Research Statement

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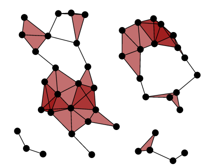
Given a set of points X sampled from a metric space M, what information can one recover about the underlying space M? One approach is to build a *Vietoris–Rips simplicial complex* $\mathrm{VR}(X;r)$ on top of the sampling X; this is a geometric construction which depends on the choice of a scale parameter $r > 0$. In nice cases the Vietoris–Rips complex of the sampling X will recover the unknown space M up to homotopy type. Vietoris–Rips complexes were first invented for use in geometric group theory and metric cohomology [14, 30], but within the last 20 years they have emerged as an important tool in applied and computational topology, including persistent homology [16, 18]. In particular, data science practitioners currently use software packages [13] to compute Vietoris–Rips complexes of datasets X as a way to approximate the shape of the dataset. However, in order to interpret their computations, there is a need to mathematically better understand the limiting object (the Vietoris–Rips complex of the underlying shape M). My recent work has extended the theory of Vietoris–Rips complexes at larger scales (§1), inspired a new definition of these thickenings allowing intuitive proofs of old and new results (§2), led to a generalization of the Borsuk–Ulam theorem for maps into higher-dimensional codomains (§3), and connected to interdisciplinary research in machine learning, sensor networks, and high-dimensional data analysis (§4).



§1. Vietoris–Rips simplicial complexes at larger scale parameters. Given a metric space X and scale parameter $r \geq 0$, the *Vietoris–Rips simplicial complex* $\mathrm{VR}(X;r)$ contains a finite subset $\sigma \subseteq X$ as a simplex if its diameter is at most r. Since we do not know a priori how to choose the scale r, the idea of persistent homology is to compute the homology of the Vietoris–Rips complex of metric space X over a large range of scale parameters r, and to trust those topological features which persist. The motivation for using Vietoris–Rips complexes are Hausmann's and Latschev's reconstruction results [24, 29]: for M a Riemannian manifold, scale r sufficiently small, and sample $X \subseteq M$ sufficiently dense, we have a homotopy equivalence $\mathrm{VR}(X;r) \simeq M$. But as the key idea of persistence is to allow scale r to vary, the assumption that scale r is kept sufficiently small does not hold in practice.

Theorem 1 ([1])**.** *As the scale r increases, the Vietoris–Rips complexes of the circle are homotopy equivalent to the circle, the 3-sphere, the 5-sphere, the 7-sphere, the 9-sphere, ...*

This is the first connected non-contractible Riemannian manifold for which the homotopy types of the Vietoris–Rips complex are known at all choices of scale. We extend our result to classify the homotopy types of Vietoris–Rips complexes of ellipses, regular polygons, n-dimensional tori, random subsets from the circle, and metric gluings [3, 4, 5, 6]. Our work has also been extended in [20, 38, 37] in order to identify the 1-dimensional persistent homology of metric graphs and geodesic spaces, and in [39] for applications to geometric group theory. However, in general little is known about Vietoris–Rips complexes at larger scales, though they arise naturally in applications of persistent homology.

Question. *What are the homotopy types of Vietoris–Rips complexes of other Riemannian manifolds, such as n-spheres?*

The above question is fundamental for applications of persistent homology to data analysis, since as finite subset $X \subseteq M$ gets denser and denser, the persistent homology of the Vietoris–Rips complex of X converges to the (unknown) persistent homology of the Vietoris–Rips complex of manifold M [17]. In the following section, we give a partial answer for n-spheres.

§2. From Vietoris–Rips complexes to metric thickenings. Though we started with a metric space X, if X is not discrete then the Vietoris–Rips complex $\mathrm{VR}(X;r)$ is not a metric space. Indeed, if $\mathrm{VR}(X;r)$ is not locally finite then it is not metrizable, meaning it is *impossible* to equip $\mathrm{VR}(X;r)$ with a metric without changing the homeomorphism type. A related problem (especially for proofs) is that for X not discrete, the inclusion $X \hookrightarrow \mathrm{VR}(X;r)$ is not continuous. In [2] we address these issues. We use optimal transport (i.e. a Wasserstein or Kantarovich metric [19, 22, 36]) to build a metric space $\mathrm{VR}^m(X;r)$ — the *Vietoris–Rips thickening* — for all $r > 0$.

