The Vietoris–Rips Complexes of Finite Subsets of an Ellipse of Small Eccentricity

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1 Introduction

Let X be a metric space and let $r > 0$ be a scale parameter. The Vietoris–Rips simplicial complex $\mathbf{VR}_<(X;r)$ (resp. $\mathbf{VR}_\leq(X;r)$) has X as its vertex set and has a finite $\sigma \subseteq X$ as a simplex whenever the diameter of σ is less than r (resp. at most r). When we do not want to distinguish between $<$ and $\leq$, we use the notation $\mathbf{VR}(X;r)$ to denote either complex. Vietoris–Rips complexes have been used to define a cohomology theory for metric spaces [14], in geometric group theory [9], and more recently in computational topology and topological data analysis [8, 6].

In the setting of computational topology, metric space X is often a finite sample from an unknown space M, and one would like to use X to recover information about M. A common technique is to use the thickening $\mathbf{VR}(X;r)$ as an approximation for M, although it is not clear how to choose scale parameter r. The idea of *persistent homology* is to study nested sequences of topological spaces, and to trust homological features that persist across several spaces as represenative of the true homological features of the underlying manifold; transient features are disregarded as noise. One common nested sequence of spaces is the *Vietoris-Rips filtration*, a sequence of Vietoris-Rips complexes for increasing scale parameter. There are several theoretical results which justify this approach:

- If M is a Riemannian manifold and scale parameter r is sufficiently small, then [11, Theorem 3.5] proves that $\mathbf{VR}_<(X;r)$ is homotopy equivalent to M.
- If M is a Riemannian manifold, scale parameter r is sufficiently small, and metric space X is sufficiently close to M in the Gromov-Hausdorff distance[1], then [13, Theorem 1.1] proves that $\mathbf{VR}_<(X;r)$ is homotopy equivalent to M.
- If $M \subseteq \mathbb{R}^d$ is compact with sufficiently large weak feature size and finite set $X \subseteq \mathbb{R}^d$ is sufficiently close to M in the Hausdorff distance, then [7, Theorem 3.6] proves that the homology groups of M can be recovered from the persistent homology of $\mathbf{VR}_<(X;r)$ at sufficiently small scale parameters.

[1] One way X can be sufficiently close to M in the Gromov-Hausdorff distance is if $X \subseteq M$ is a sufficiently dense.

Persistent homology is frequently used for topological data analysis, in which a finite sample X from an unknown manifold M is used to study the topology of the underlying manifold. In low dimensions, we can create visual representations of the data, and take advantage of our natural ability to recognize patterns. However, as dimensions increase, creating useful such representations quickly becomes infeasible, thus necessitating the development of mathematical techniques. One common problem with persistent homology in practice is that one does not know the underlying manifold M and thus cannot ensure that the scale parameters are sufficiently small (so as to be able to take advantage of the above results). As such, practitioners often allow the scale parameter to range from small to large, often going far beyond "sufficiently small" values. Hence Vietoris–Rips complexes with large scale parameters arise commonly in applications of persistent homology, even though little is known about their behavior.

In [2] this problem is studied in the specific case of S^1, the circle. Adamaszek and Adams demonstrate that as r increases, $\mathbf{VR}_<(S^1; r)$ obtains the homotopy types S^1, S^3, S^5, …, until it is finally contractible. Here we mean S^k to denote the k-sphere. They also show that for sufficiently dense finite samples $X \subseteq S^1$, $\mathbf{VR}(X; r) \simeq \mathbf{VR}(S^1; r)$. To our knowledge this is the only connected non-contractible manifold whose Vietoris–Rips complex homotopy type is known for all scale parameters. This example agrees with a conjecture of Hausmann [11, (3.12)] that for M a compact Riemannian manifold, the connectivity of $\mathbf{VR}_<(M; r)$ is a non-decreasing function of r.

In this paper we study Vietoris–Rips complexes of ellipses equipped with the Euclidean metric. We restrict our attention to ellipses $Y = \{(x, y) \in \mathbb{R}^2 \mid (x/a)^2 + y^2 = 1\}$ with semi-major axis length a satisfying $1 < a \leq \sqrt{2}$, henceforth referred to as ellipses of *small eccentricity*. The threshhold $a \leq \sqrt{2}$ guarantees that all metric balls in Y are connected. Our main results are the following, where $r_1 = \frac{4\sqrt{3}a}{a^2+3}$ and $r_2 = \frac{4\sqrt{3}a^2}{3a^2+1}$:

Theorem 6.1. *Let Y be an ellipse of small eccentricity. For any sufficiently dense finite sample $X \subseteq Y$, and $r_1 < r < r_2$, we have that $\mathbf{VR}(X; r) \simeq \bigvee^i S^2$ for some $i \geq 1$.*

Theorem 6.2. *Let Y be an ellipse of small eccentricity. For any $r_1 < r < r_2$, $\epsilon > 0$, and $m \geq 2$, there exists an ϵ-dense finite subset $X \subseteq Y$ such that $\mathbf{VR}(X; r) \simeq \bigvee^{m-1} S^2$.*

This shows that even for an arbitrarily dense subset $X \subseteq Y$, the homotopy types of $\mathbf{VR}(X; r)$ and $\mathbf{VR}(Y; r)$ need not agree. Hence the behavior of

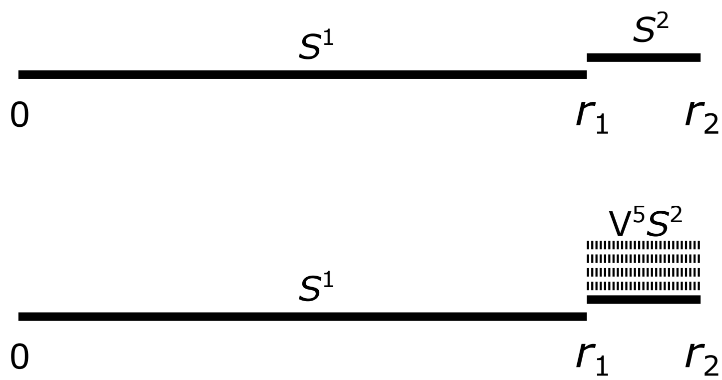

Figure 1: Schematic diagrams of the persistent homology of $\mathbf{VR}_<(Y; r)$ (top) and $\mathbf{VR}_\leq(Y; r)$ (bottom) in homological dimensions 1 and 2. See Theorem 6.3 and Corollary 6.3 for precise statements.

the ellipse is quite different from that of the circle with regard to finite approximations. Indeed, any sufficiently dense sample of the circle recovers the homotopy type of $\mathbf{VR}_<(S^1; r)$ [2, Section 5]. In the particular case when the Riemannian manifold is the circle, an approximation result similar to [13, Theorem 1.1] is true not only for small scale parameters r but also for arbitrary scale parameters. However, Theorem 6.2 demonstrates that such an approximation result does not hold at higher scale for the ellipse, which is not a Riemannian manifold when equipped with the Euclidean metric.

In an upcoming paper [5] we generalize the approach presented herein to handle the infinite case, with the following results:

Theorem 6.3. *Let* $1 < a \leq \sqrt{2}$, $r_1 = \frac{4\sqrt{3}a}{a^2+3}$, *and* $r_2 = \frac{4\sqrt{3}a^2}{3a^2+1}$. *Then*

$$\mathbf{VR}_<(Y; r) \simeq \begin{cases} S^1 & \text{for } 0 < r \leq r_1 \\ S^2 & \text{for } r_1 < r \leq r_2 \end{cases}$$

$$\text{and} \quad \mathbf{VR}_\leq(Y; r) \simeq \begin{cases} S^1 & \text{for } 0 < r < r_1 \\ S^2 & \text{for } r = r_1 \text{ or } r_2 \\ \bigvee^5 S^2 & \text{for } r_1 < r < r_2. \end{cases}$$

These results are shown in Figure 1.

Theorem 6.3. *The 1-dimensional persistent homology of* $\mathbf{VR}_<(Y;r)$ *(resp.* $\mathbf{VR}_\leq(S^1;r)$*) consists of a single interval* $(0,r_1]$ *(resp.* $[0,r_1)$*).*

The 2-dimensional persistent homology of $\mathbf{VR}_<(Y;r)$ *consists of a single interval* $(r_1,r_2]$*. The 2-dimensional persistent homology of* $\mathbf{VR}_\leq(S^1;r)$ *consists of the interval* $[r_1,r_2]$*, as well as a point* $[r,r]$ *on the diagonal with multiplicity four for every* $r_1 < r < r_2$*.*

In Section 2 we provide preliminary concepts and notation on graphs, simplicial complexes, and Vietoris–Rips complexes. In Section 3 we review finite cyclic graphs from [1, 2, 3, 4], and in Section 4 we introduce finite dynamical systems and their relation to finite cyclic graphs. Section 5 contains necessary geometric lemmas about ellipses. Finally, in Section 6 we use the theory of cyclic dynamics to prove our results about the homotopy types of finite subsets of Vietoris–Rips complexes of small eccentricity ellipses.

