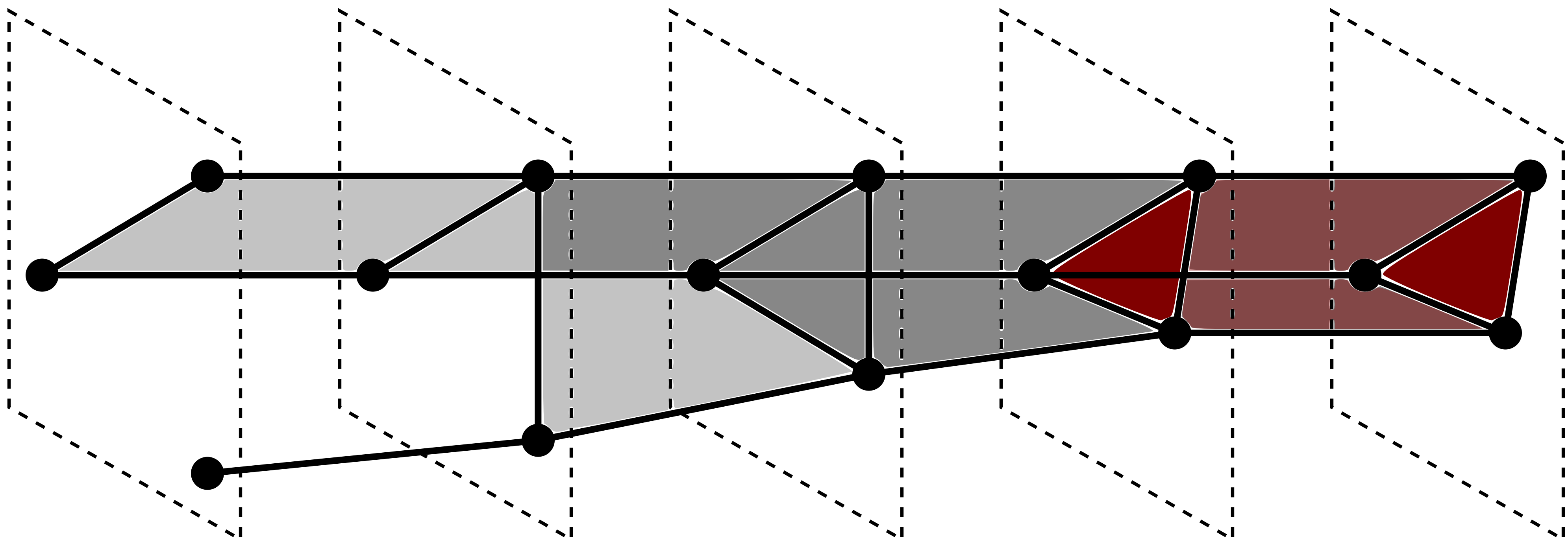

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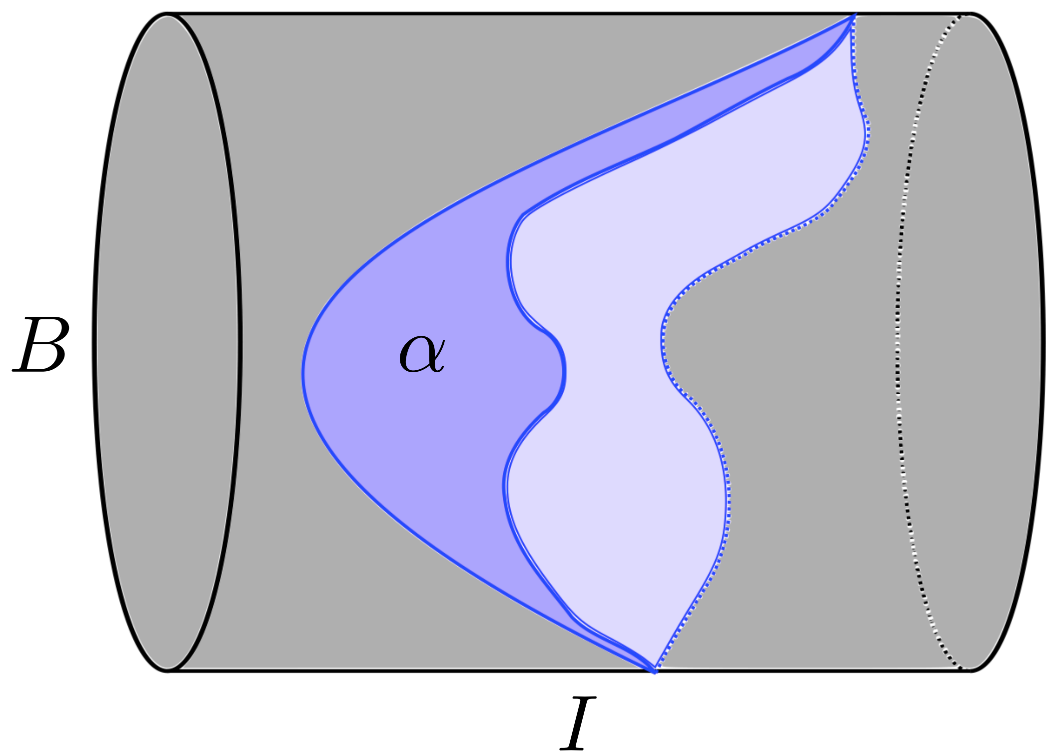

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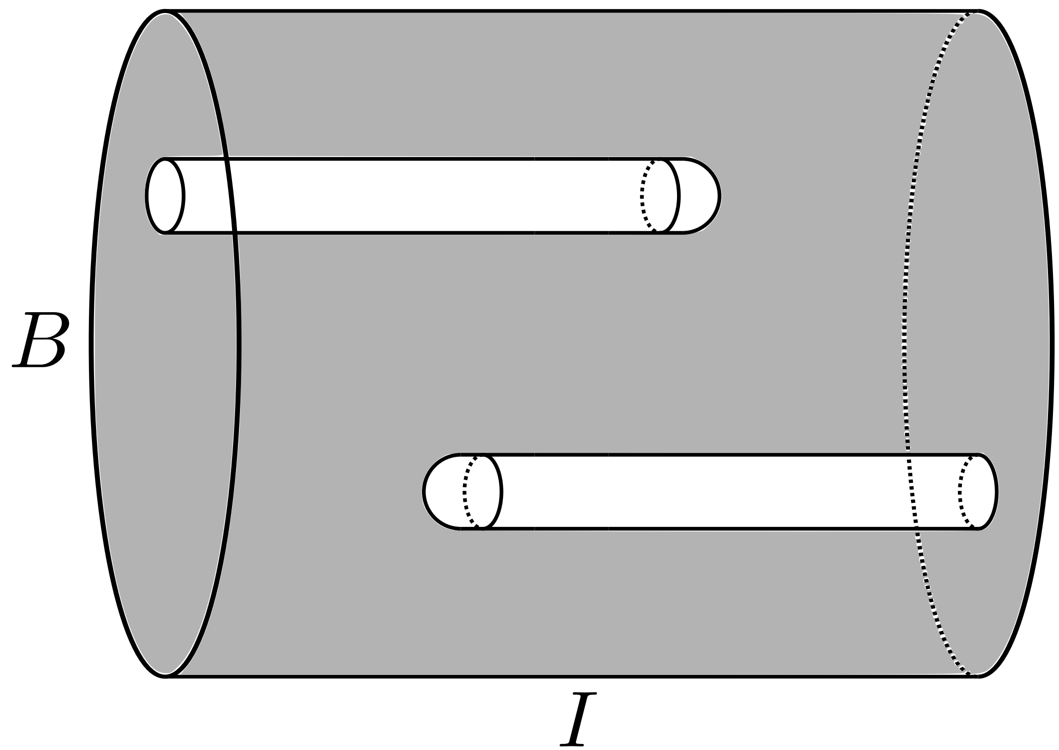

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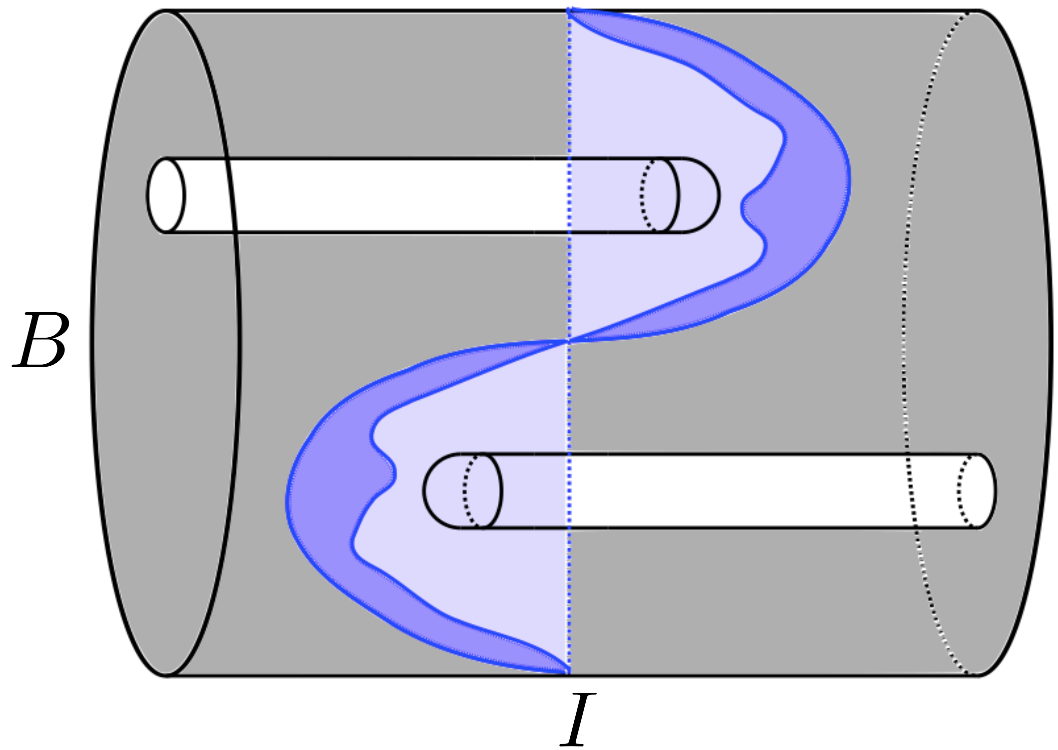

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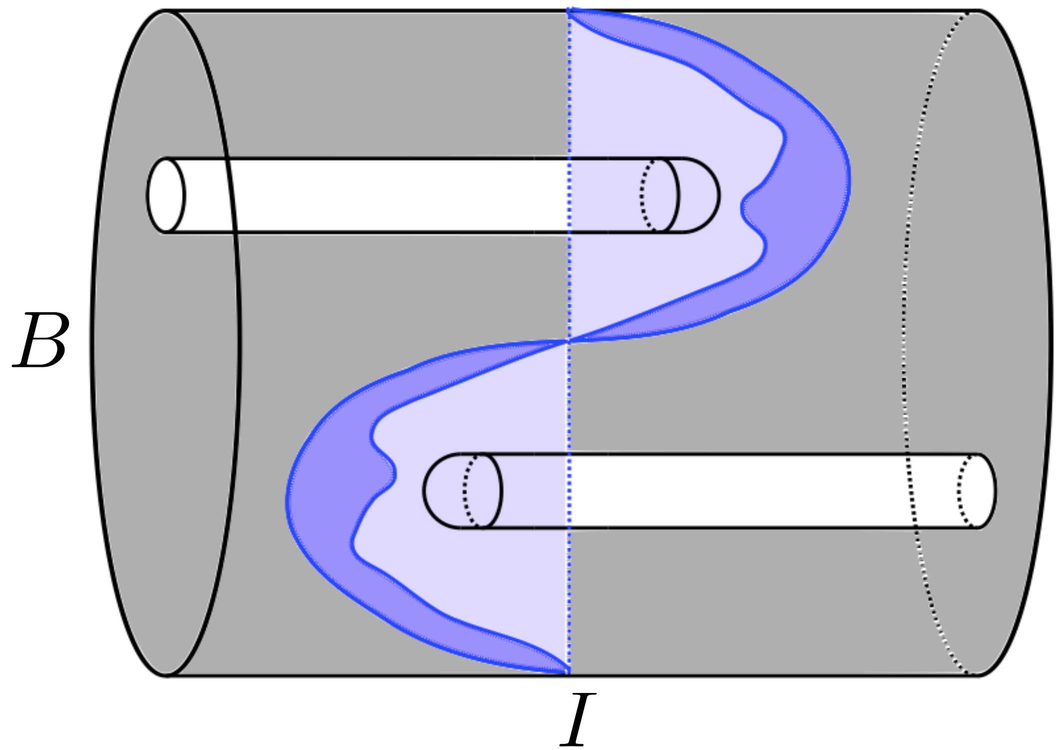

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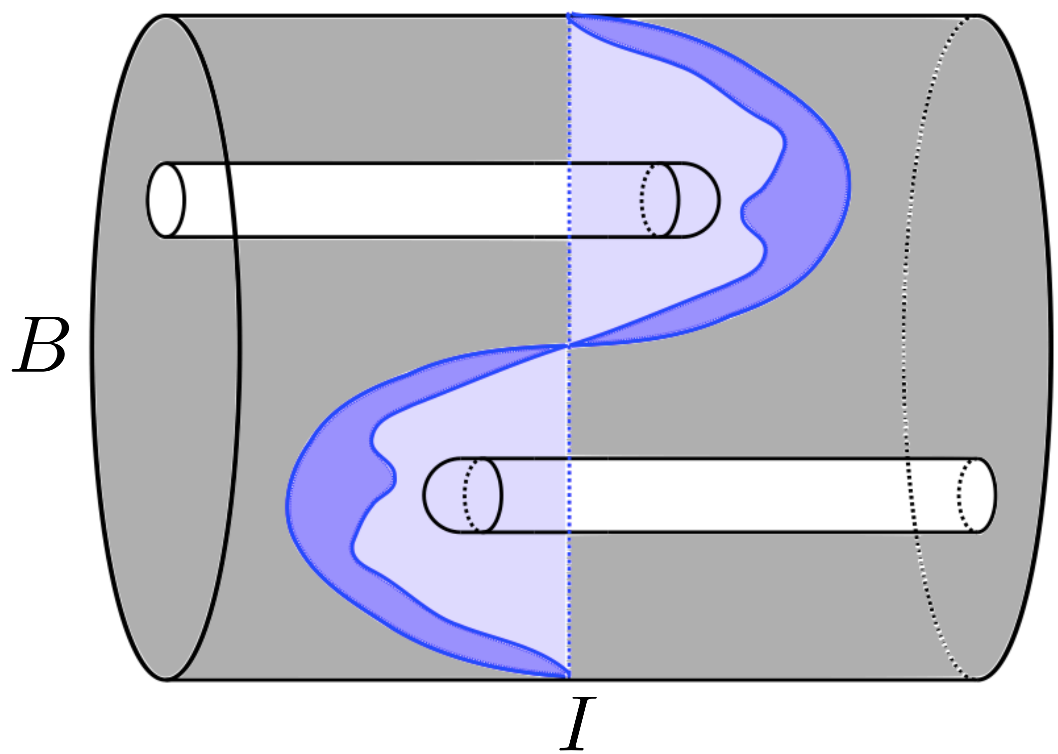

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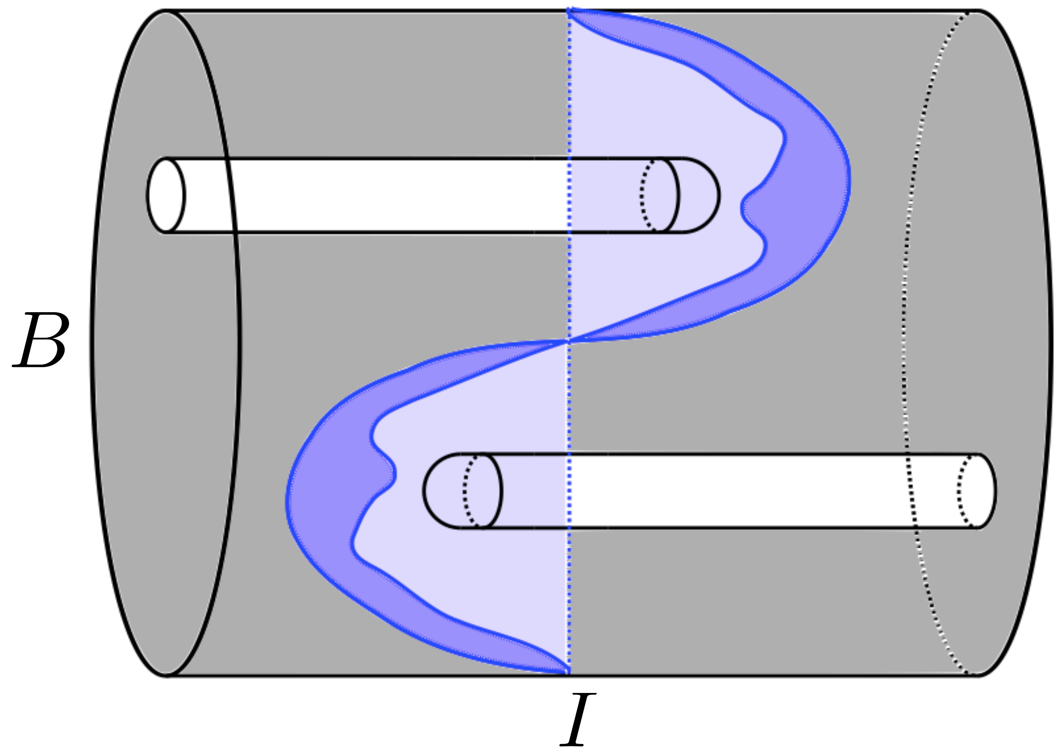

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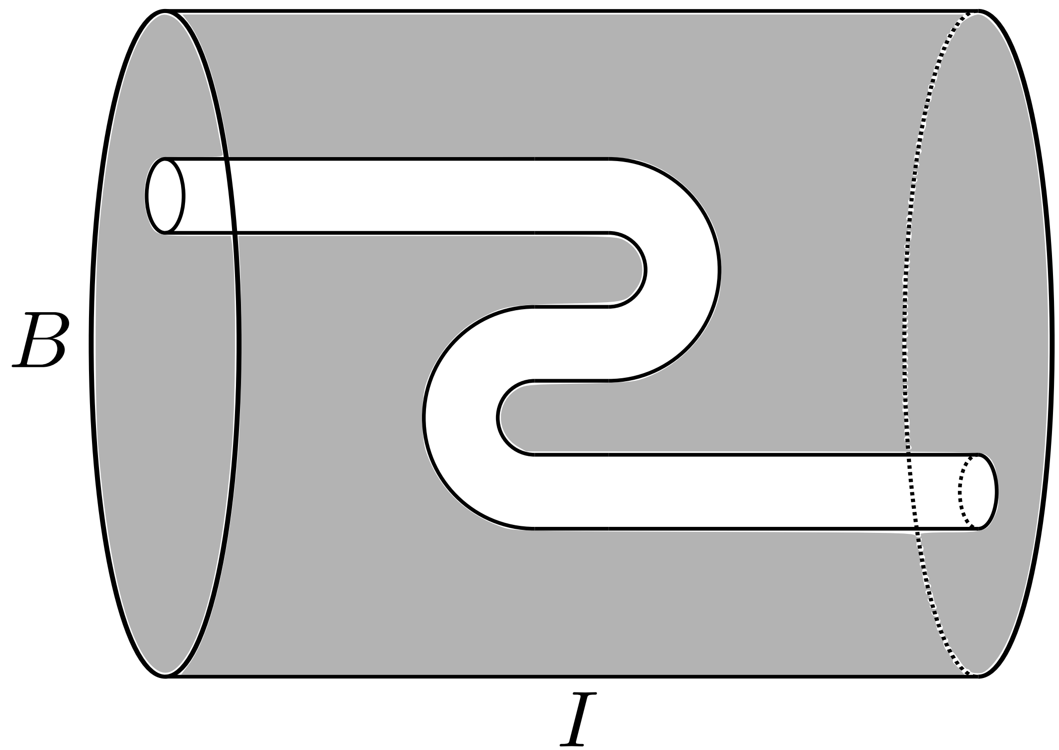

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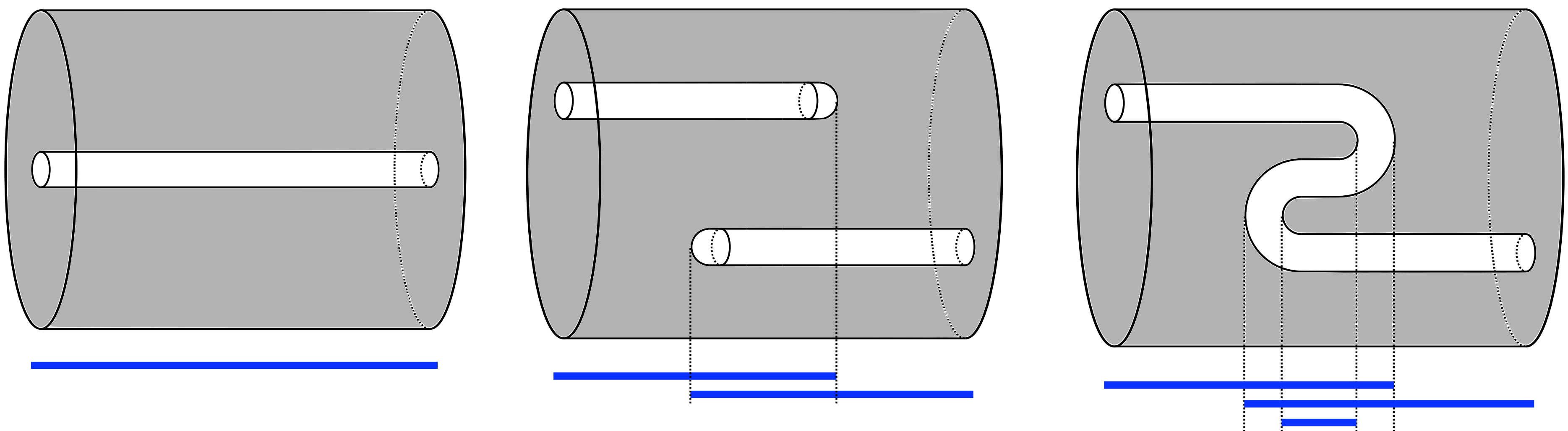

Zigzag Persistence by G. Carlsson and V. de Silva

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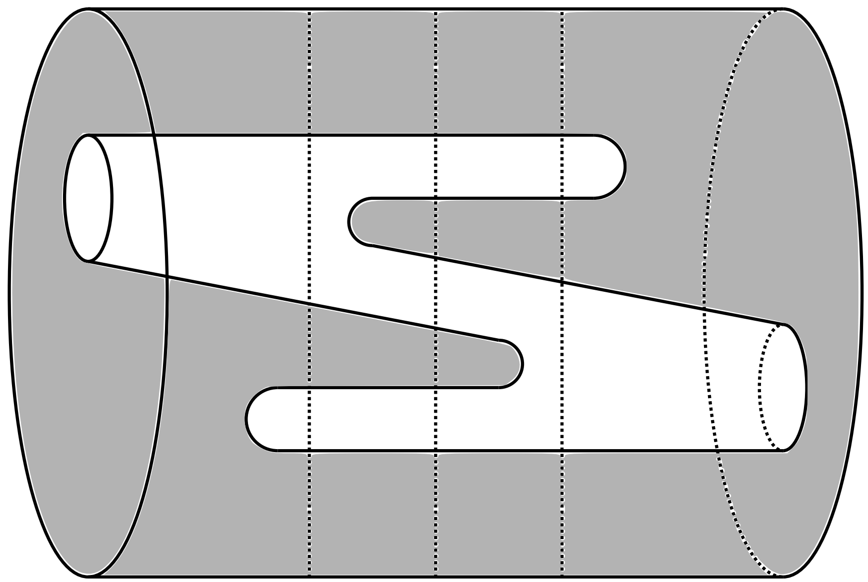