There is strong evidence that when X is not a discrete metric space, then the metric thickening $\mathrm{VR}^m(X;r)$ is a more natural object than the simplicial complex $\mathrm{VR}(X;r)$. Consider for example Hausmann's theorem [24], which states that if M is a compact Riemannian manifold, then for small enough scale parameters $r > 0$, the complex $\mathrm{VR}(M;r)$ is homotopy equivalent to M. Though this theorem is foundational for the subject, its proof is quite complicated. Indeed, Hausmann's homotopy equivalence $T\colon \mathrm{VR}(M;r) \to M$ is *extremely non-canonical*; it depends on the arbitrary choice of a fixed total ordering of all of the points in manifold M, and different choices of total orderings will produce different maps $T\colon \mathrm{VR}(M;r) \to M$. Furthermore, since the manifold M is not a discrete metric space, the inclusion $M \hookrightarrow \mathrm{VR}(M;r)$ is not continuous and hence cannot be a homotopy inverse for T. Indeed, Hausmann does not write down an explicit homotopy inverse for T. By contrast, when using metric Vietoris–Rips thickenings instead of simplicial complexes, we give a natural proof of a metric analogue of Hausmann's theorem.

Theorem 2 ([2])**.** *For M a Riemannian manifold of bounded curvature, the Vietoris–Rips metric thickening $\mathrm{VR}^m(M;r)$ is homotopy equivalent to M for r sufficiently small.*

In our proof we use Karcher means [25] to define a map $g\colon \mathrm{VR}^m(M;r) \to M$ which is a homotopy equivalence for r sufficiently small. Key features of our proof are that we give a canonical choice for map g, which has the (now continuous) inclusion $\iota\colon M \hookrightarrow \mathrm{VR}^m(M;r)$ as a homotopy inverse, as proven by writing down a simple linear homotopy from the composition $\iota \circ g$ to the identity map.

Using metric thickenings, in [2] we are able to identify the first new homotopy type in the Vietoris–Rips thickening of the n-sphere, namely $\mathrm{VR}^m(S^n;r) \simeq \Sigma^{n+1}\frac{\mathrm{SO}(n+1)}{A_{n+2}}$, with alternating group A_{n+2} arising as the rotational symmetries of an inscribed regular $(n+1)$-simplex. Extensions to larger scale parameters will relate to Lovász' strongly self-dual polytopes [32].

Our work in [2] has been extended to connect Vietoris–Rips complexes to the filling radius [31, 34], as studied by Gromov [21, 23] and Katz [27, 28, 26]. In his PhD thesis [33], my former student Joshua Mirth has developed a Morse theory for Vietoris–Rips thickenings, providing tools towards understanding the homotopy types of thickenings at all scales, without having to invent new tools for each new space.

§3. Borsuk–Ulam theorems into higher-dimensional codomains. In [9] we use Vietoris–Rips thickenings of spheres to produce generalizations of the Borsuk–Ulam theorem for maps $f\colon S^n \to \mathbb{R}^k$ from the n-sphere into higher-dimensional codomains $\mathbb{R}^k$ with

$k \geq n$. Instead of finding a point $x \in S^n$ with $f(x) = f(-x)$ (which may not exist for $k > n$), we find a set $X \subseteq S^n$ of bounded diameter such that the convex hull of $f(X)$ intersects the convex hull of $f(-X)$. Furthermore, we show that our diameter bounds are sharp when $n = 1$, or alternatively when $k = n+1, n+2$. Michael Crabb at the University of Aberdeen has recently used characteristic classes to extend our work to provide sharp diameter bounds, for select values of n, into even higher-dimensional codomains $k = n+3, \ldots$, but going further will require a better understanding of the homotopy types of Vietoris–Rips thickenings of spheres. My PhD student Johnathan Bush and I are using this generalized Borsuk–Ulam theorem to derive generalized versions of the ham sandwich theorem, in which there are k masses in $\mathbb{R}^n$ to equipartition, now with $k \geq n$, but more than a single cut is allowed.

§4. Interdisciplinary applications. I have worked on applications of Vietoris–Rips complexes to machine learning (our persistence image paper [12] is the next most popular technique behind landscapes [15] for transforming persistence barcodes into machine learning input vectors), to sensor networks [11], and to high-dimensional data analysis [7, 8, 10]. I am also a coauthor of the tutorial for Javaplex [35], a software package for persistent homology, which contains exercises, solutions, and examples on real life data. I help advertise and share my broader research area in my role as Co-Director of the Applied Algebraic Topology Research Network, with over 700 members. Our network hosts an online research seminar, with over 135 hour-long research seminar recordings posted to our YouTube Channel, which has over 950 subscribers and over 1,700 hours watched in the last year.

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