2 Preliminaries

Directed graphs

A *directed graph* is a pair $G=(V,E)$ with V the set of vertices and $E\subseteq V\times V$ the set of directed edges, where we require that there are no loops and that no edges are oriented in both directions. The edge (v,w) will also be denoted by $v\to w$. A *homomorphism of directed graphs* $h\colon G\to H$ is a vertex map such that for every edge $v\to w$ in G, either $f(v)=f(w)$ or there is an edge $f(v)\to f(w)$ in H. For a vertex $v\in V$ we define the out- and in-neighborhoods

$$N^+(G,v)=\{w \mid v\to w\},\quad N^-(G,v)=\{w \mid w\to v\},$$

as well as their respective closed versions

$$N^+[G,v]=N^+(G,v)\cup\{v\},\quad N^-[G,v]=N^-(G,v)\cup\{v\}.$$

For $V'\subseteq V$ we let $G[V']$ denote the induced subgraph on the vertex set V'. An undirected graph is a graph in which the orientations on the edges are omitted.

Conventions for the circle

Let S^1 be the circle of unit circumference equipped with the arc-length metric. If $p_1, p_2, \ldots, p_s \in S^1$ then we write

$$p_1 \preceq p_2 \preceq \cdots \preceq p_s \preceq p_1$$

to denote that $p_1, \ldots, p_s$ appear on S^1 in this clockwise order, allowing equality. We may replace $p_i \preceq p_{i+1}$ with $p_i \prec p_{i+1}$ if in addition $p_i \neq p_{i+1}$. For $p, q \in S^1$ we denote the clockwise distance from p to q by $\vec{d}(p,q)$, and we denote the closed clockwise arc from p to $q \neq p$ by $[p,q]_{S^1} = \{z \in S^1 \mid p \preceq z \preceq q \preceq p\}$. Open and half-open arcs are defined similarly and denoted $(p,q)_{S_1}$, $(p,q]_{S_1}$, or $[p,q)_{S_1}$.

For a fixed choice of $0 \in S^1$ each point $x \in S^1$ can be identified with the real number $\vec{d}(0,x) \in [0,1)$, and this will be our coordinate system on S^1.

Simplicial Complexes

A Vietoris–Rips complex is an example of a simplicial complex. To define the simplicial complex, we must first define the *simplex*, the basic unit upon which simplicial complexes are built. Let $u_0, \ldots, u_k \in \mathbb{R}^{k+1}$ be affinely independent, or equivalently, let $u_1 - u_0, \ldots, u_k - u_0$ be linearly independent. Then a k-simplex is a set of the form

$$\sigma = \left\{ t_0 u_0 + \ldots t_k u_k \;\middle|\; 0 \leq i \leq k, t_i \geq 0, \sum_{i=0}^{k} t_i = 1 \right\}.$$

Alternatively, we may simply let σ be the convex hull of $u_0, \ldots, u_k$ (the smallest convex set containing the points). Intuitively, we may think of the k-simplex as the simplest k dimensional polyhedron: the 0-simplex is a vertex, the 1-simplex is an edge, the 2-simplex is a triangle, and the 3-simplex is a tetrahedron.

Notice that the boundary of each simplex is composed of lower dimensional simplices: vertices form the boundary of an edge, edges and vertices form the boundary of a triangle, and triangles, edges, and vertices form the boundary of a tetrahedron. We refer to these lower dimensional simplices as *faces* of the higher dimensional simplex. More formally, we may define the faces of a simplex as the convex hull some subset of its vertices (the $u_0, \ldots, u_k$).

We use these simplices as the basic building block for more complicated topological structures, called *simplicial complexes*.

Definition 2.1. *A* geometric simplicial complex *is the union of a finite set K of simplices with the property that for any two simplices $\sigma, \tau \in K$, either $\sigma \cap \tau = \varnothing$ (they are disjoint) or $\sigma \cap \tau$ is a simplex of K.*

Sometimes it is helpful to abstract from this geometric definition, so here we provide a purely combinatorial definition of the simplical complex.