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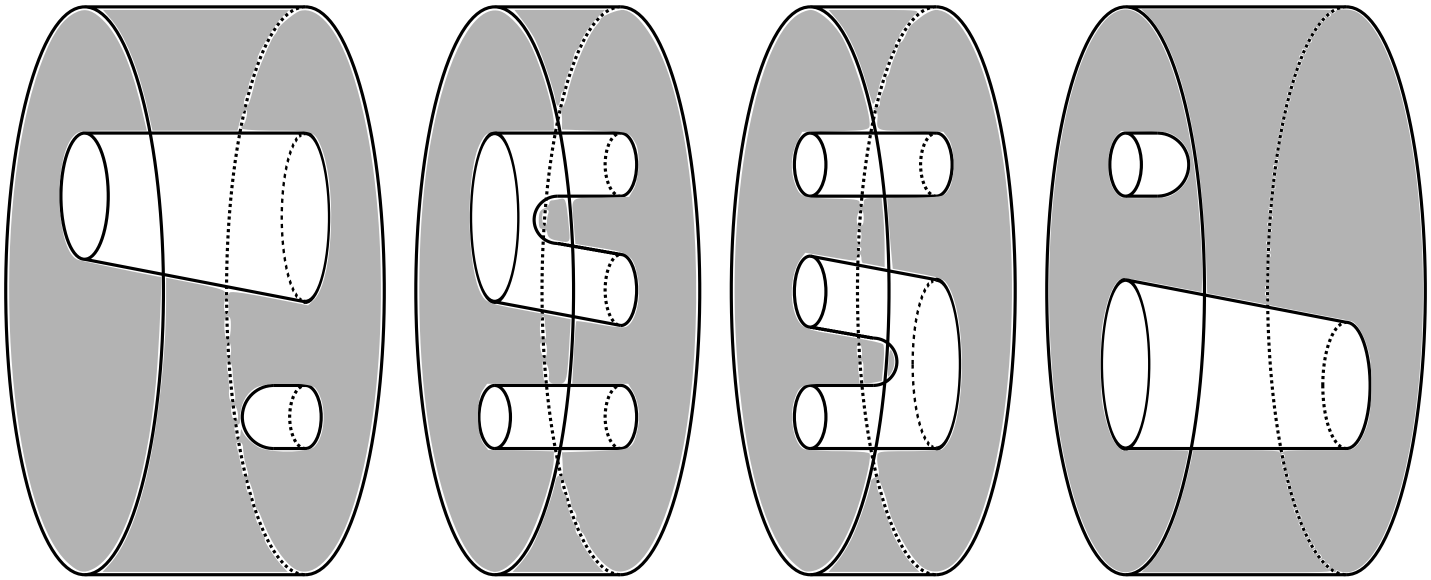

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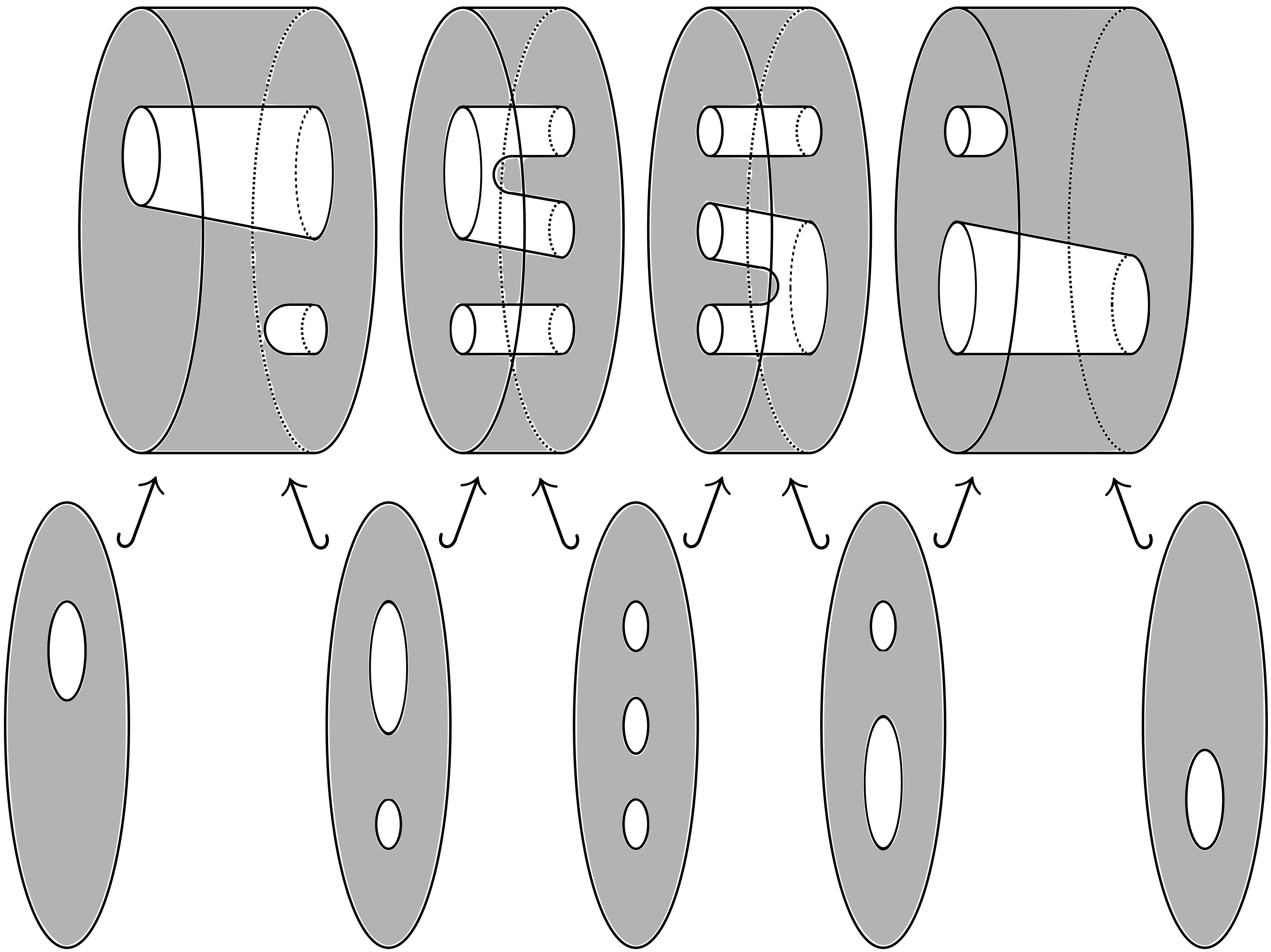

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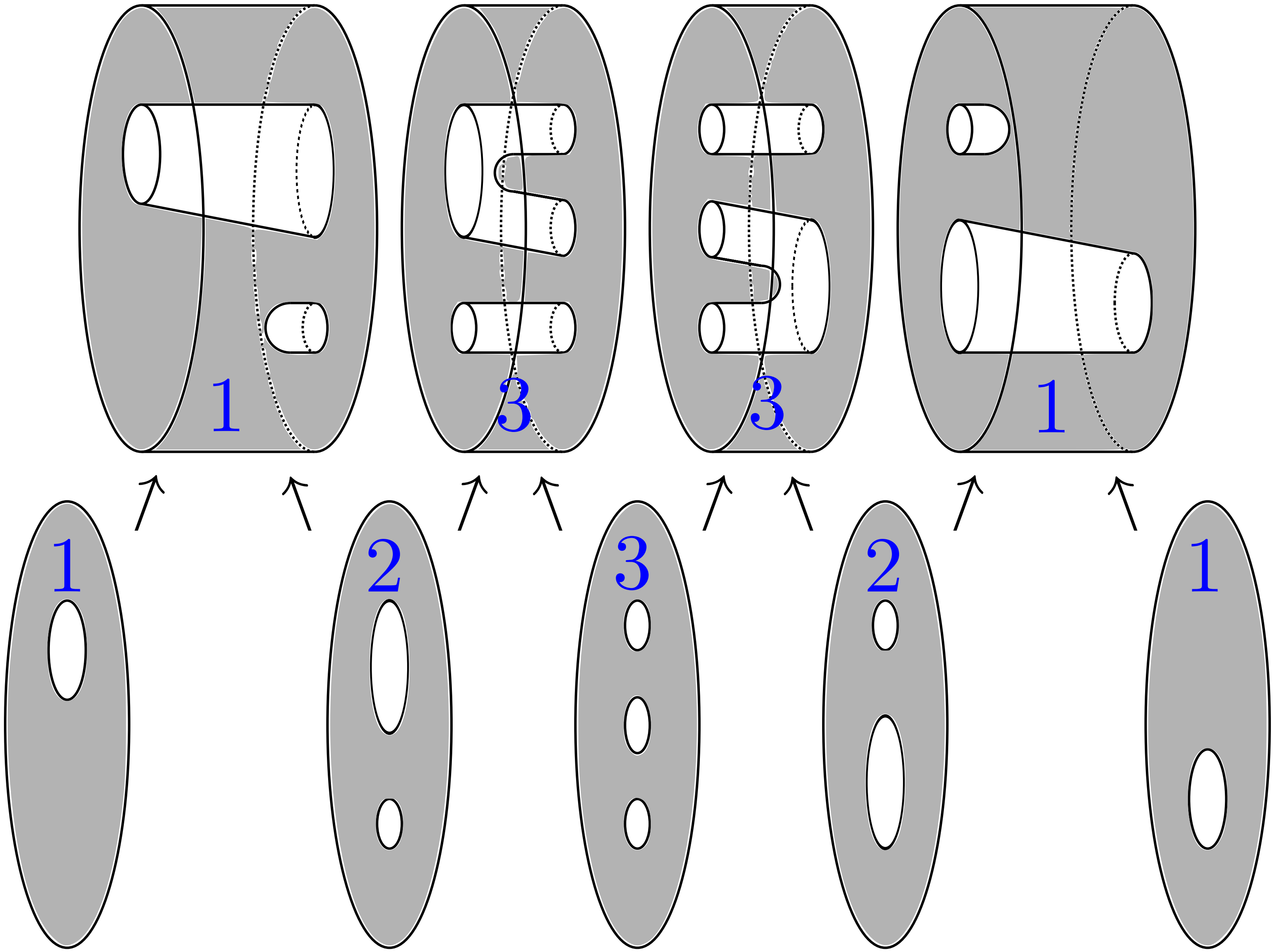

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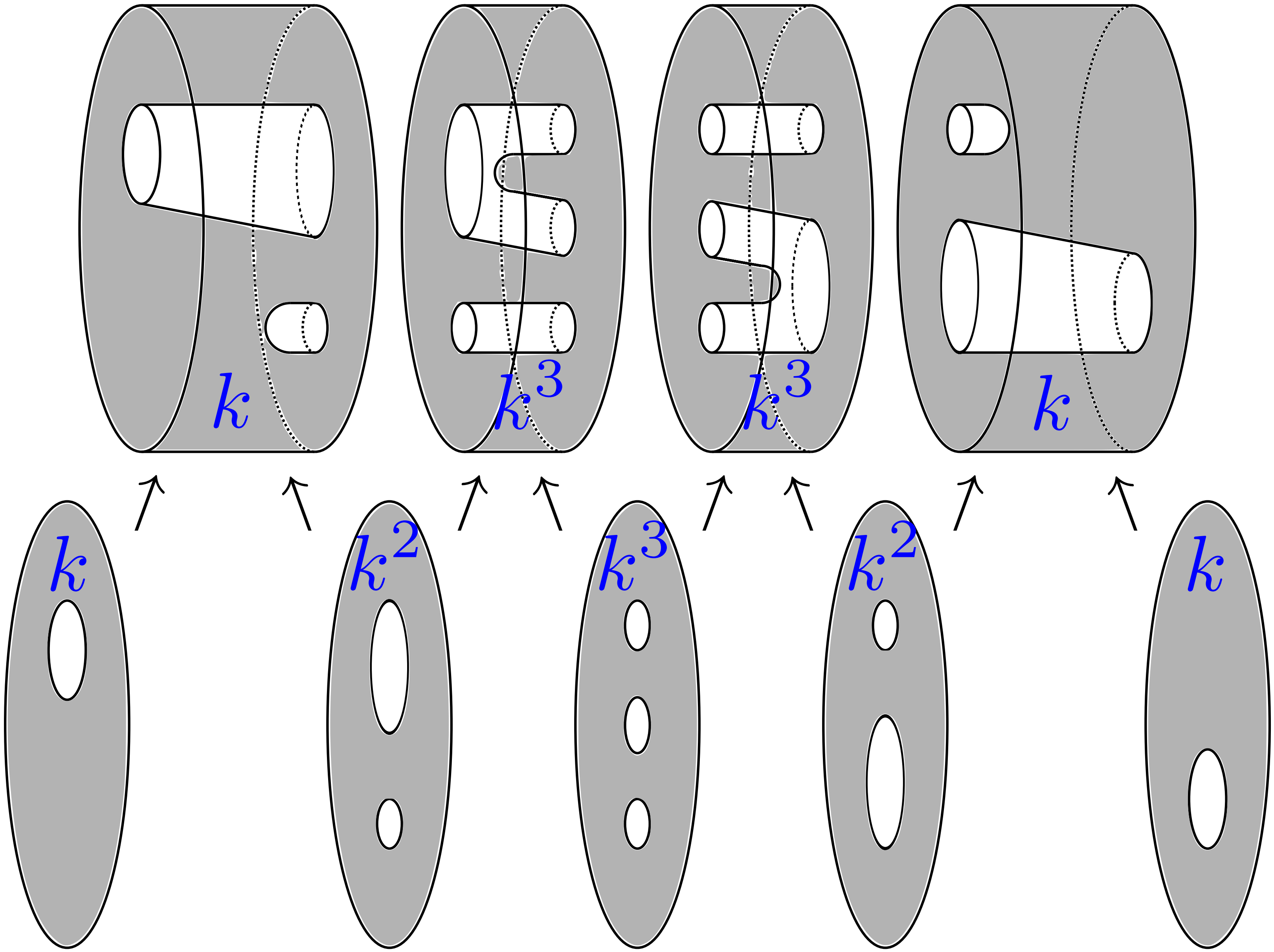

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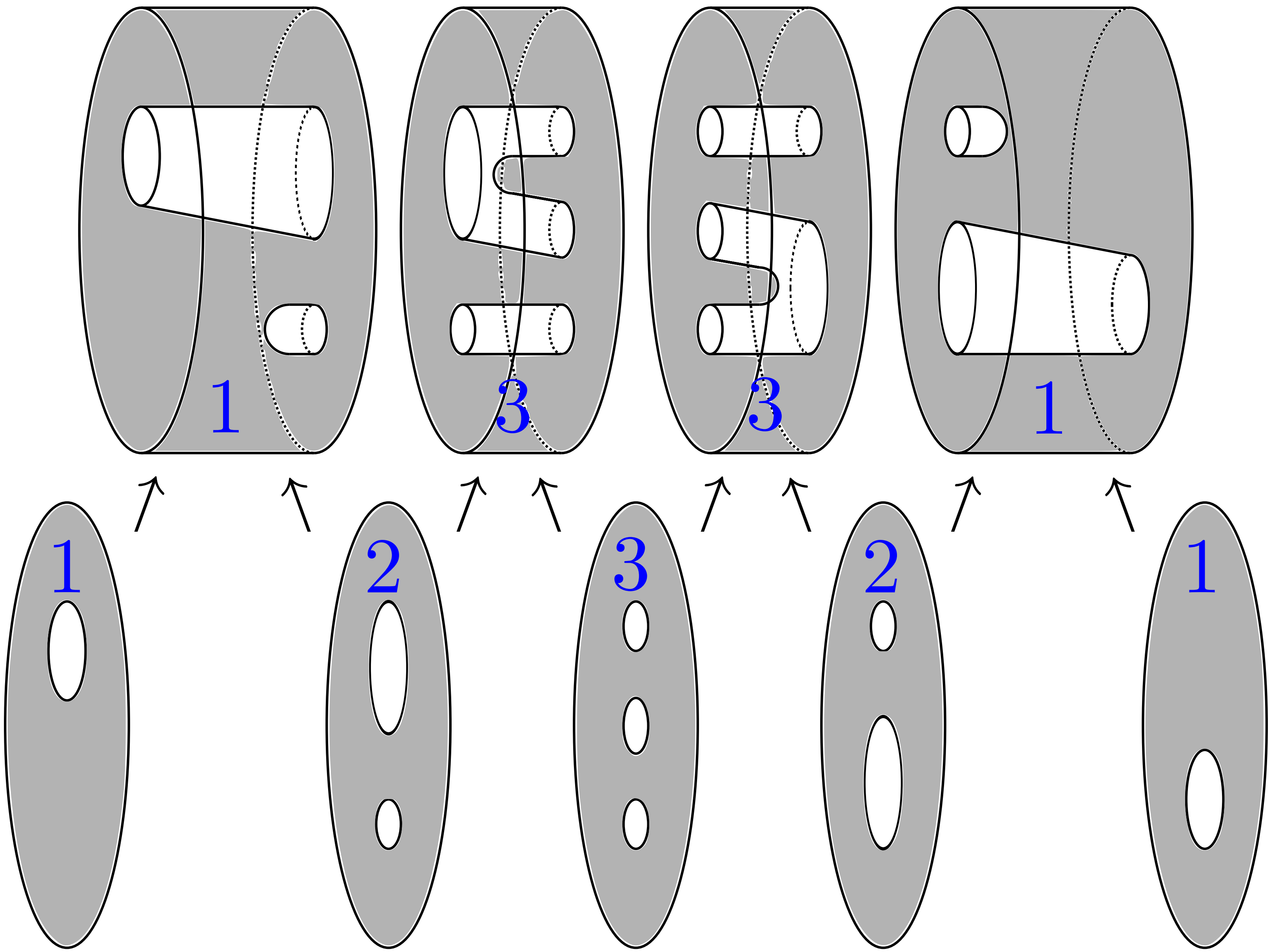

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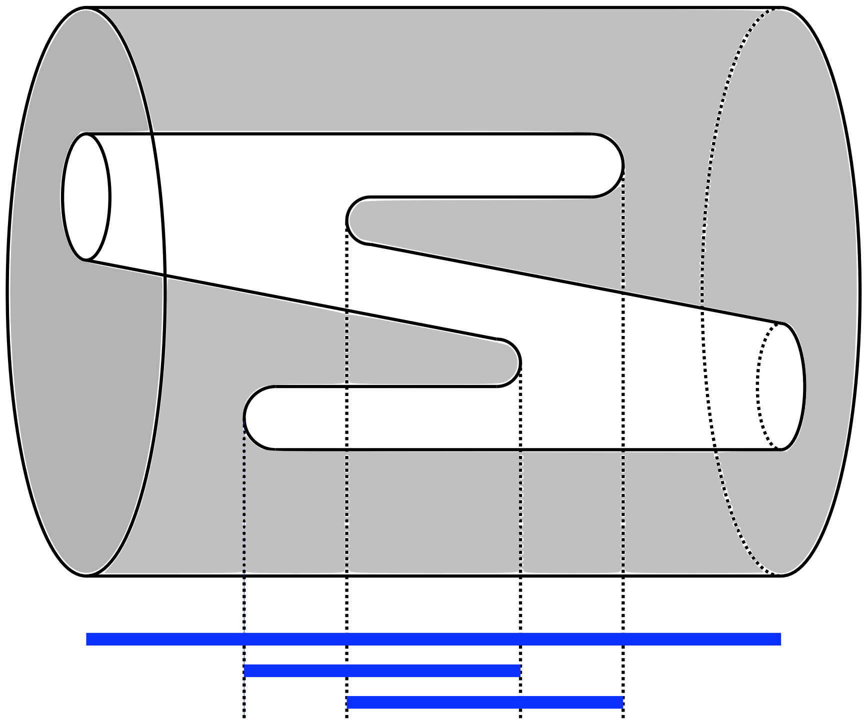

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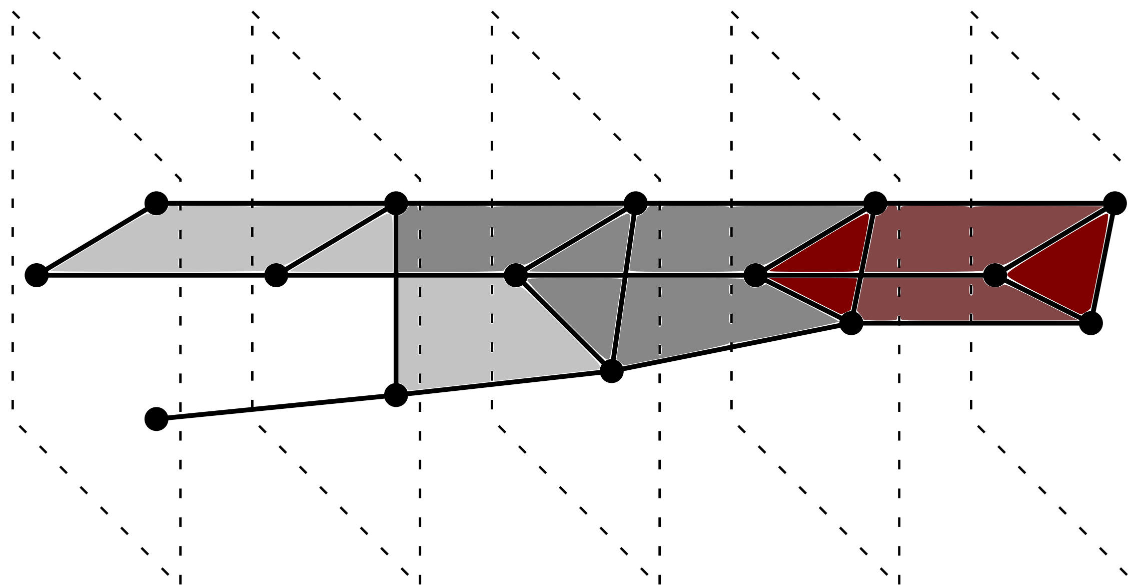

