A Few Putnam Problems Involving Power Series

CSU Putnam Club

October [redacted], 2019

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1. (2014 A1) Prove that every nonzero coefficient of the Taylor series of $(1 - x - x^2)e^x$ about $x = 0$ is a rational number whose numerator (in lowest terms) is either 1 or a prime number.

2. (2017 B3) Suppose $f(x) = \sum_{i=0}^{\infty} c_i x^i$ is a power series such that each coefficient c_i is either 0 or 1. Show that, if $f(2/3) = 3/2$, then $f(1/2)$ cannot be rational.

3. (2015 A2) Let $a_0 = 1$, $a_1 = 2$, and $a_n = 4a_{n-1} - a_{n-2}$ for $n \geq 2$. Find an odd prime factor of a_{2015}.

4. (2002 A1) Let k be a fixed positive integer. The nth derivative of $\frac{1}{x^k-1}$ has the form

$$\frac{P_n(x)}{(x^k-1)^{n+1}}$$

where $P_n(x)$ is a polynomial. Find $P_n(1)$.

2010 A2 Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f'(x) = \frac{f(x+n) - f(x)}{n}$$

for all real numbers x and all positive integers n.

1984 **A-2.** Express $\sum_{k=1}^{\infty} \frac{6^k}{(3^{k+1} - 2^{k+1})(3^k - 2^k)}$ as a rational number

2008 A1 Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that $f(x,y) + f(y,z) + f(z,x) = 0$ for all real numbers x, y, and z. Prove that there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x,y) = g(x) - g(y)$ for all real numbers x and y.

2008 A2 Alan and Barbara play a game in which they take turns filling entries of an initially empty 2008×2008 array. Alan plays first. At each turn, a player chooses a real number and places it in a vacant entry. The game ends when all the entries are filled. Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?