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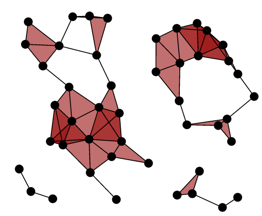
Given a set of points X sampled from a metric space M, what information can one recover about the underlying space M? One approach is to build a *Vietoris–Rips simplicial complex* $\mathrm{VR}(X;r)$ or a *Čech simplicial complex* $\check{\mathrm{C}}(X;r)$ on top of the sampling X; these geometric constructions depend on the choice of a scale parameter $r > 0$. When the scale parameter r is sufficiently small depending on the curvature of M, and when X is a sufficiently nice sampling from M, then the Vietoris–Rips and Čech simplicial complexes recover the space M up to homotopy type [23, 15]. However, very little is known about these geometric complexes at larger scale parameters, which is an important question for applications of topology using persistent homology [16, 18], where one allows the scale parameter r to vary from small to large. My recent work has extended the theory of Vietoris–Rips and Čech complexes (§1), inspired a new definition of these thickenings allowing cleaner proofs of old and new results (§2), and connected the mathematics to interdisciplinary research in machine learning, sensor networks, computer vision, and high-dimensional data analysis (§3).

§1. Vietoris–Rips and Čech simplicial complexes

Definition. *Given a metric space X and scale parameter $r \geq 0$, the* Vietoris–Rips simplicial complex $\mathrm{VR}(X;r)$ *contains a finite subset $\sigma \subseteq X$ as a simplex if its diameter is at most r.*



Since we do not know a priori how to choose the scale r, the idea of persistent homology is to compute the homology of the Vietoris–Rips complex of metric space X over a large range of scale parameters r, and to trust those topological features which persist. The motivation for using Vietoris–Rips complexes are Hausmann's and Latschev's reconstruction results [21, 23]: for M a Riemannian manifold, scale r sufficiently small, and sample $X \subseteq M$ sufficiently dense, we have a homotopy equivalence $\mathrm{VR}(X;r) \simeq M$. But as the key idea of persistence is to allow scale r to vary, the assumption that scale r is kept sufficiently small does not hold in practice.

Theorem 1 ([1])**.** *As the scale r increases, the Vietoris–Rips and Čech complexes of the circle are homotopy equivalent to the circle, the 3-sphere, the 5-sphere, the 7-sphere, ...*

This is the only connected non-contractible Riemannian manifold for which the homotopy types of the Vietoris–Rips complex are known at all choices of scale. Furthermore, we extend our result to classify the homotopy types of Vietoris–Rips complexes of ellipses [6], regular polygons [13], n-dimensional tori with the l_∞ metric [1], random subsets from the circle [5], and metric gluings [4]. The proof of Theorem 1 relies upon homotopy types of nerve complexes of circular arcs [3], which has applications to the Nerve Lemma for covers which are not good, to cyclic polytopes, to sizes of gaps between roots of trigonometric polynomials, and to the Lovász bound on the chromatic number of circular complete graphs.

Question. *What are the homotopy types of Vietoris–Rips and Čech complexes of Riemannian manifolds? (The Čech complex is the nerve of all geodesic balls of a fixed radius.)*

The above question is fundamental for applications of persistent homology to data analysis, since as finite subset $X \subseteq M$ gets denser and denser, the persistent homology of the Vietoris–Rips complex of X converges to the (unknown) persistent homology of the Vietoris–Rips complex of manifold M [17]. In the following section, we give a partial answer for n-spheres.

§2. From Vietoris–Rips and Čech complexes to metric thickenings

Though we started with a metric space X, if X is not discrete then the Vietoris–Rips complex $\mathrm{VR}(X;r)$ does not come equipped with a natural choice of metric. Indeed, if $\mathrm{VR}(X;r)$ is not locally finite then it is not metrizable, meaning it is *impossible* to equip $\mathrm{VR}(X;r)$ with a metric without changing the homeomorphism type. A related problem is that if X is not discrete, then the inclusion $X \hookrightarrow \mathrm{VR}(X;r)$ is not continuous. In [2] we address these issues. We use optimal transport (i.e. a Wasserstein or Kantaroivch metric [27, 19, 20]) to build a metric space $\mathrm{VR}^m(X;r)$ – the *Vietoris–Rips thickening* – for all $r > 0$. These metric thickenings also contain abstract convex combinations of points in X of diameter at most r, just as in the simplicial complex $\mathrm{VR}(X;r)$. However, if $\mathrm{VR}(X;r)$ is not locally finite then necessarily $\mathrm{VR}^m(X;r)$ has a different (metrizable) homeomorphism type, and on rare occasions even a different homotopy type.

There is strong evidence that when X is not a discrete metric space, then the metric thickening $\mathrm{VR}^m(X;r)$ is a more natural object than the simplicial complex $\mathrm{VR}(X;r)$. Consider for example Hausmann's theorem [21], which states that if M is a compact Riemannian manifold, then for small enough scale parameters $r > 0$, the complex $\mathrm{VR}(M;r)$ is homotopy equivalent to M. Though this theorem is foundational for the subject, its proof is quite complicated. Indeed, Hausmann's homotopy equivalence $T\colon \mathrm{VR}(M;r) \to M$ is extremely non-canonical; it depends on the arbitrary choice of a fixed total ordering of all of the points in manifold M, and different choices of total orderings will produce different maps $T\colon \mathrm{VR}(M;r) \to M$. Furthermore, since the manifold M is not a discrete metric space, the inclusion $M \hookrightarrow \mathrm{VR}(M;r)$ is not continuous and hence cannot be a homotopy inverse for T. Indeed, Hausmann does not write down an explicit homotopy inverse for T.

By contrast, when using metric Vietoris–Rips thickenings instead of simplicial complexes, we give a much nicer proof of a metric analogue of Hausmann's theorem.

Theorem 2 ([2])**.** *If M is a complete Riemannian manifold with bounded curvature, then the Vietoris–Rips and Čech metric thickenings* $\mathrm{VR}^m(M;r)$ *and* $\check{\mathrm{C}}(M;r)$ *are homotopy equivalent to M for r sufficiently small.*

In our proof we use Karcher means [22] to define a map $g\colon \mathrm{VR}^m(M;r) \to M$ which is a homotopy equivalence for r sufficiently small. Key features of our proof are that we give a canonical choice for map g, which has the (now continuous) inclusion $M \hookrightarrow \mathrm{VR}^m(M;r)$ as a homotopy inverse, as proven by writing down linear homotopy equivalences from the compositions to the corresponding identity maps.

In [14] my student and I extend Hausmann's theorem to work not only for Riemannian manifolds, but also for submanifolds of Euclidean space.

Using metric thickenings, in [2] we are able to identify the first new homotopy type in the Vietoris–Rips thickening of the n-sphere, namely $\mathrm{VR}^m(S^n;r) \simeq \Sigma^{n+1}\frac{\mathrm{SO}(n+1)}{A_{n+2}}$, with alternating group A_{n+2} arising as the rotational symmetries of an inscribed regular $(n+1)$-simplex.

§3. Interdisciplinary applications

I have worked on applications of Vietoris–Rips and Čech complexes to machine learning [12], sensor networks [11, 8, 7], computer vision [10], and high-dimensional data analysis [9]. Many of my research projects are with undergraduates [6, 7, 13, 24, 25]. I am also a coauthor of the tutorial for Javaplex [26], a software package for persistent homology, which contains exercises, solutions, and examples on real life data sets.

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