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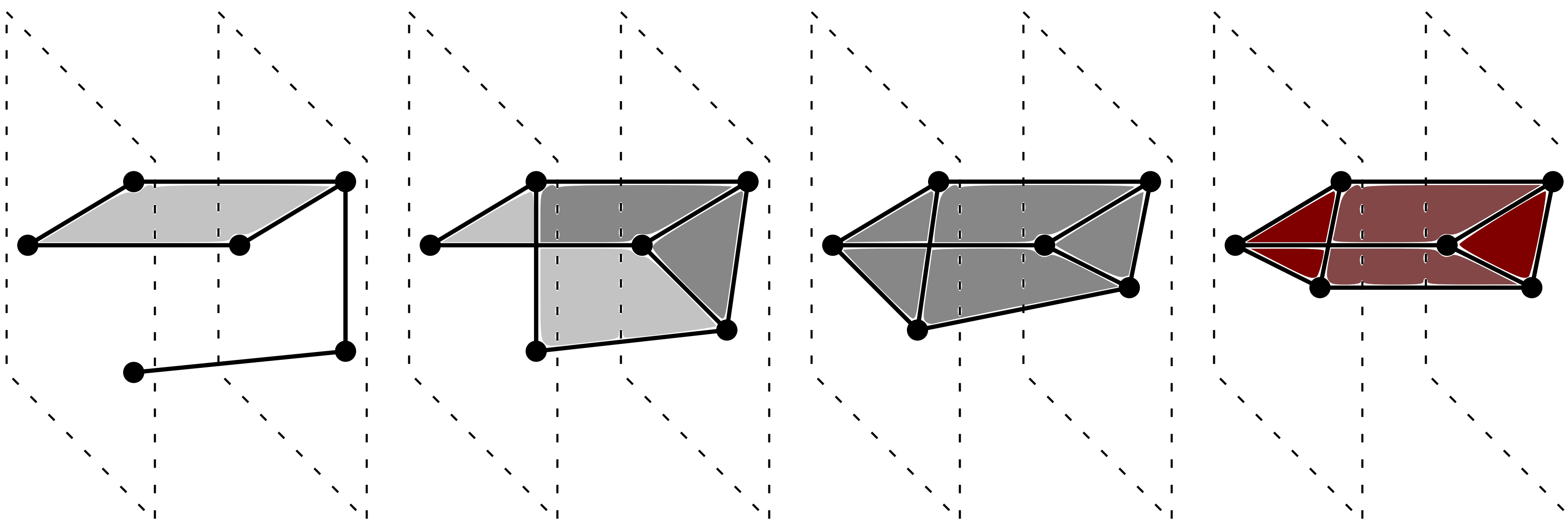

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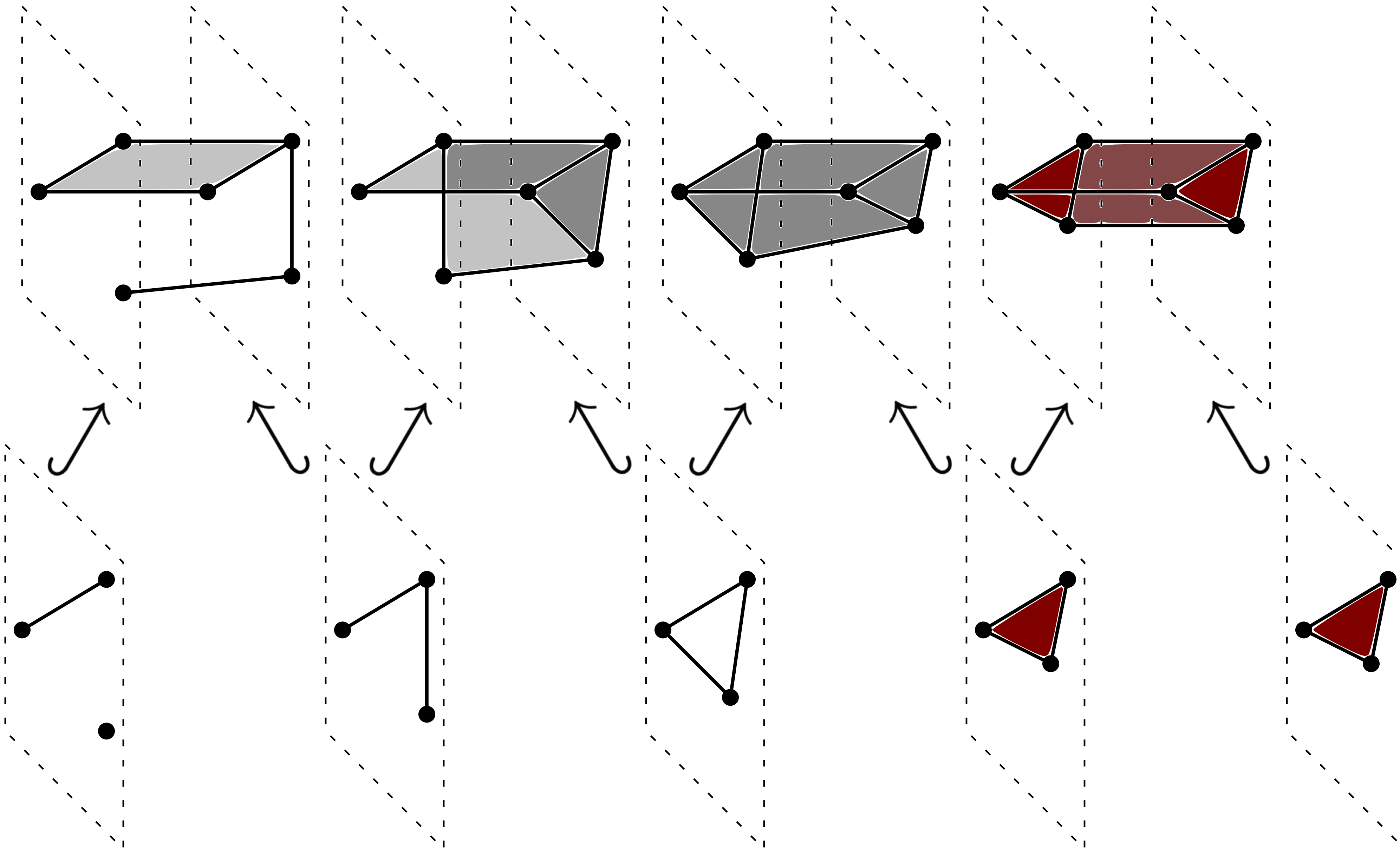

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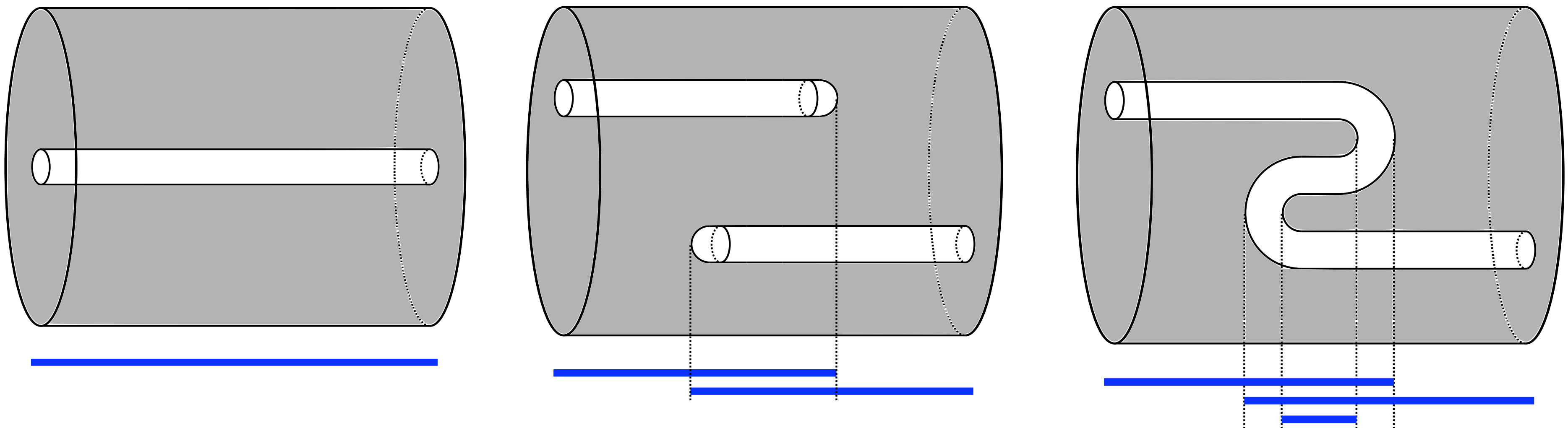

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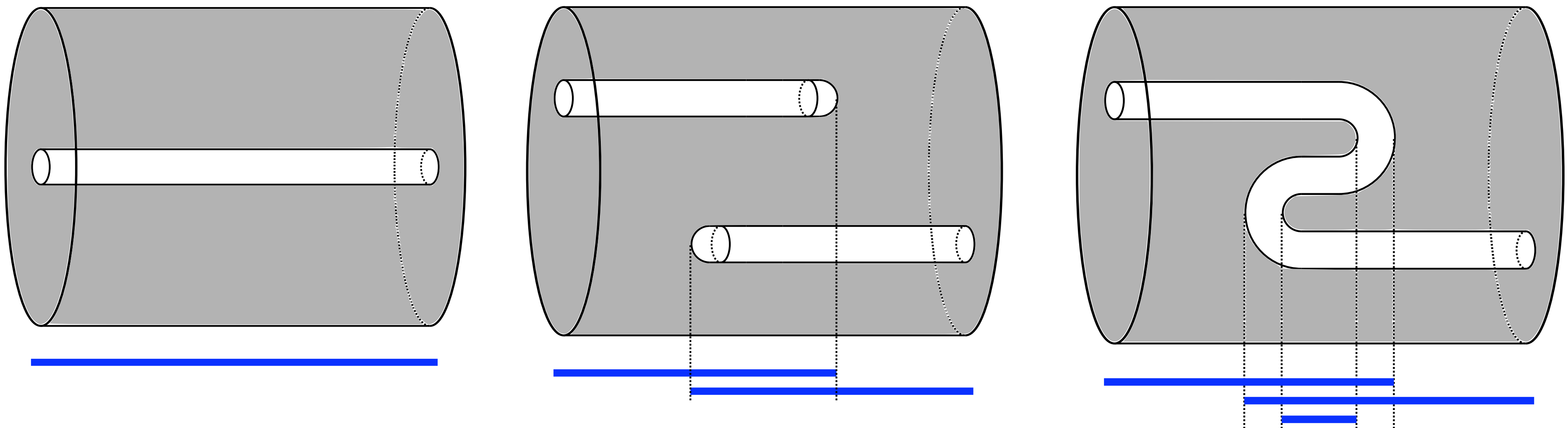

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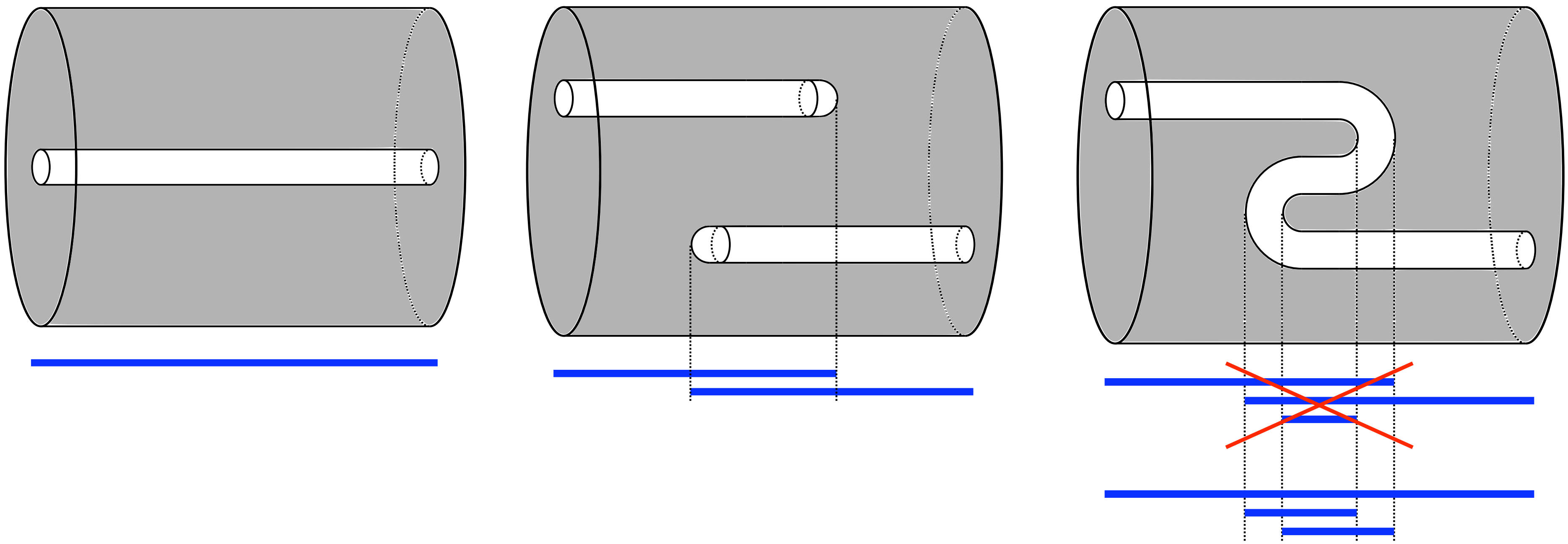

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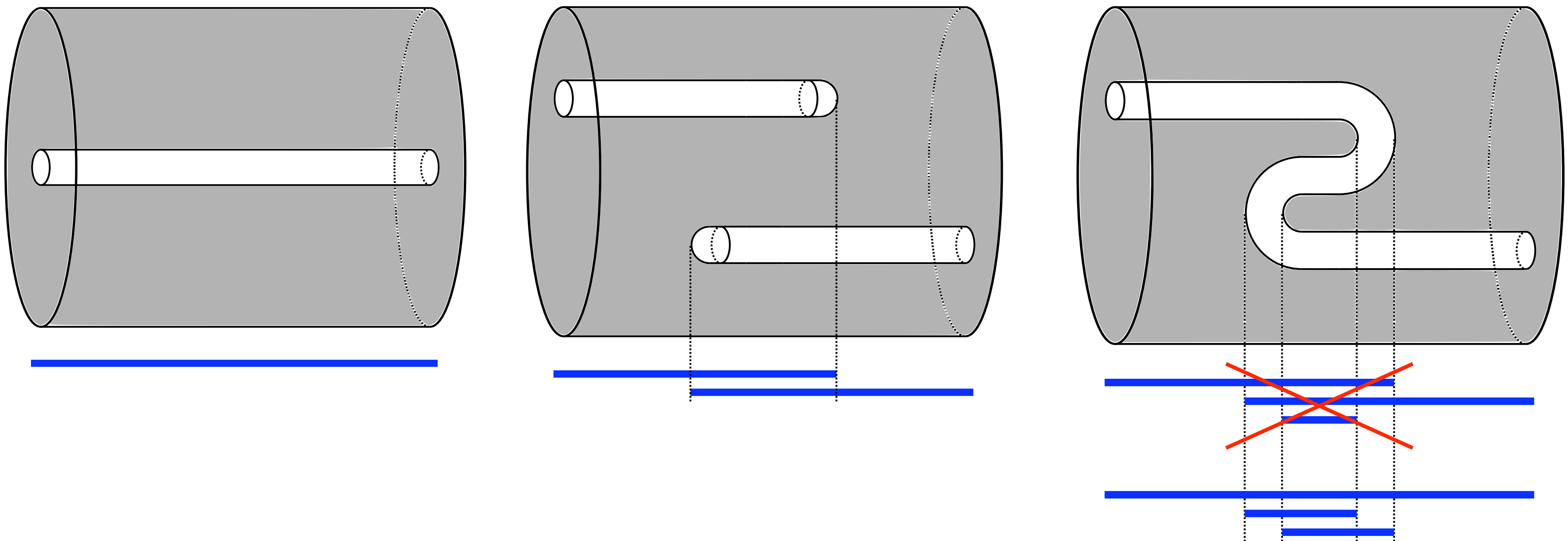

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- Streaming computation.

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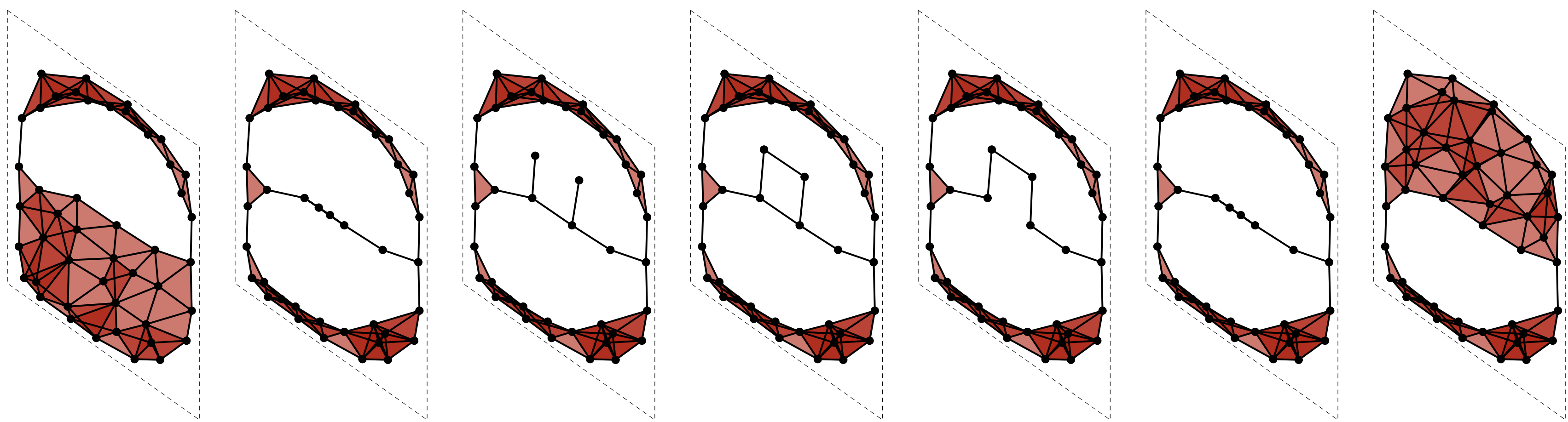
- The time-varying Čech complex of X does not determine if an evasion path exists!



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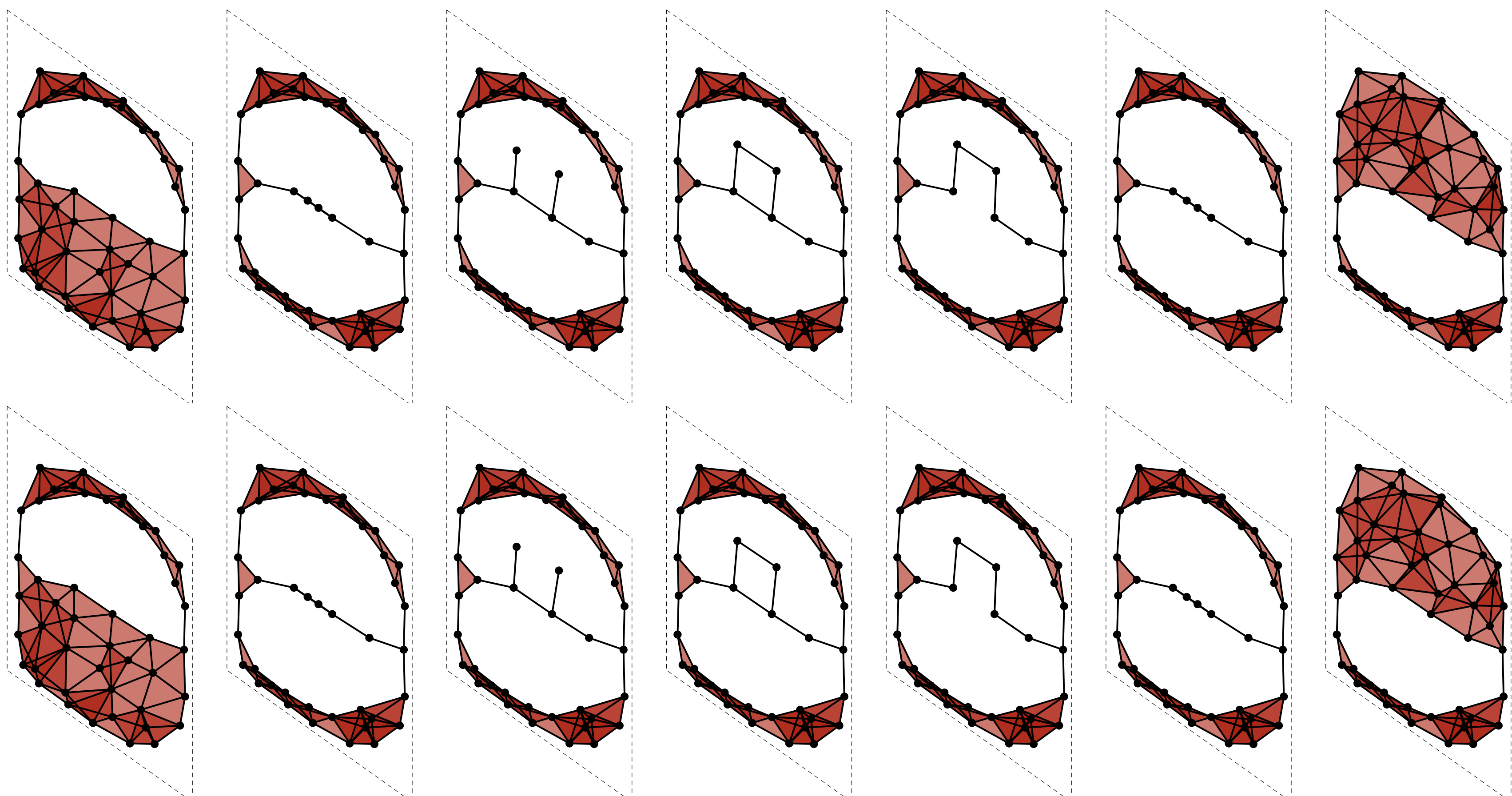

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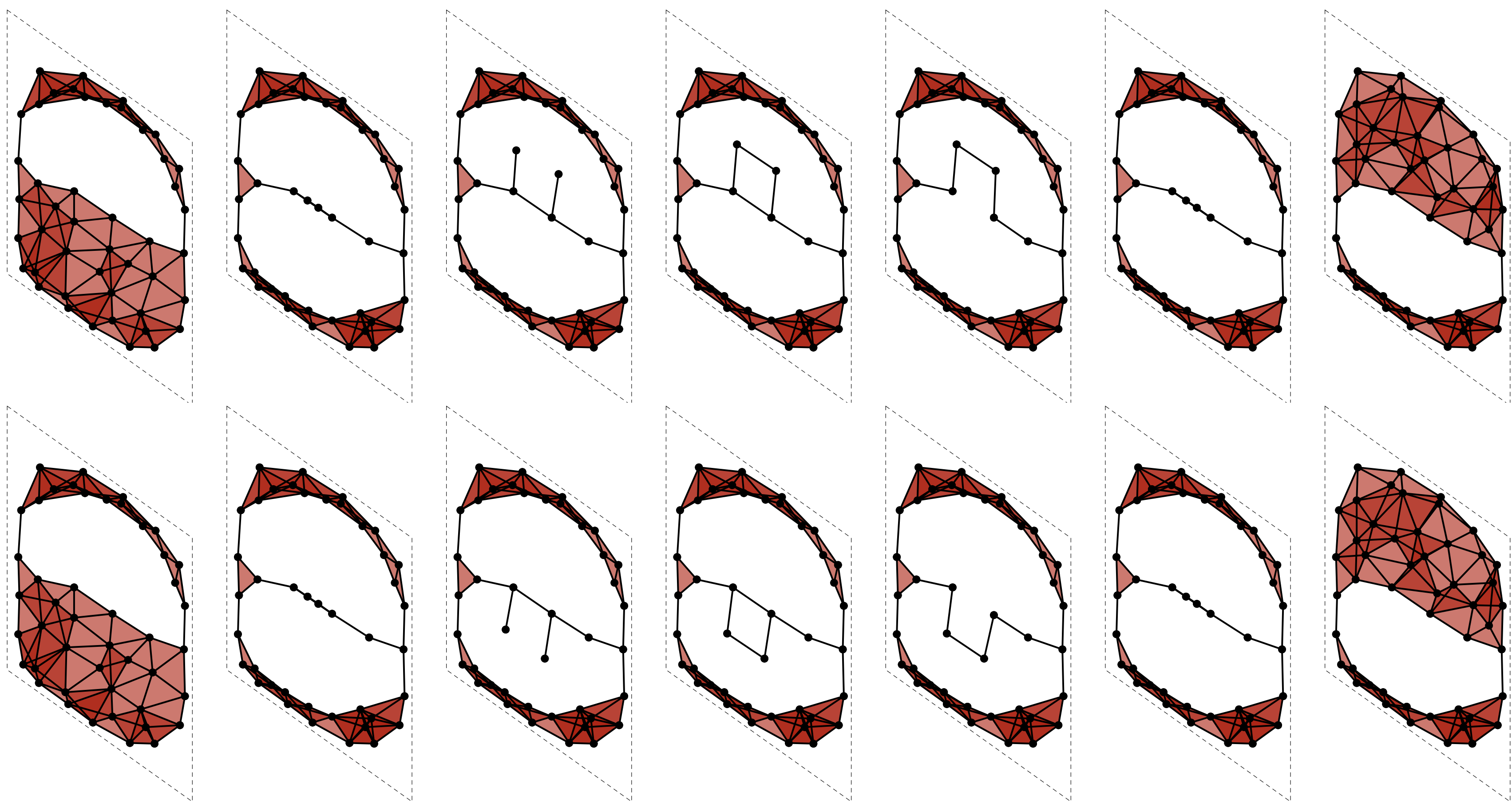

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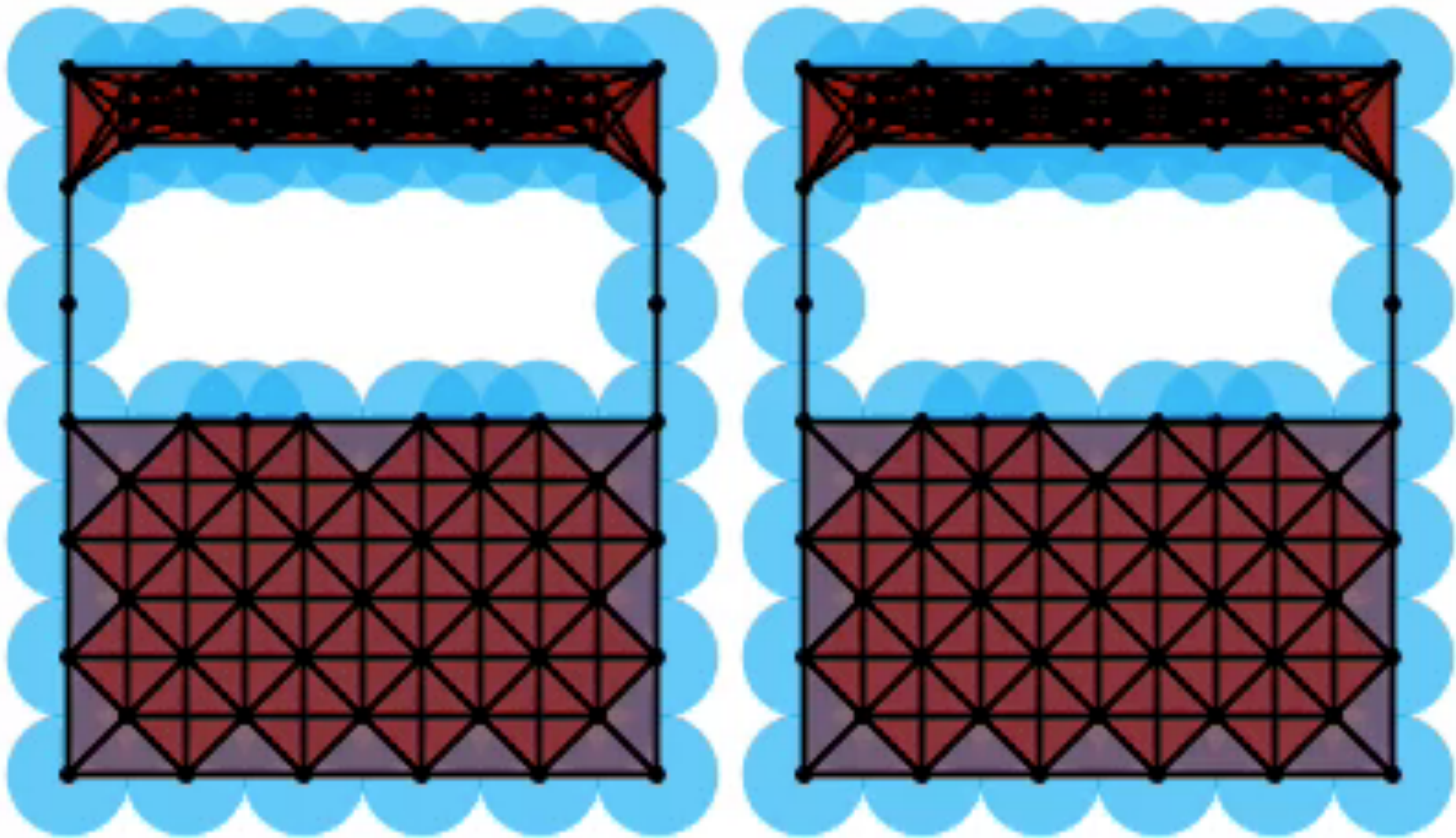

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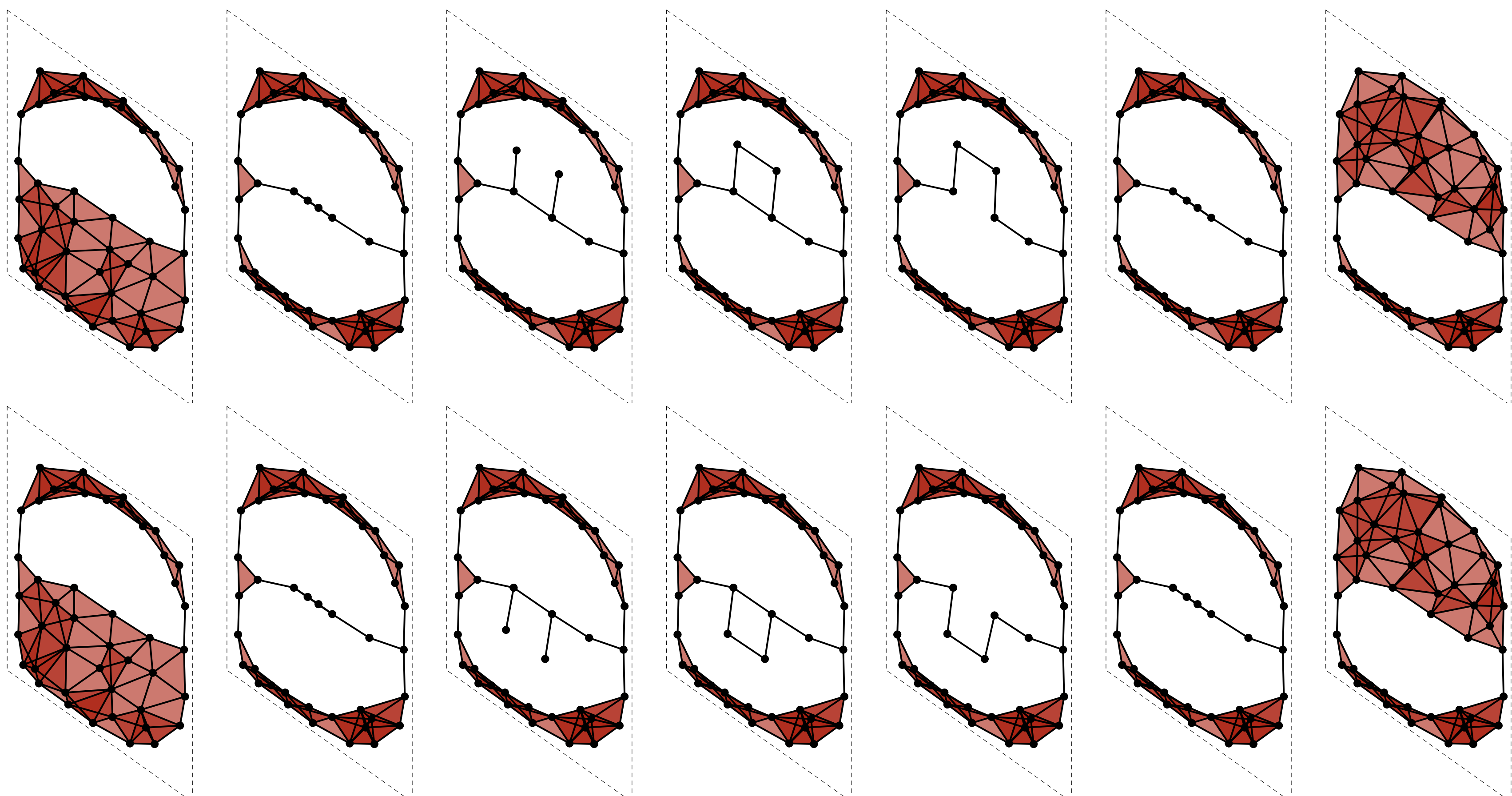

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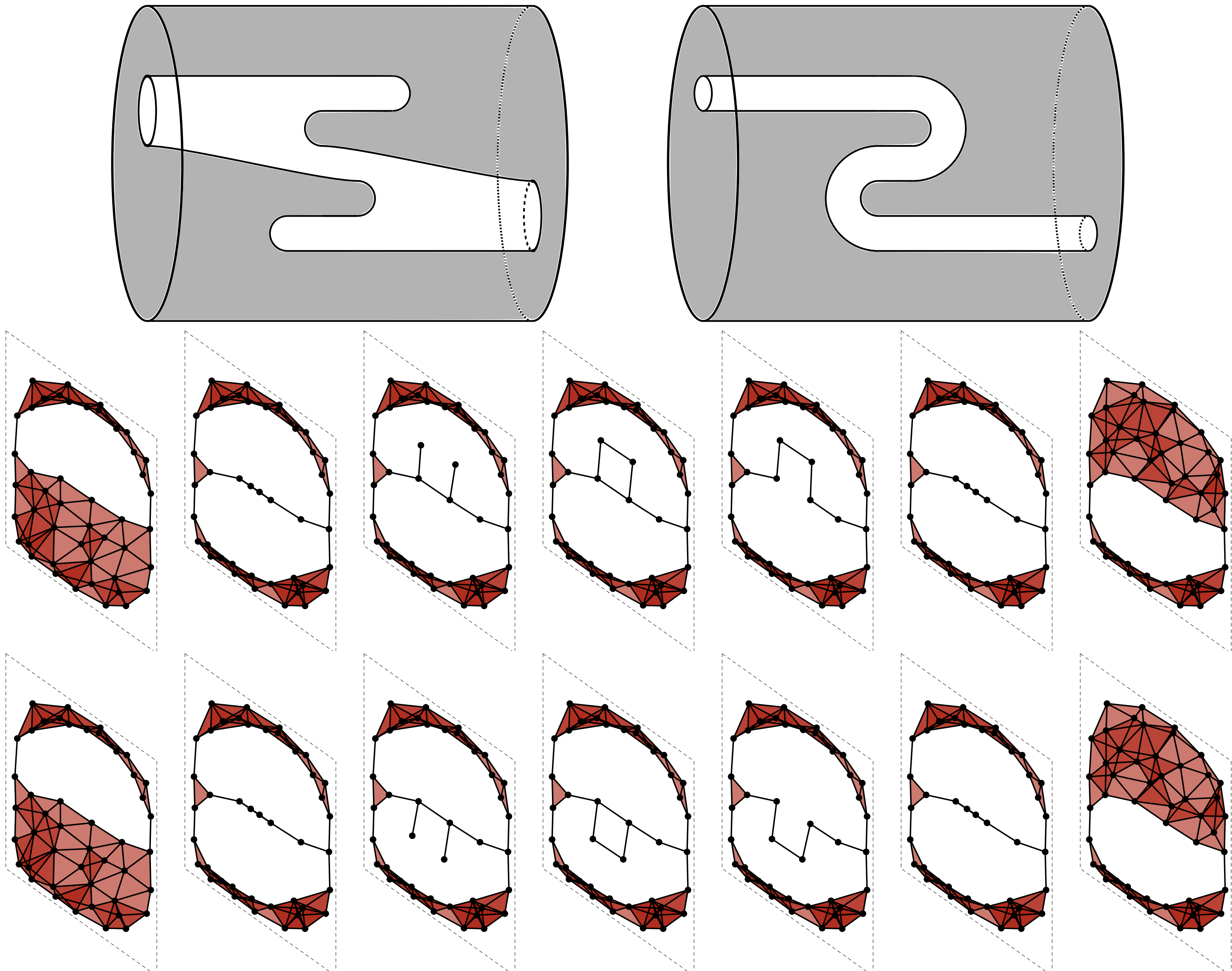

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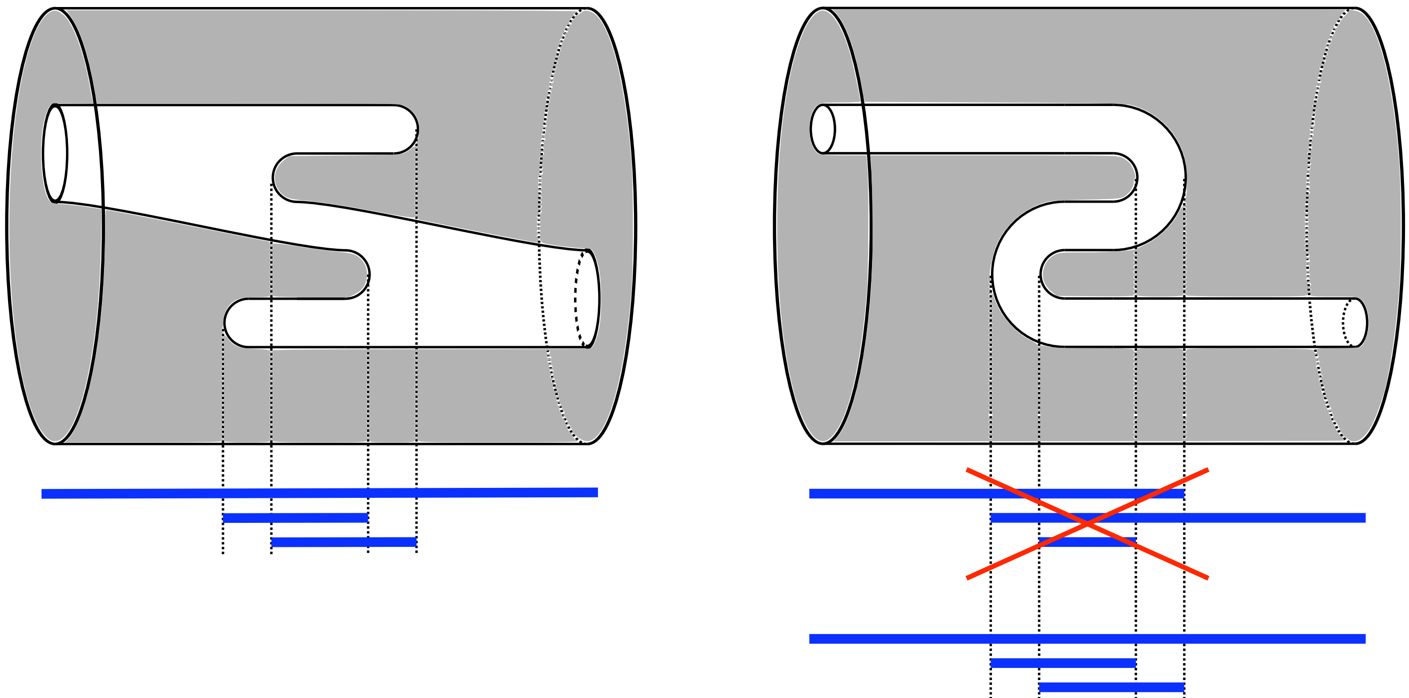

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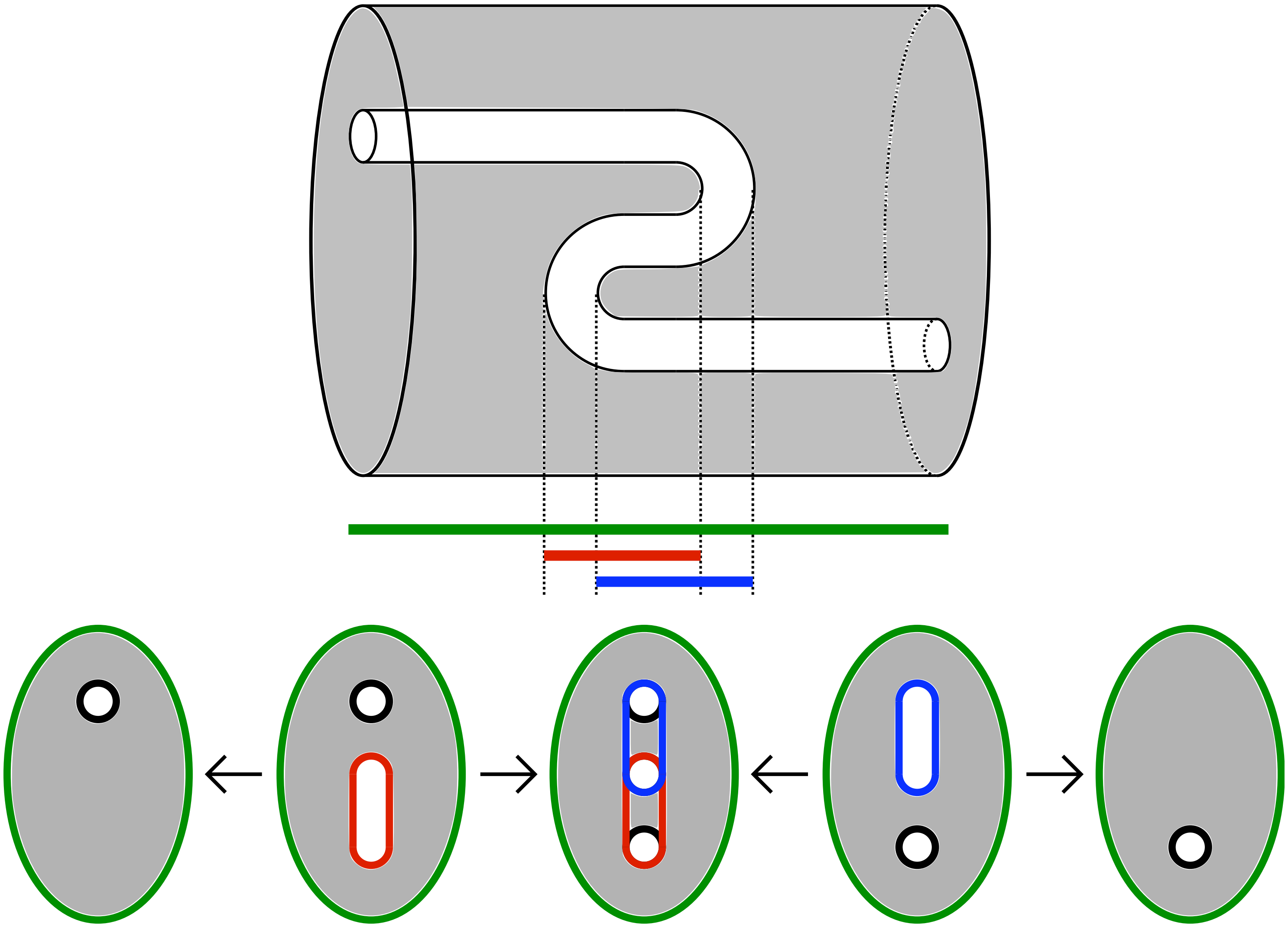

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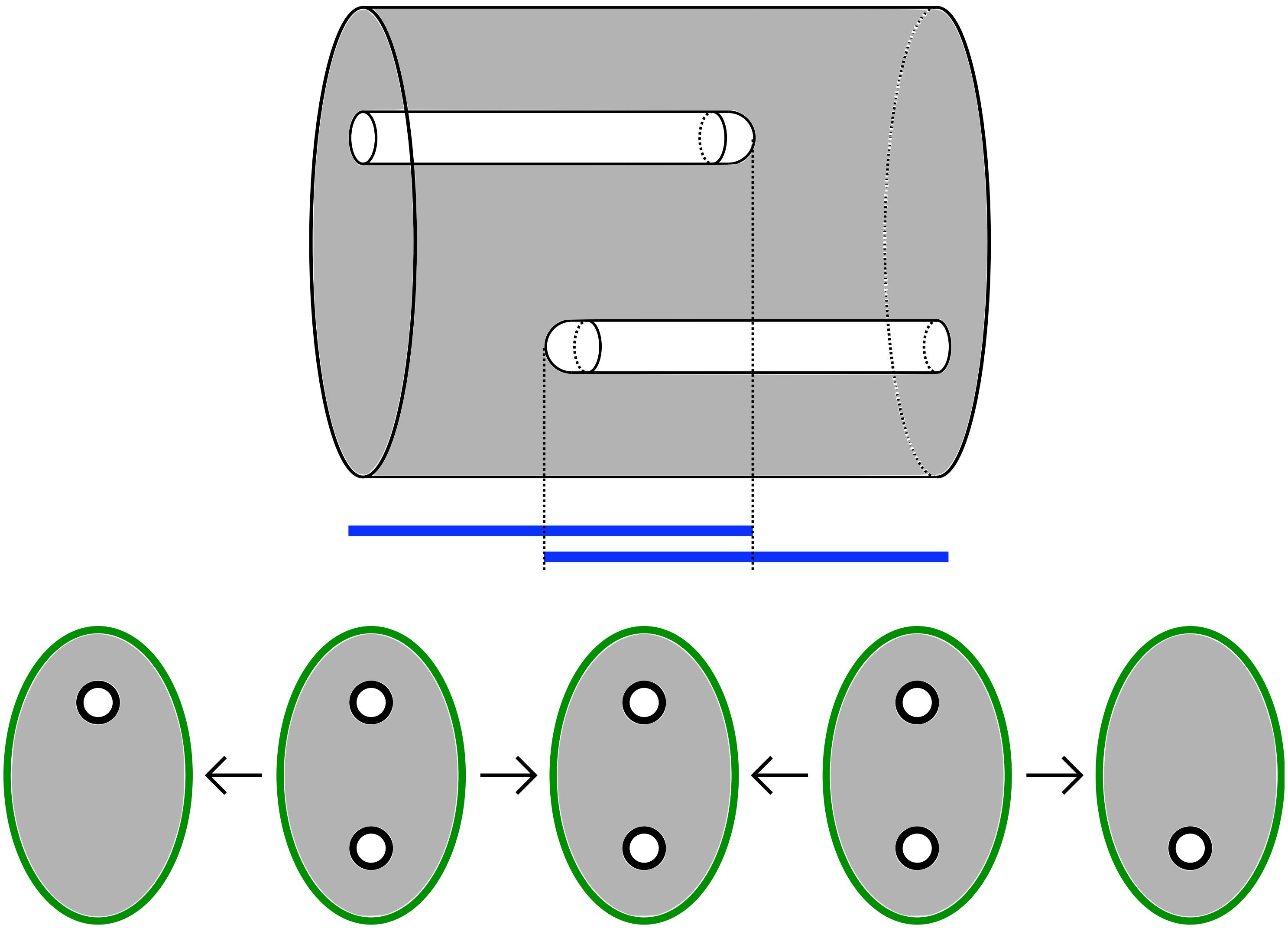

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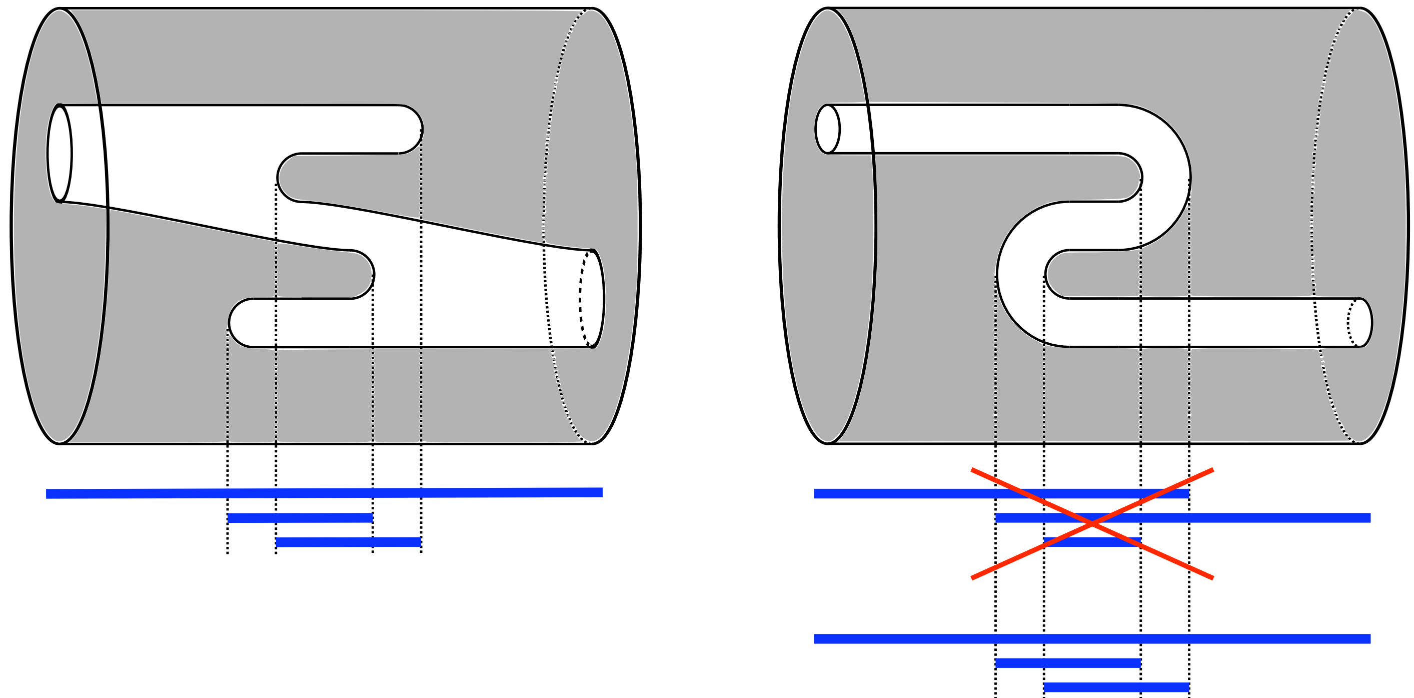

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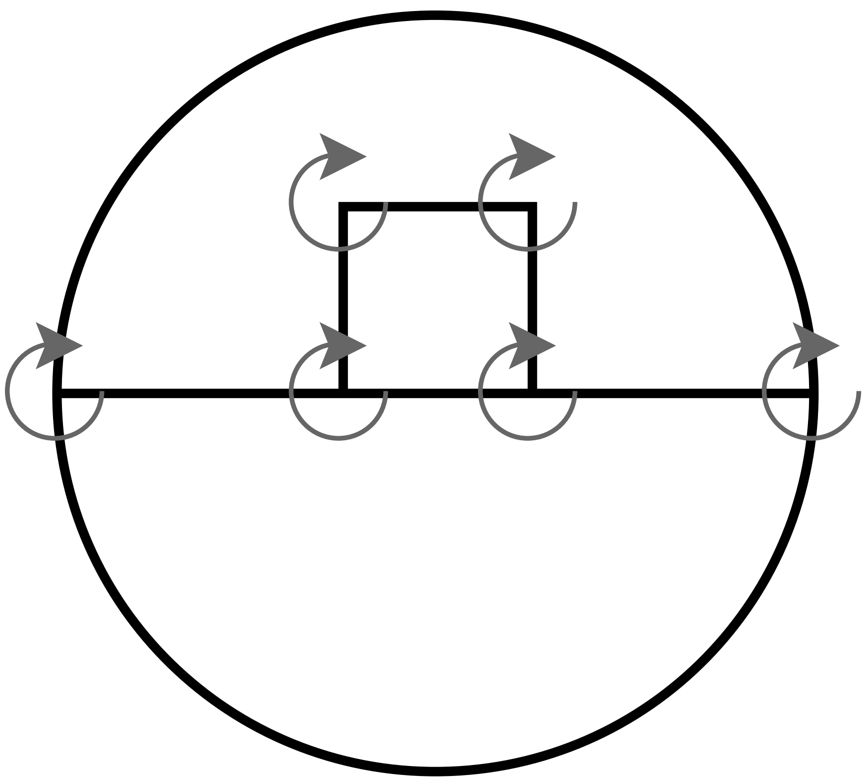
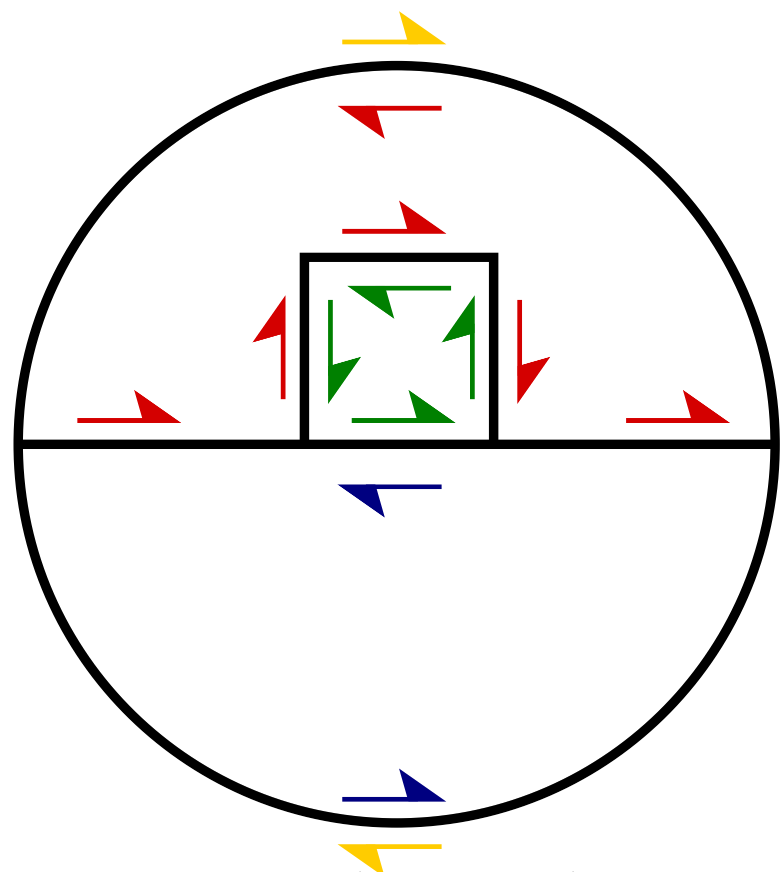
Fat graphs

- A fat graph structure specifies a cyclic ordering of edges about each vertex (left).
- Equivalent to a set of boundary cycles (right).



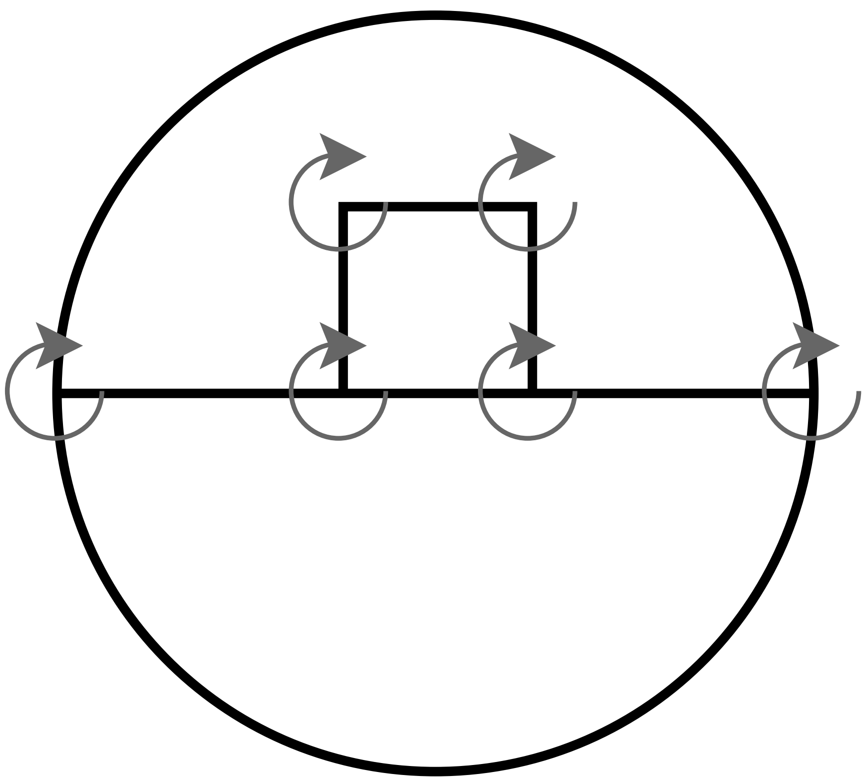
Cyclic orderings



Boundary cycles

Planar sensors measuring cyclic orders

- Theorem. In a planar sensor network that remains connected, the time-varying alpha complex with rotation information determines if an evasion path exists.



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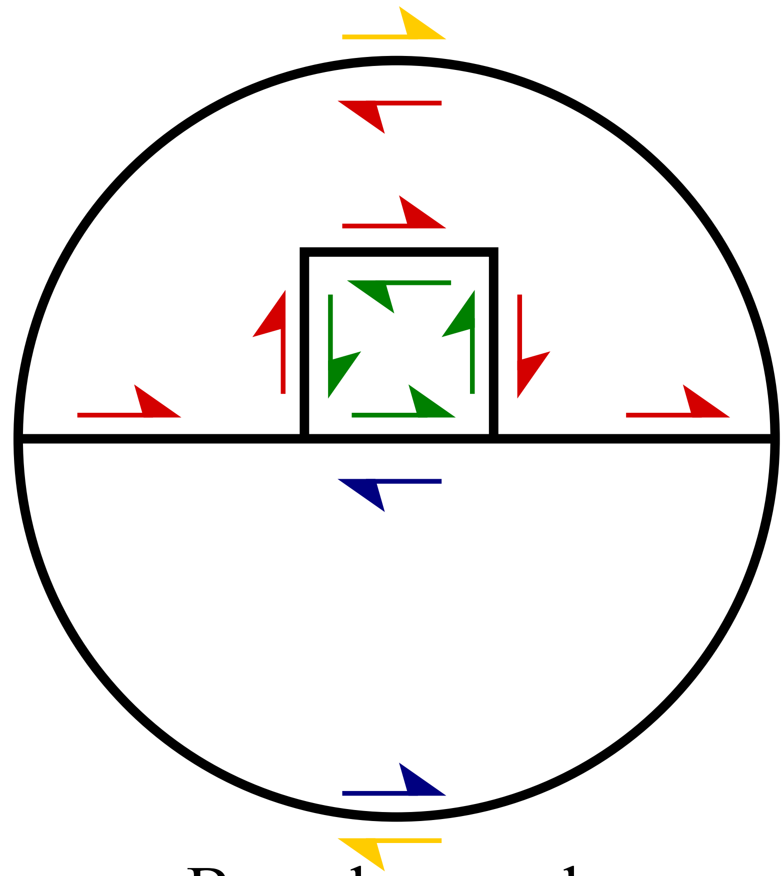

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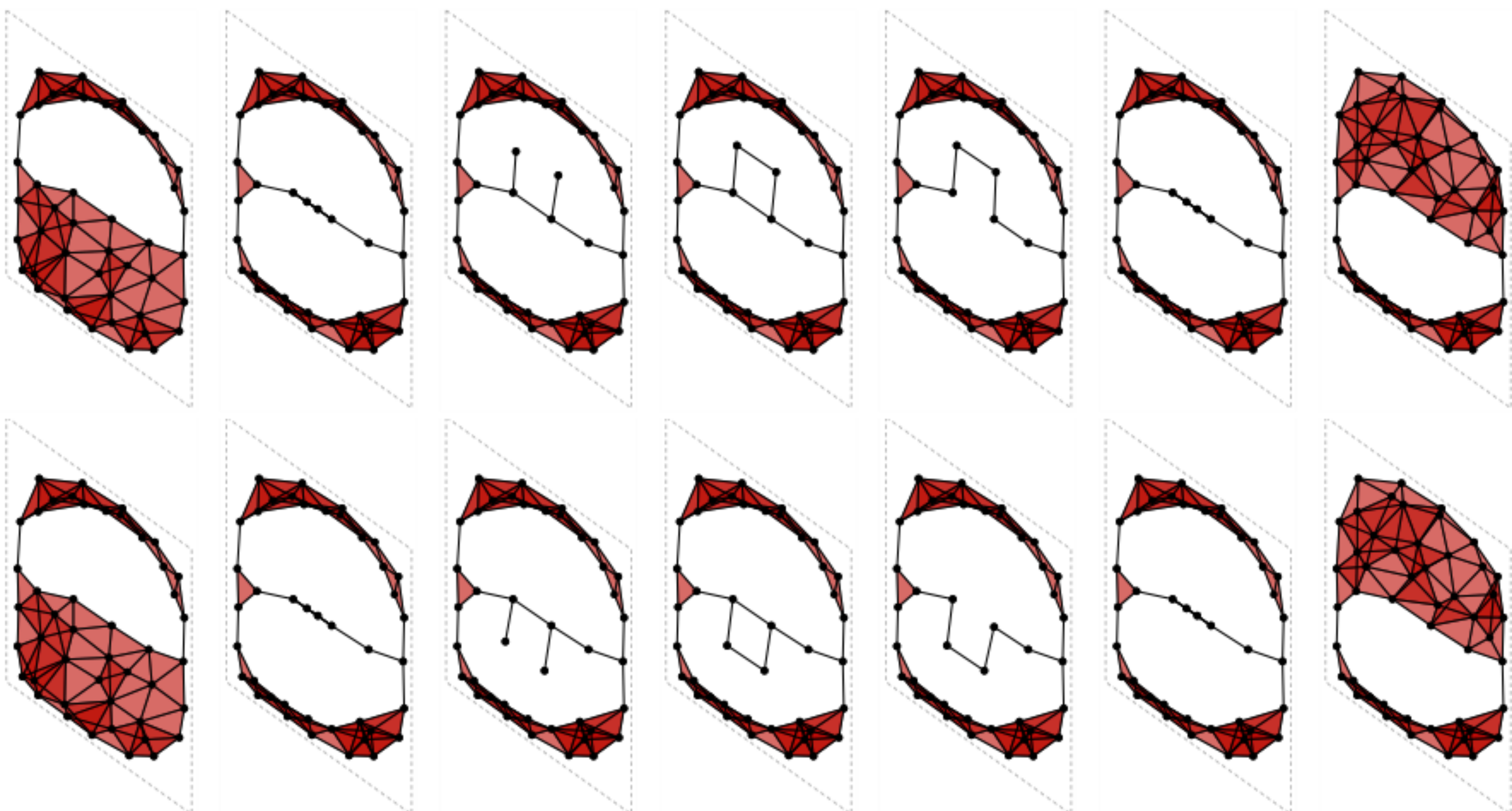

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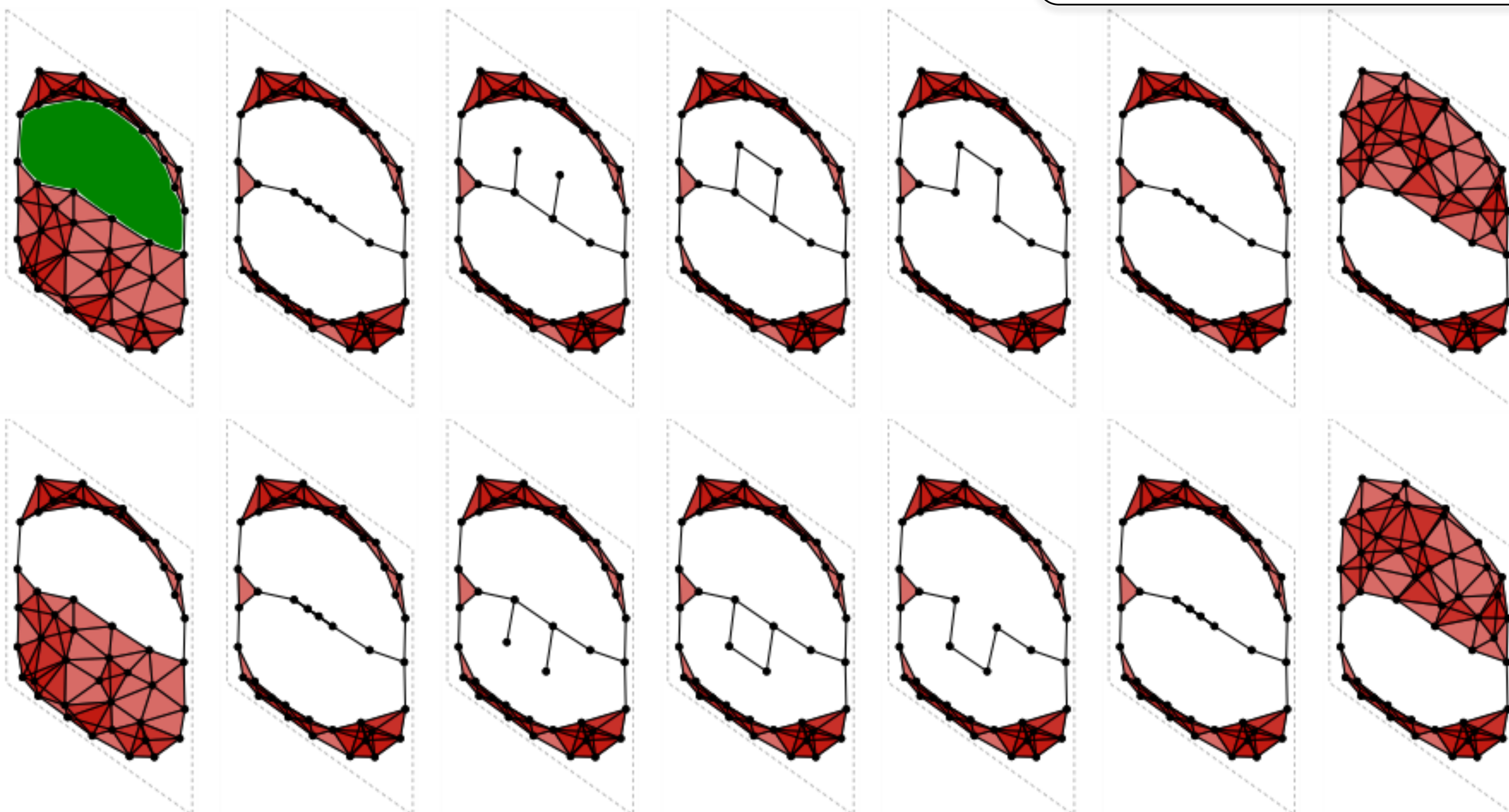
May contain an intruder



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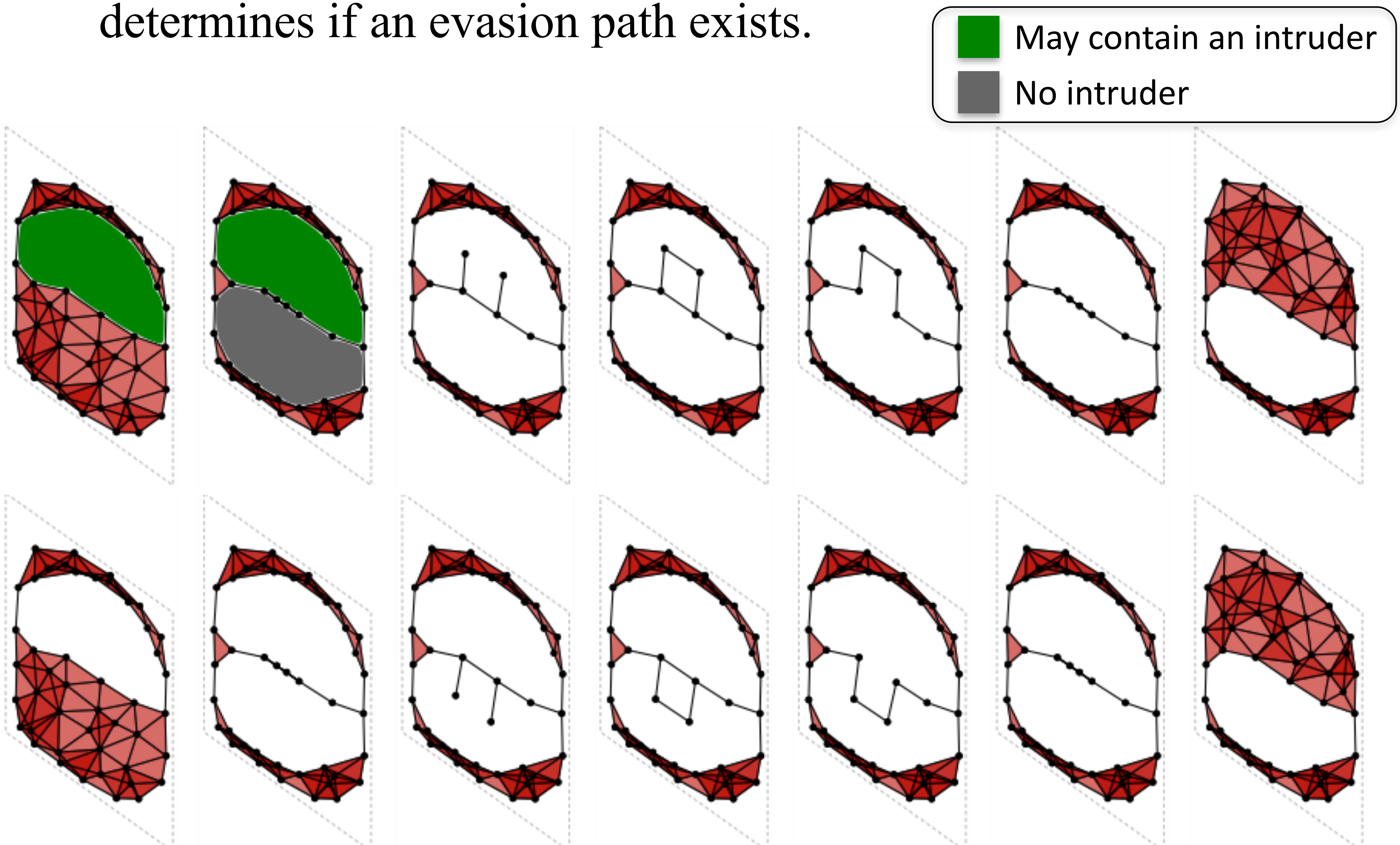

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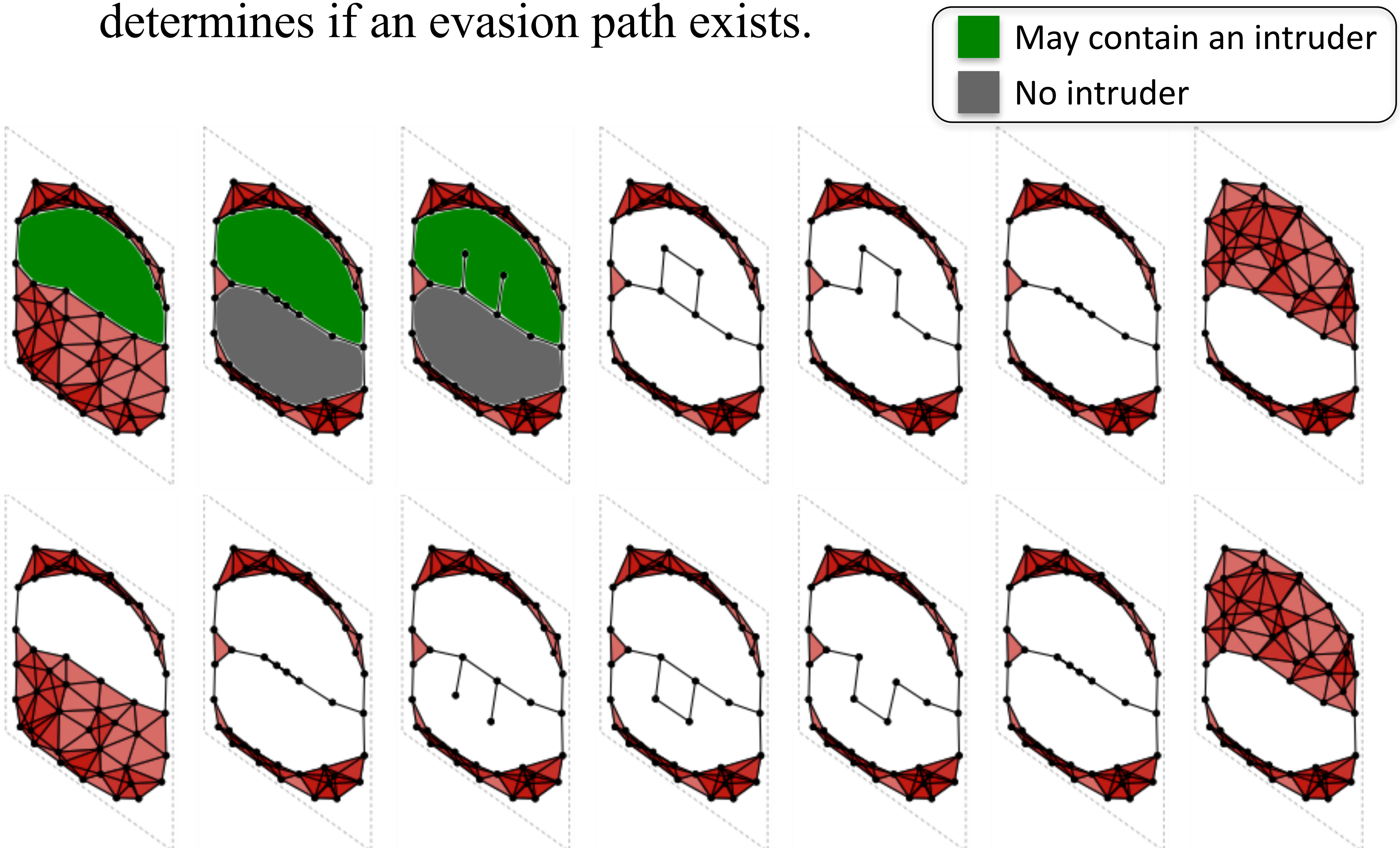

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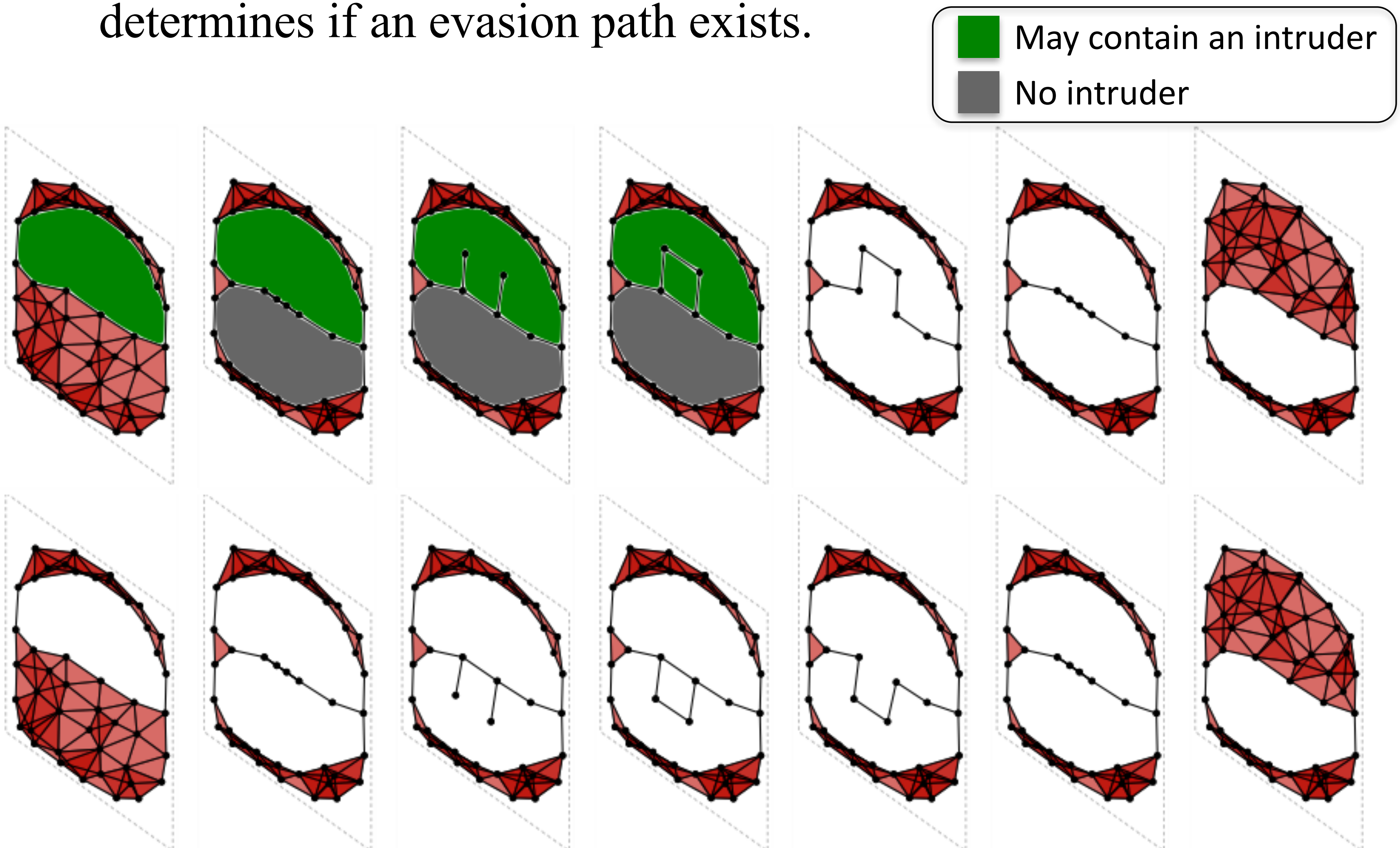

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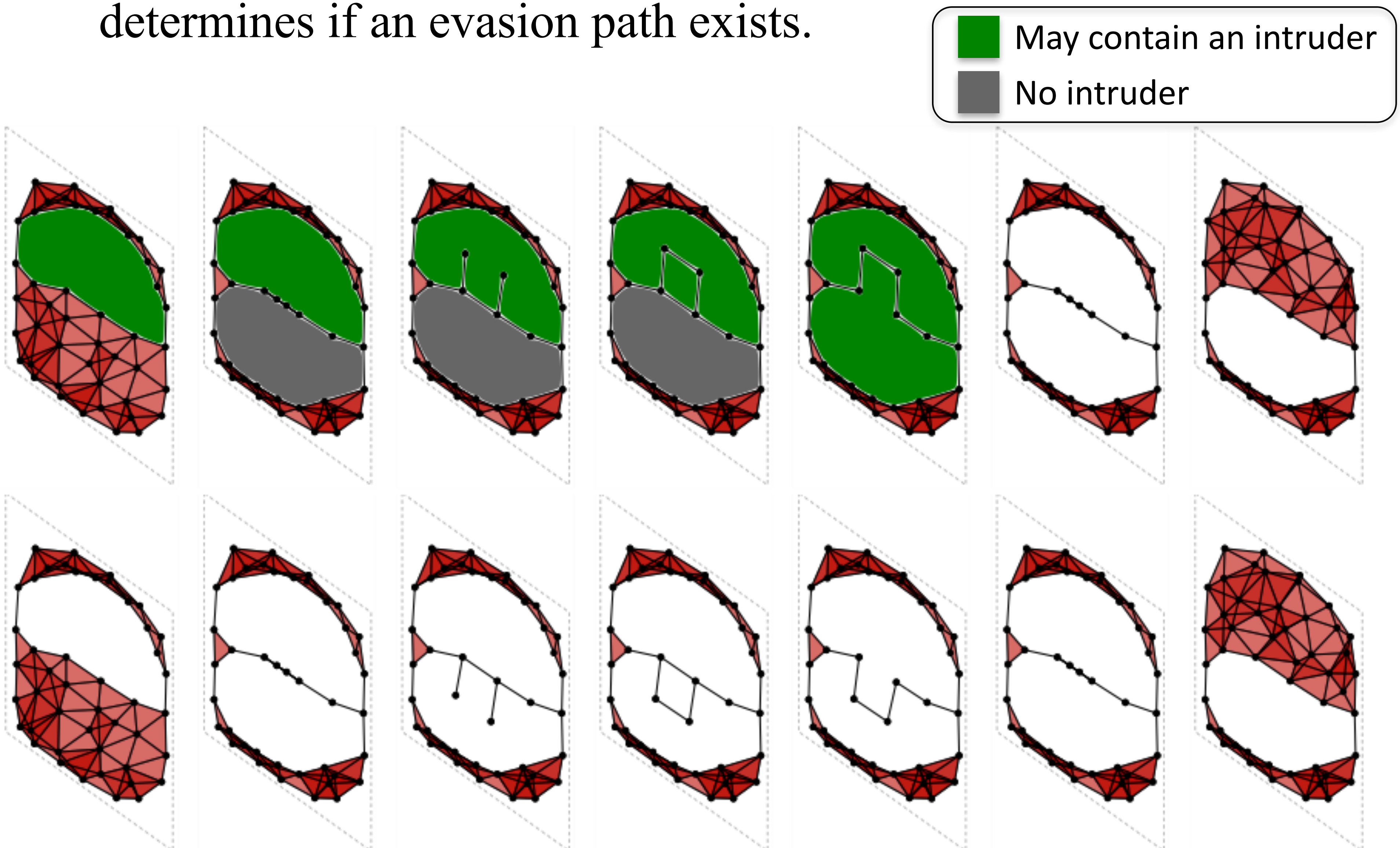

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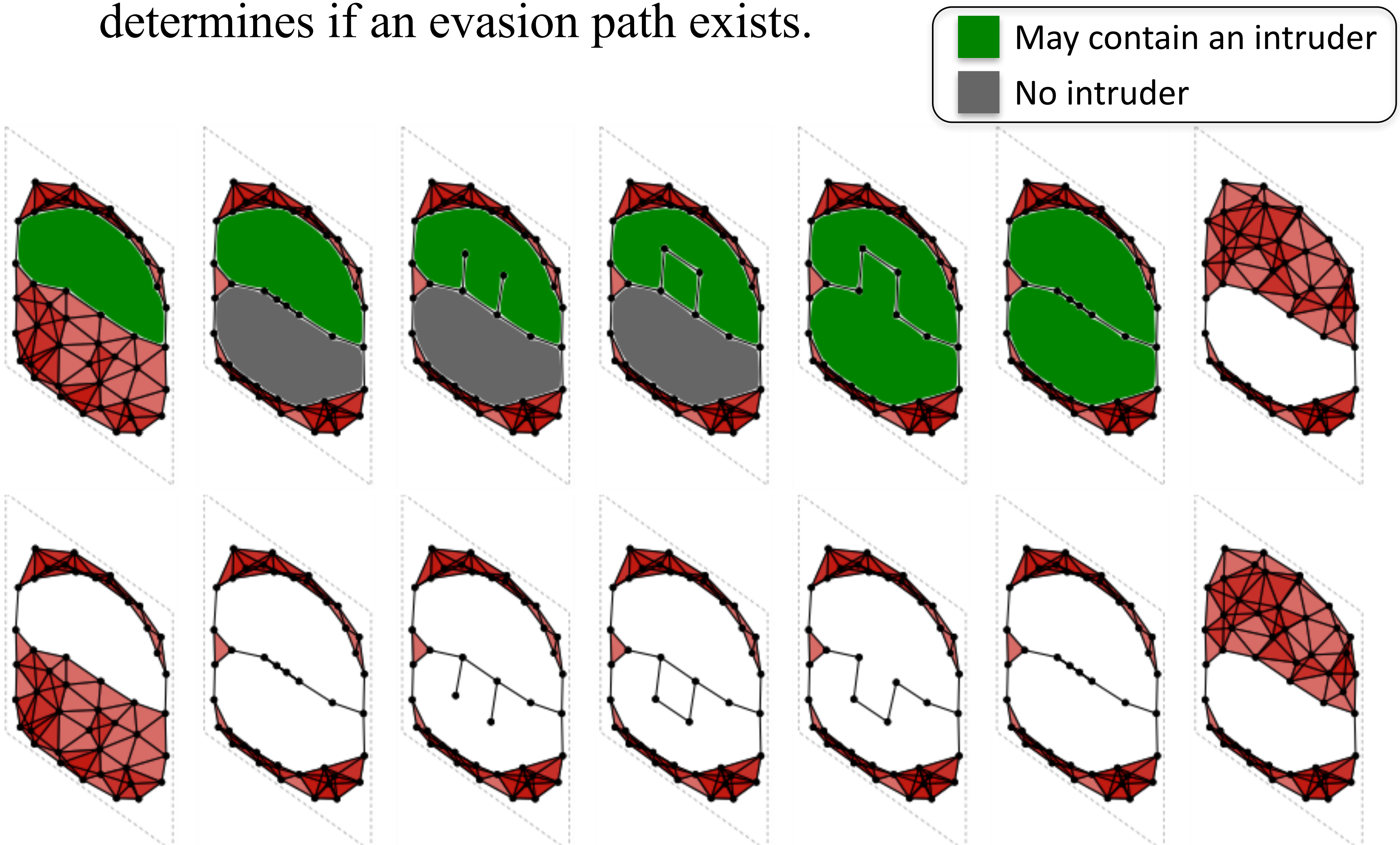

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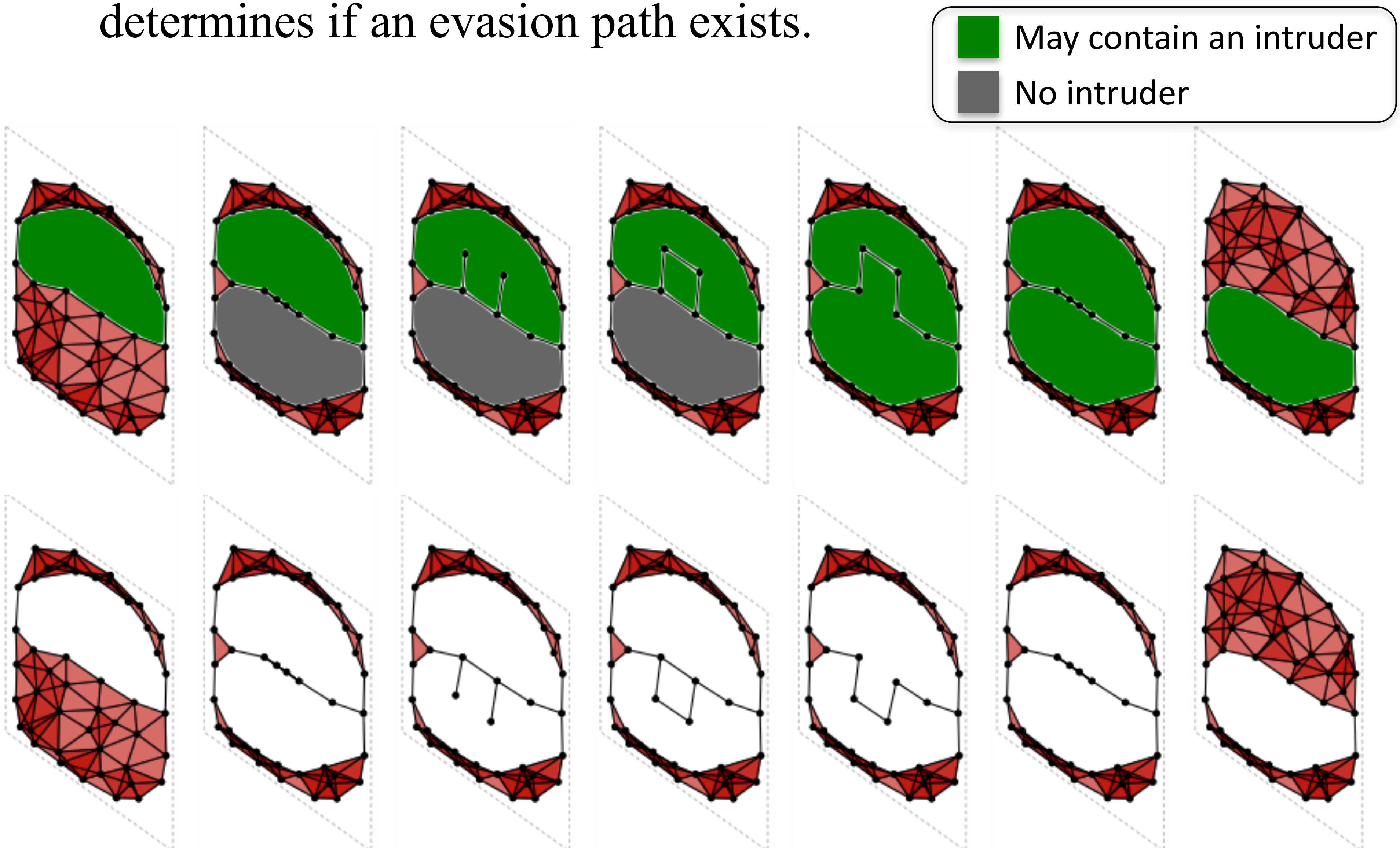

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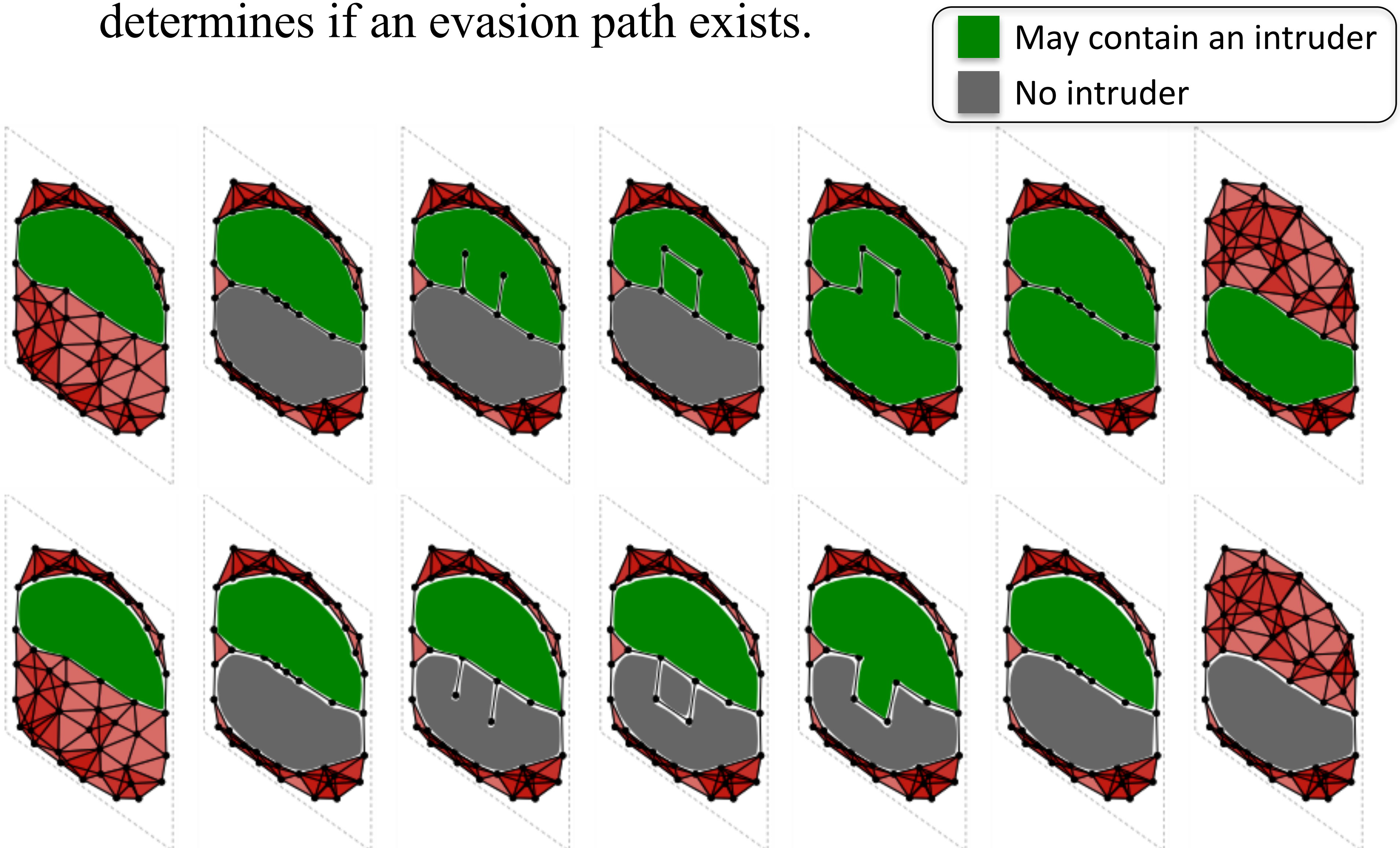

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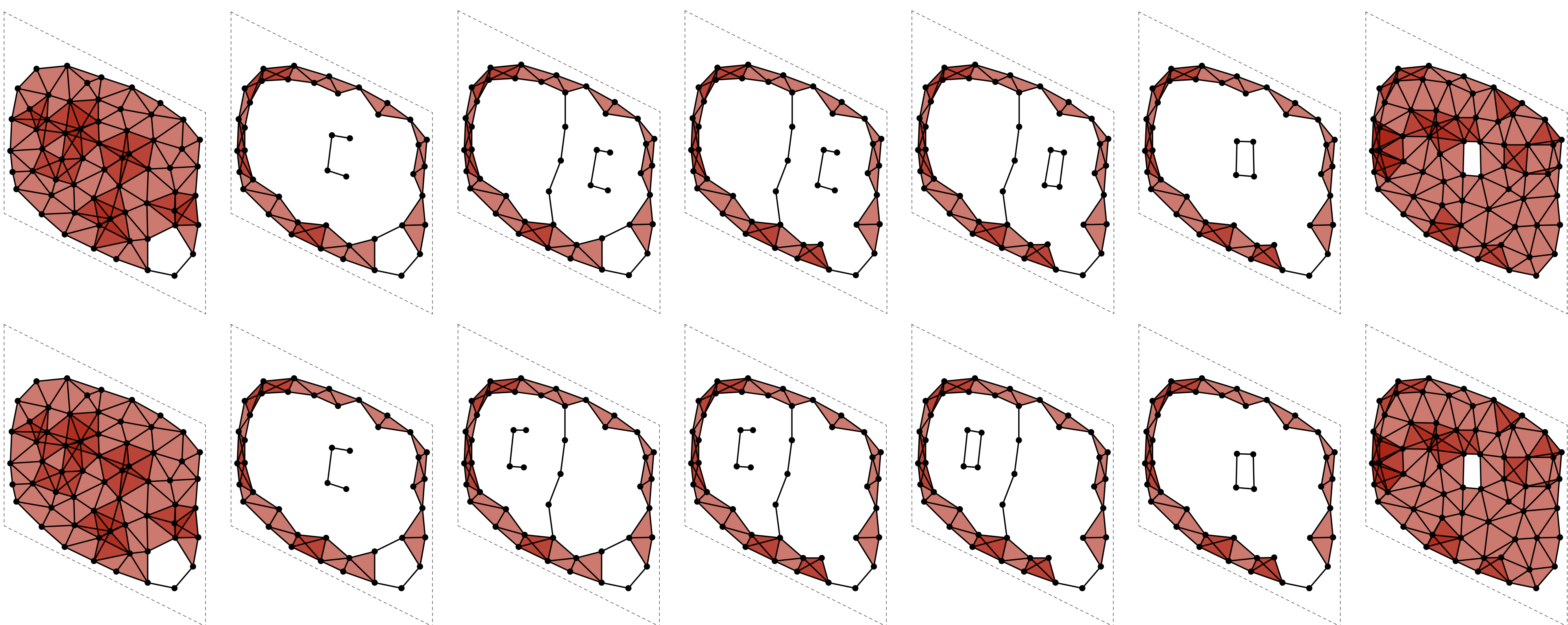
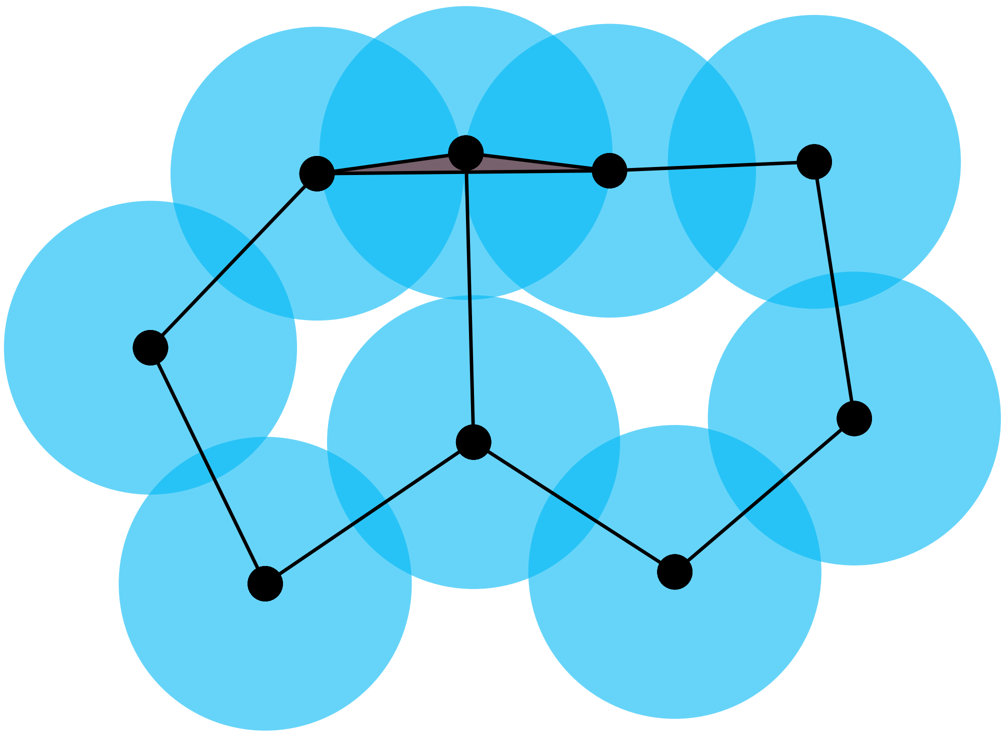
May contain an intruder

No intruder

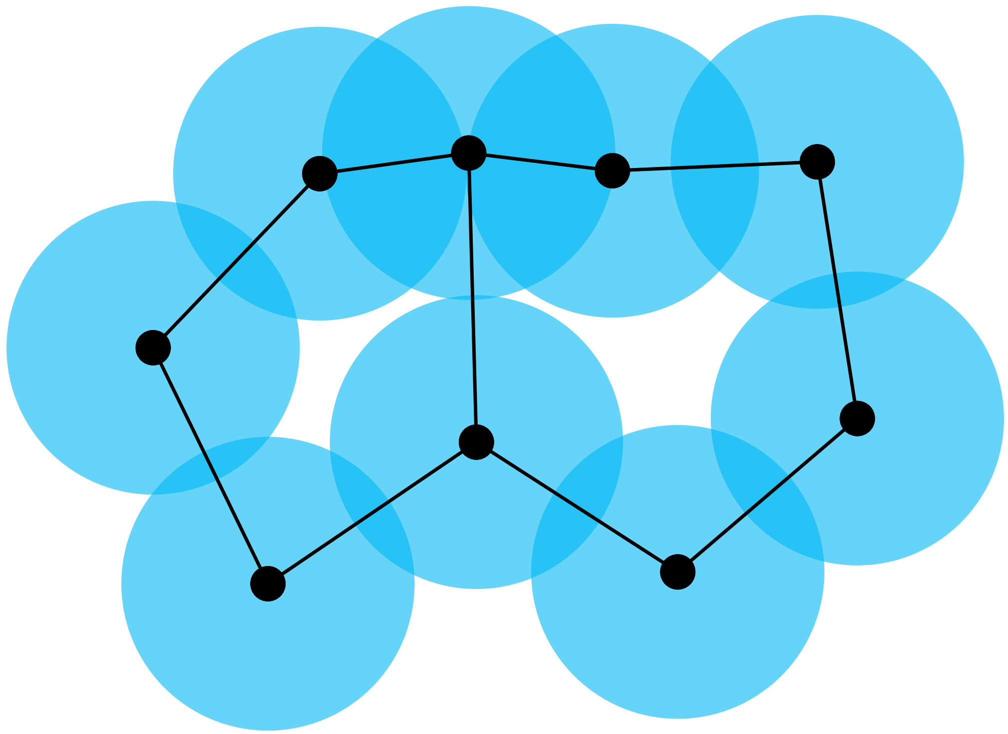


Planar sensors measuring cyclic orders

- Theorem. In a planar sensor network that remains connected, the time-varying alpha complex with rotation information determines if an evasion path exists.



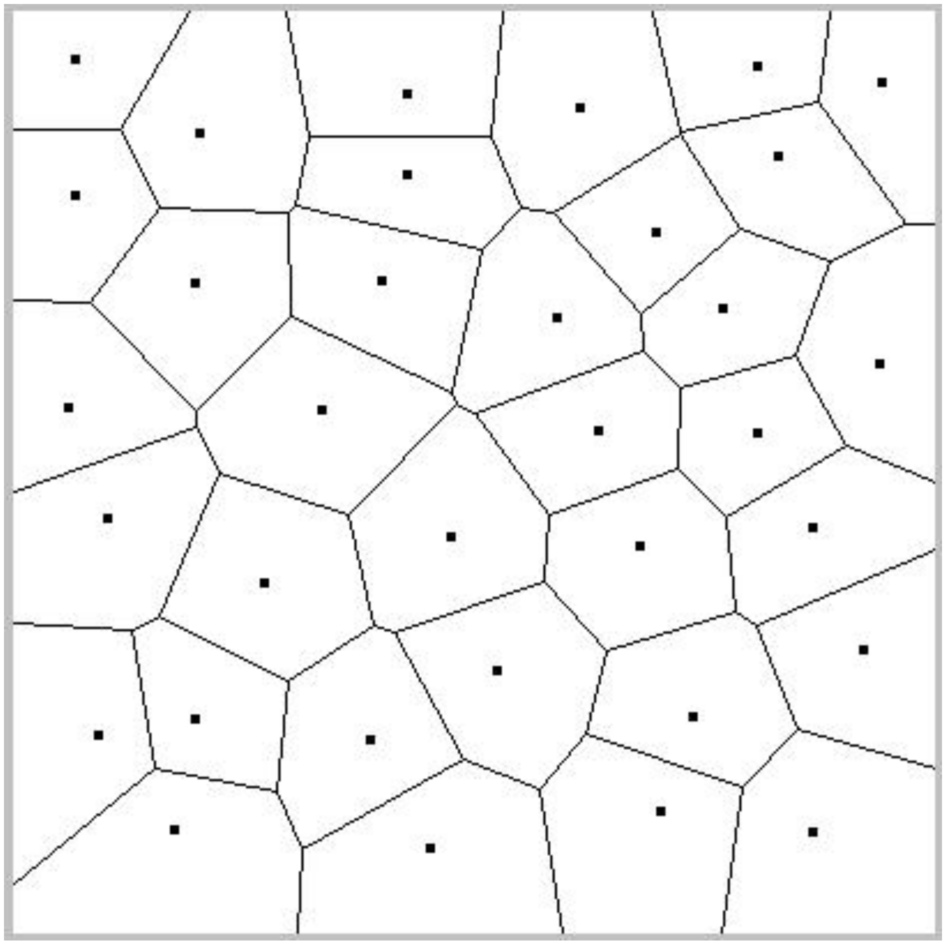
Čech



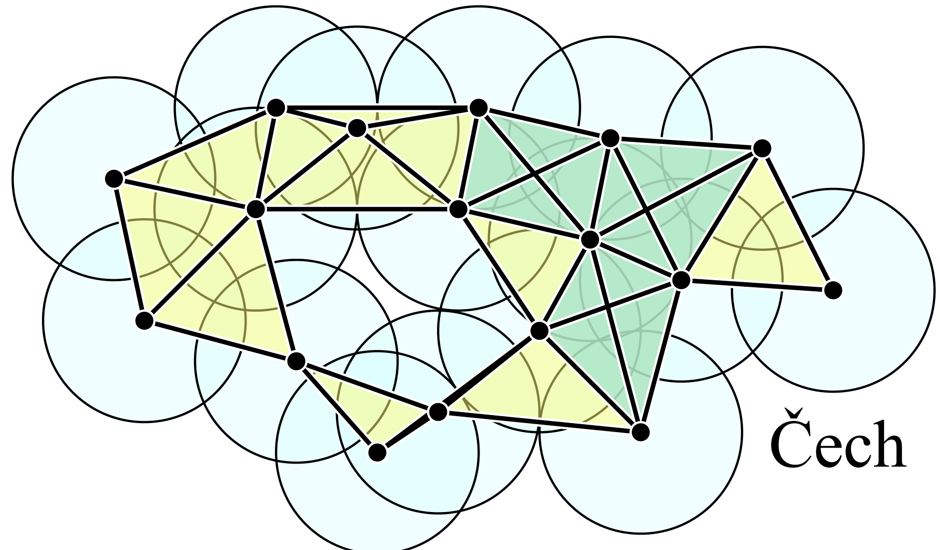
Alpha

Planar sensors measuring cyclic orders

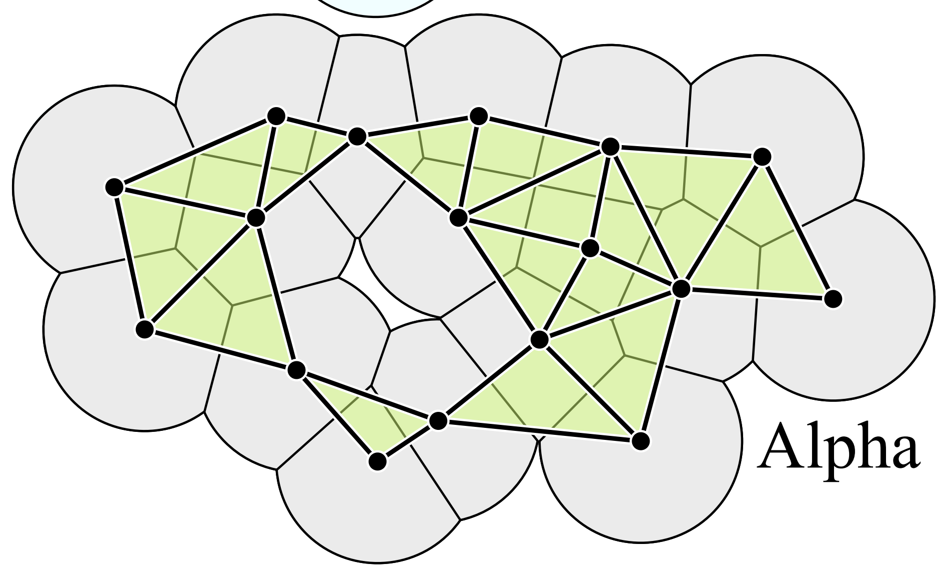
- <u>Theorem.</u> In a planar sensor network that remains connected, the time-varying alpha complex with rotation information determines if an evasion path exists.



Voronoi regions



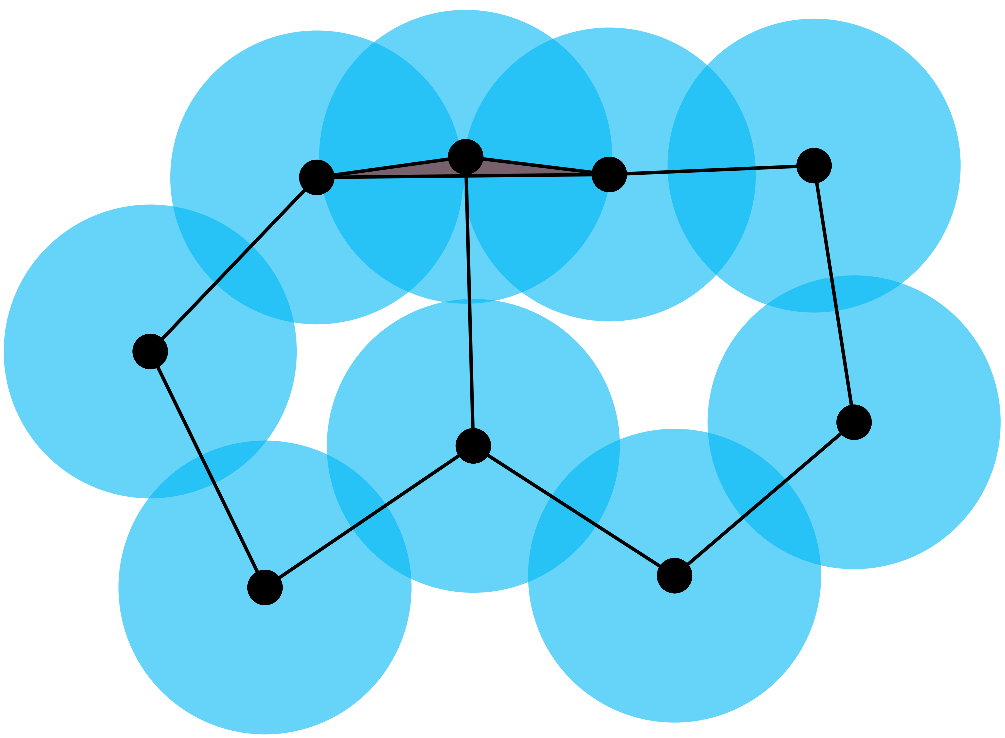
Čech



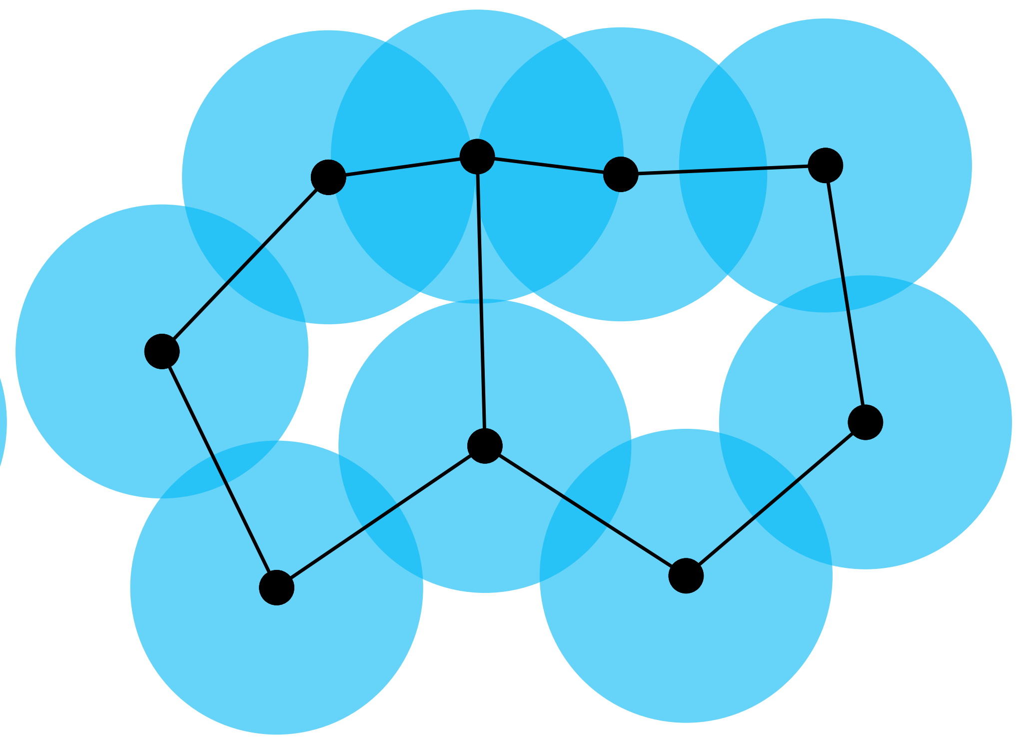
Alpha

Planar sensors measuring cyclic orders

- <u>Theorem.</u> In a planar sensor network that remains connected, the time-varying alpha complex with rotation information determines if an evasion path exists.
- <u>Open question.</u> Is the Čech complex with rotation information sufficient?



Čech



Alpha

Planar sensors measuring cyclic orders

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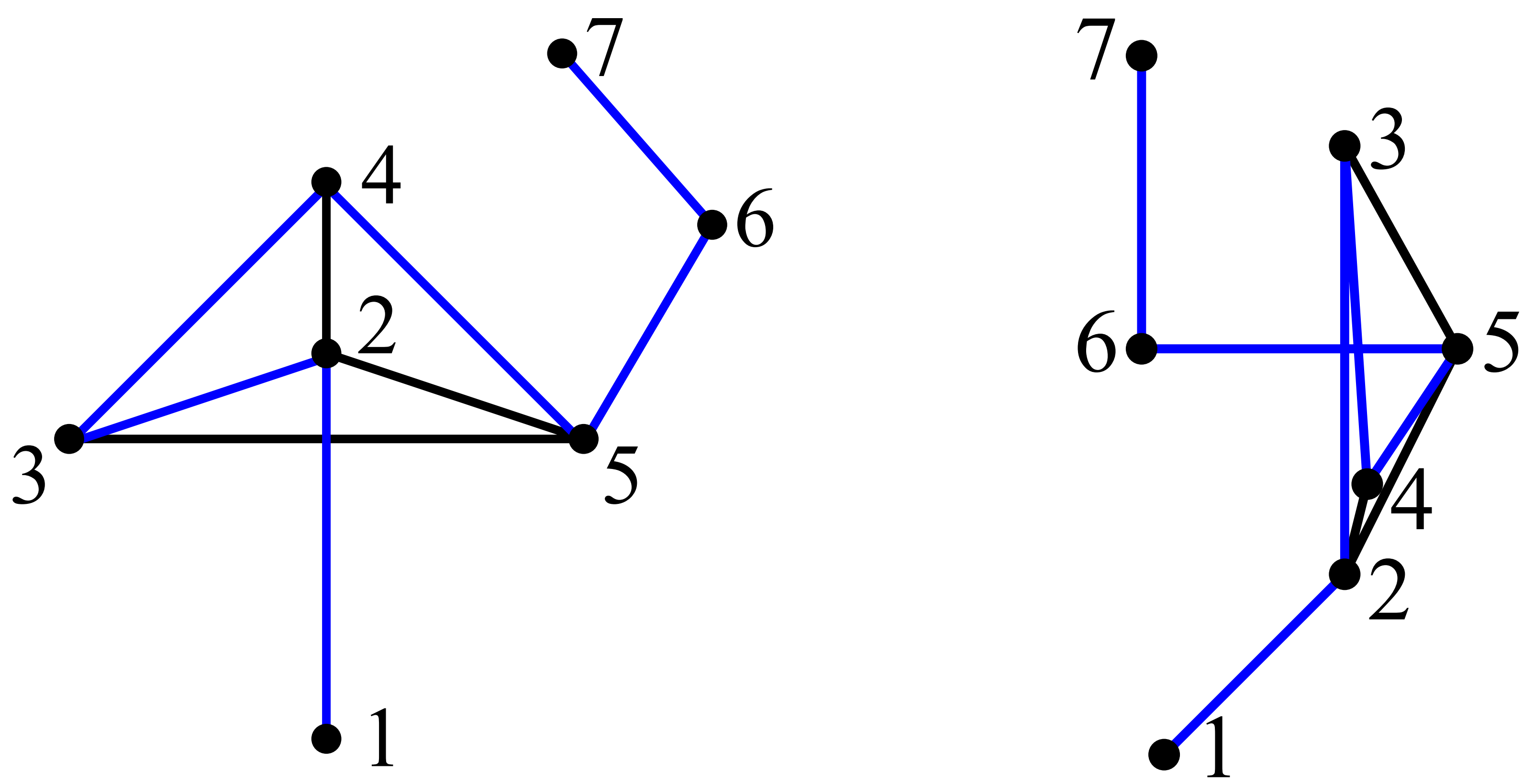

Planar sensors measuring cyclic orders

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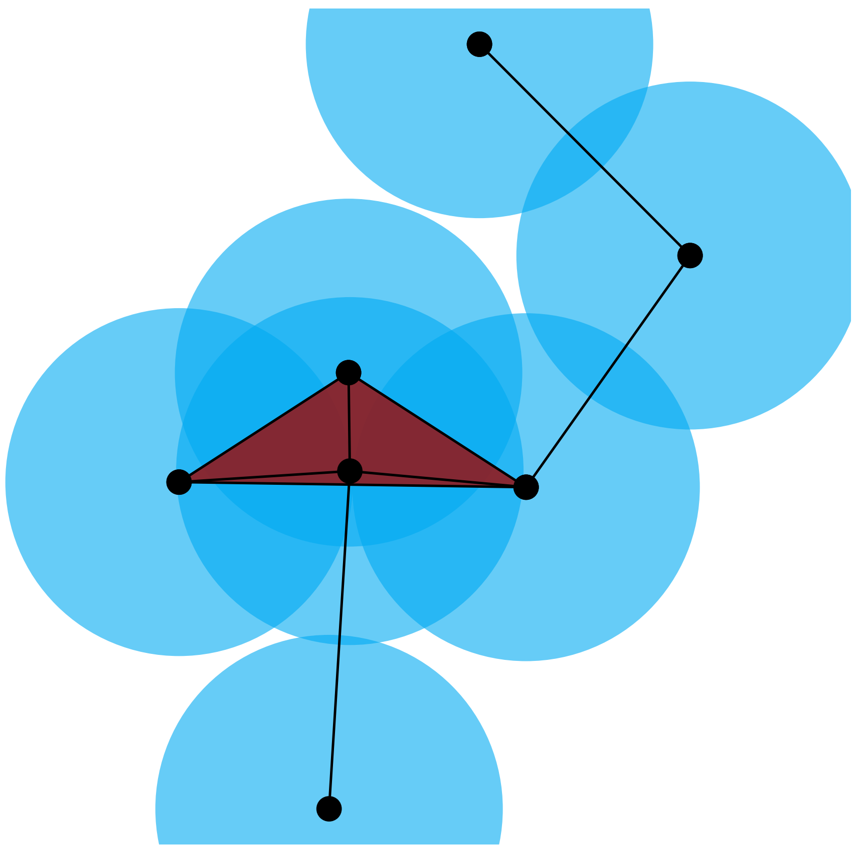
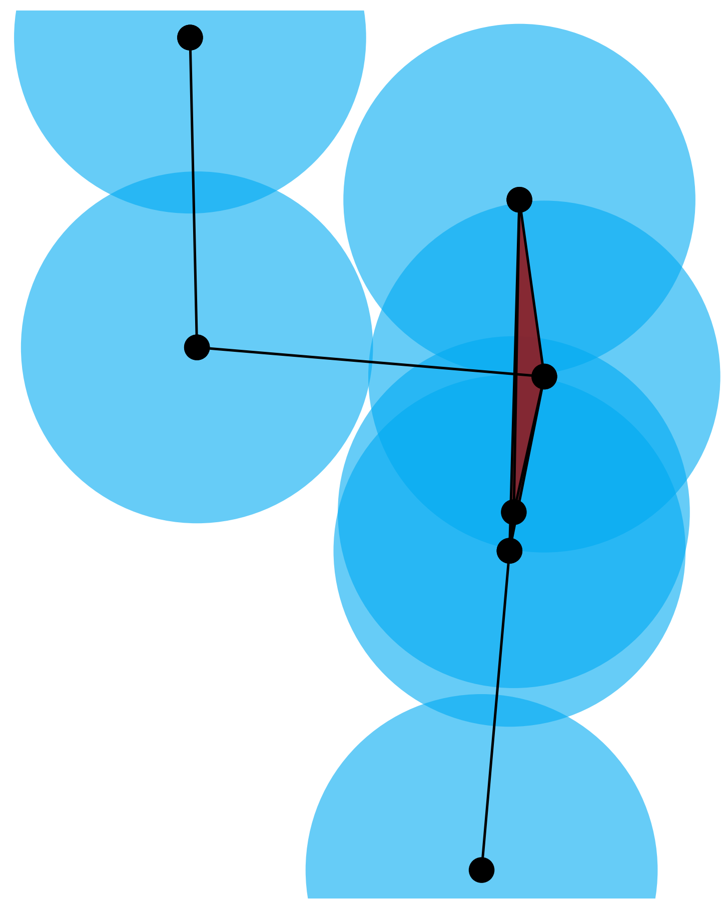
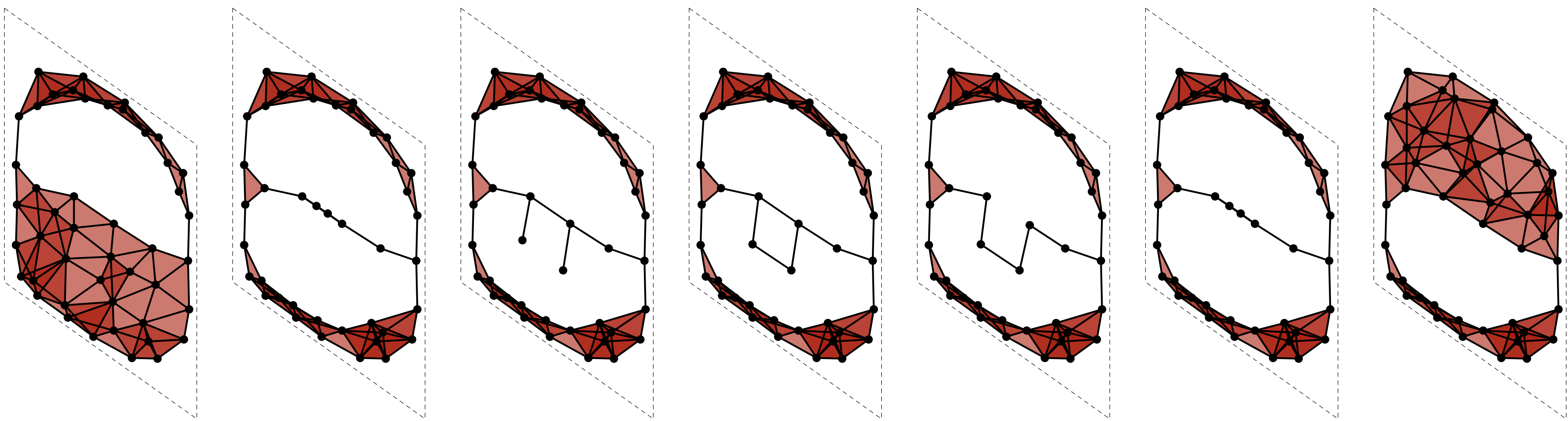




Conclusions for Part III

- There is a streaming one-sided criterion for the evasion problem using zigzag persistence.
- Cech complex insufficient.
  Alpha complex with rotation information suffices.
  What about the Čech complex with rotation information?



Where can I find resources if I am interested in applied topology?

- You may be interested in the Applied Algebraic Topology Research Network. Become a member to receive email invites to the online research seminars. Recorded talks are available at the YouTube Channel. There is also a forum.
- Another source of applied topology news is appliedtopology.org.
- A second online research seminar is GEOTOP-A: Applications of Geometry and Topology.
- Mailing lists with announcements in applied topology include WinCompTop and ALGTOP-L.

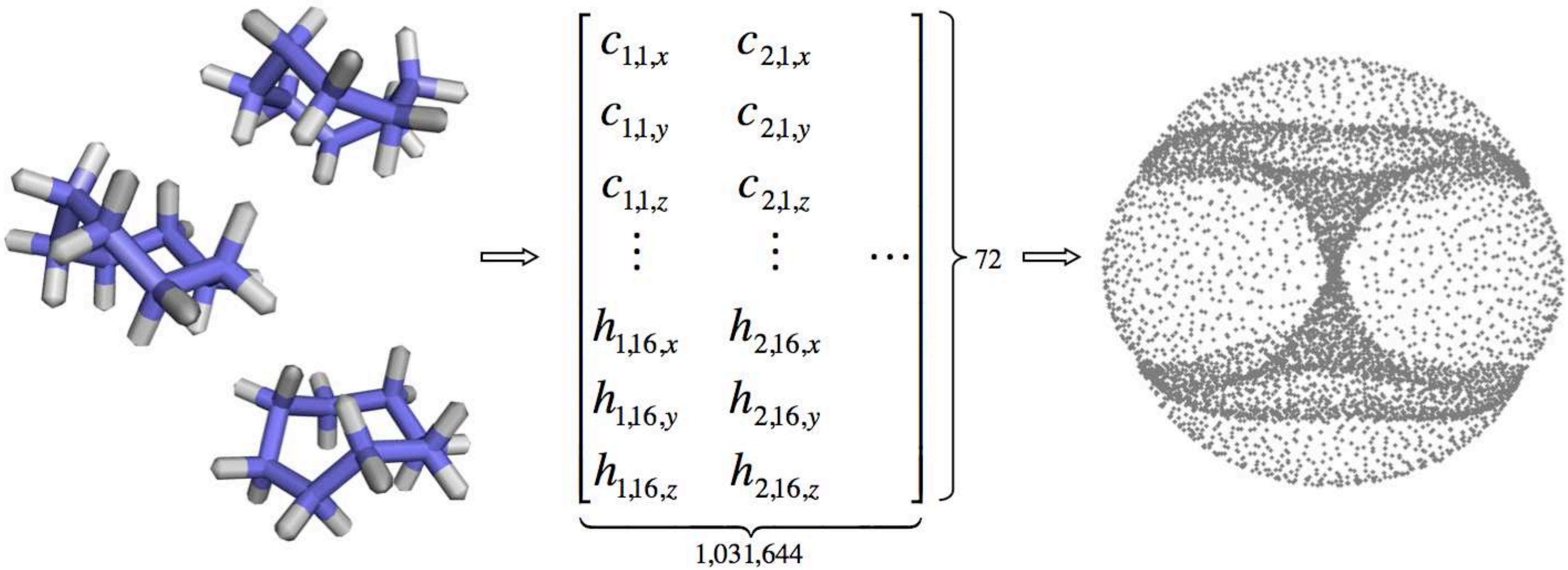
https://www.math.colostate.edu/~adams/advising

Persistent homology software tutorial

https://github.com/henryadams/Leiden-PersistentHomology

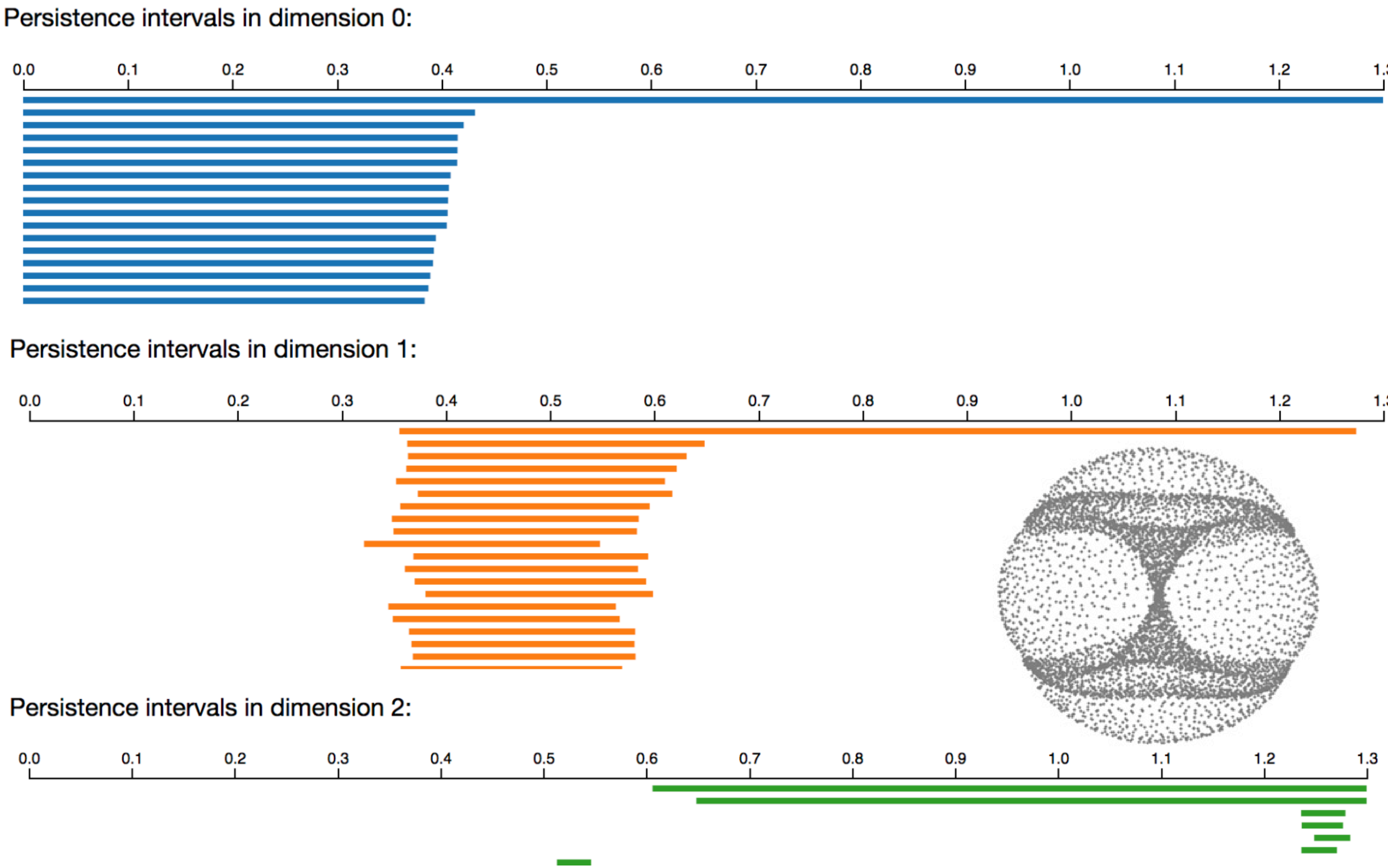
- Examples using Ripser-live in your html browser.
- Cyclo-octane molecule, optical images

Cyclo-Octane (C_8H_{16}) data



Non-Manifold Surface Reconstruction from High Dimensional Point Cloud Data
by Shawn Martin and Jean-Paul Watson, 2010.

Cyclo-Octane (C_8H_{16}) data



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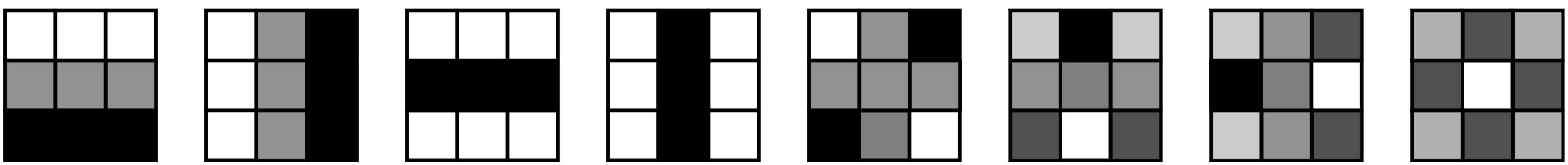
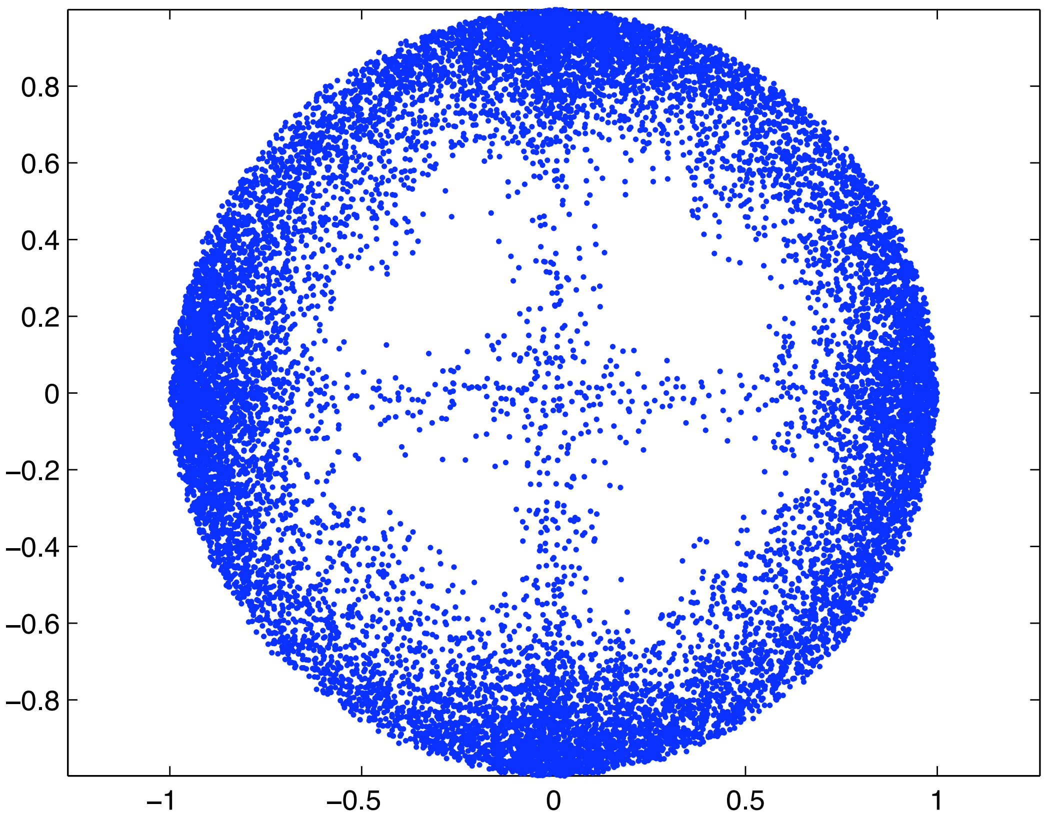
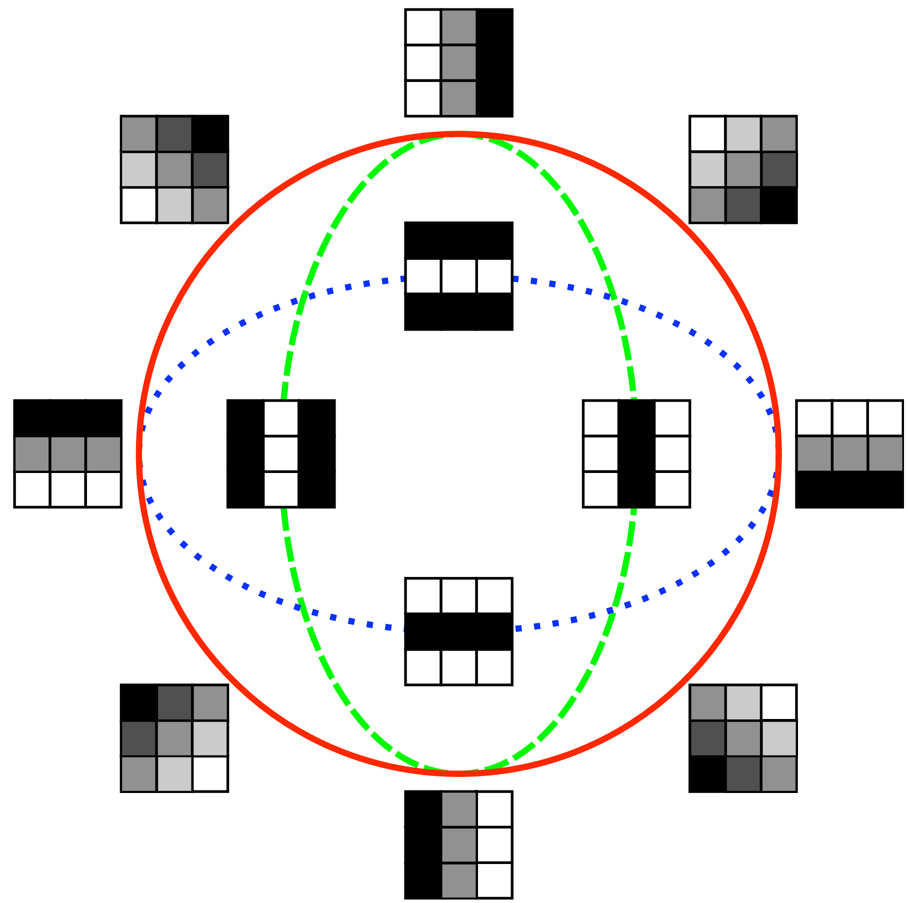
3x3 High-contrast patches from images



On the Local Behavior of Spaces of Natural Images by Gunnar Carlsson, Tigran Ishkhanov, Vin de Silva, and Afra Zomorodian, 2008.

3x3 High-contrast patches from images





Interpretation: nature prefers horizontal and vertical directions

Devanagari Example

The following image of Devanagari script, which is used in India and Nepal, is from wikipedia (https://en.wikipedia.org/wiki/Devanagari). We will use a dataset with 36 different characters.

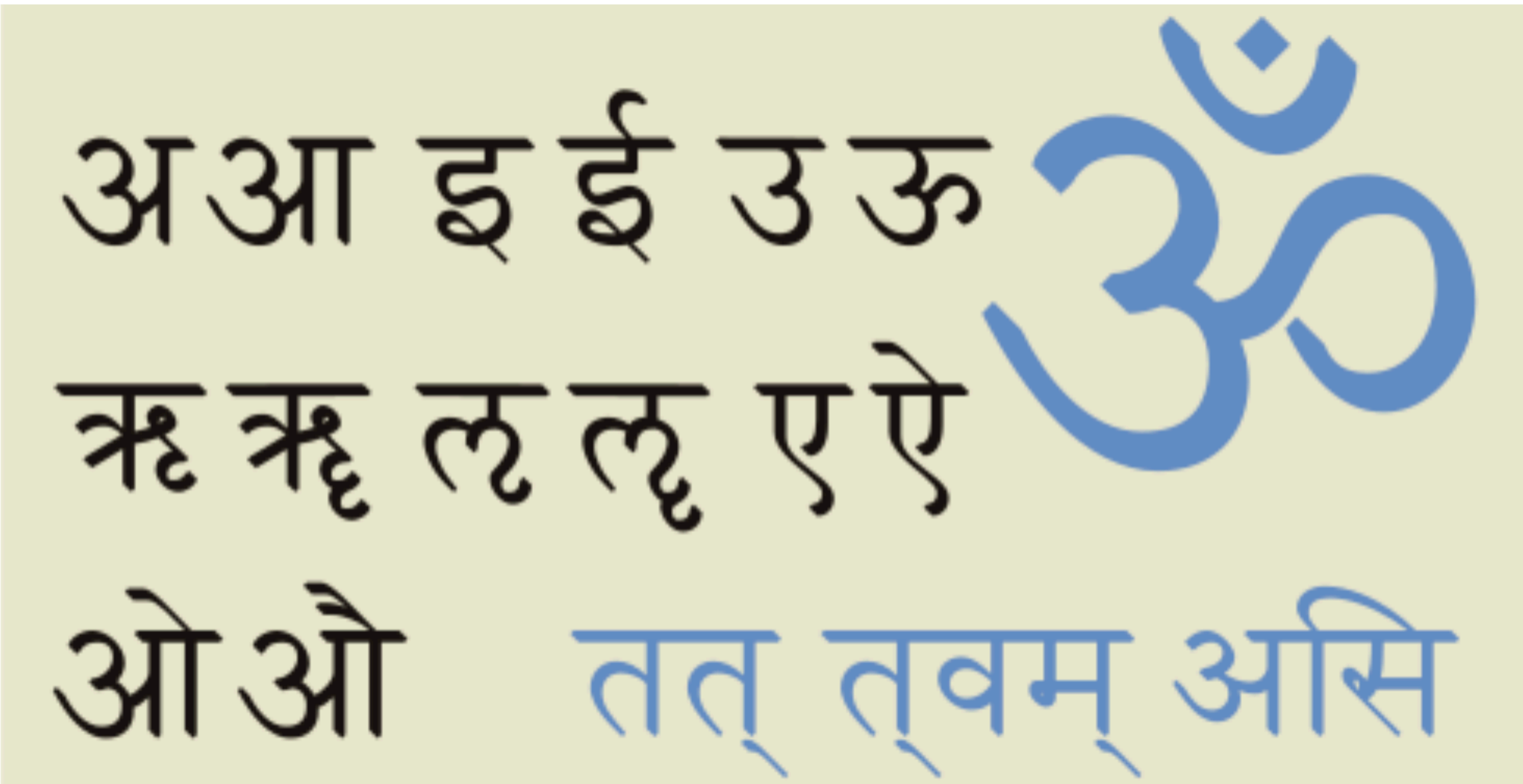


Image: Wikipedia

What applied topology software options do I have?

This list undoubtably has unintentional omissions; please email Henry with updates to this list!

https://www.math.colostate.edu/~adams/advising

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https://github.com/henryadams/Leiden-PersistentHomology

- Examples using Ripser-live in your html browser.
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