Definition 2.2. *An* abstract simplicial complex *is a set K of nonempty finite subsets of some finite set V, with the property that for every $\sigma \in K$, and every nonempty $\tau \subseteq \sigma$, we also have $\tau \in K$.*

Upon first examination this definition appears substantially different from the geometric definition provided above. This relationship becomes clearer when we associate a geometric simplicial complex with abstract simplicial complex, and vice-versa. For a given geometric simplicial complex K, we define its associated abstract simplicial complex K^* as

$$K^* = \{\sigma \mid \sigma \text{ is the set of all vertices of some simplex in } K\}.$$

Since every simplex in K^* contains its faces, we see that K^* has the required subset property to be an abstract simplicial complex. Now for a given abstract simplicial complex K^* on the vertex set V with $|V| = n$, we relabel V with affinely independent points $u_1, \dots, u_n \in \mathbb{R}^n$ and let

$$K = \bigcup_{[u_{i_0}, \dots, u_{i_k}] \in K^*} \operatorname{conv}(u_{i_0}, \dots, u_{i_k})$$

where $\operatorname{conv}(u_0, \dots, u_k)$ denotes the convex hull of $u_0, \dots, u_k$. We shall henceforth use the term simplicial complex to refer interchangeably to either an abstract simplicial complex or to its geometric realization.

Vietoris–Rips Complexes

We now define the simplicial complex of interest to this paper, the *Vietoris–Rips complex*. However, we must first define the *Vietoris–Rips graph*, to which the Vietoris–Rips complex is closely related.

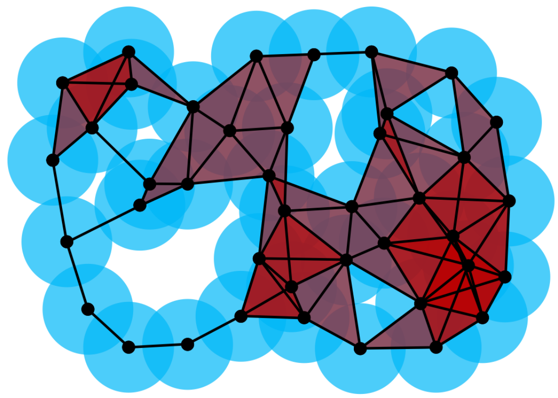

Figure 2: The blue balls are of radius r, the scale-parameter. Two points are connected if they are within r of each other. Finally we take the clique complex of the graph, to arrive at the simplicial complex pictured.

Definition 2.3. *Given a metric space X and a real scale parameter $r > 0$, the Vietoris–Rips graph $\mathrm{VR}_<(X;r)$ (respectively $\mathrm{VR}_\leq(X;r)$) is the graph whose vertices are the points in X, and which contains edges between vertices $u, v \in X$ if and only if $d(u,v) < r$ (respectively $d(u,v) \leq r$).*

The definition of the Vietoris–Rips complex also depends upon the notion of a *clique* within graphs.

Definition 2.4. *A clique of a graph $G = (V, E)$ is a subset of vertices $C \subseteq V$ such that for every $u, v \in C$, $(u, v) \in E$.*

Definition 2.5. *A* clique complex *of a graph is the (abstract) simplicial complex on vertex set V with a simplex for every subset of vertices forming a clique.*

Definition 2.6. *The Vietoris–Rips complex $\mathbf{VR}_<(X;r)$ (respectively $\mathbf{VR}_\leq(X;r)$) is the clique complex of $\mathrm{VR}_<(X;r)$ (respectively $\mathrm{VR}_\leq(X;r)$), where X is a metric space and $r > 0$ is a real scale parameter are as above.*

When statements are true for a fixed choice of either subscript ($<$ or $\leq$), we will denote the Vietoris–Rips graph and complex as $\mathrm{VR}(X;r)$ and $\mathbf{VR}(X;r)$, respectively. An example of a Vietoris–Rips complex is pictured in Figure 2

The particular aspect of Vietoris–Rips complexes that we will study is homotopy type.

Homotopy

Definition 2.7. *Two continuous functions $f, g : X \to Y$ are* homotopic *if there exists a continuous function $h : X \times [0,1] \to Y$ such that $h(x,0) = f(x)$, and $h(x,1) = g(x)$ for any $x \in X$.*

Interpreting the second parameter of h as time, we may view h as a continuous deformation from f to g. This intuition carries into the definition of homotopy equivalent spaces:

Definition 2.8. *Two spaces X and Y are* homotopy equivalent *(denoted $\simeq$) if there exist continuous functions $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ is homotopic to id_Y and $g \circ f$ is homotopic to id_X.*

This corresponds to the notion that X can be continuously deformed (bent, shrunk, or stretched) into Y, and vice-versa. In this paper we will express the homotopy types of the various Vietoris–Rips complexes in terms of their homotopy equivalence to spheres (ex. S^5) or wedge sums of spheres (ex. $S^2 \vee S^2$), where the wedge sum is a one point union (glue the spheres together at a single point).

For a more detailed coverage of topology and combinatorial topology, we refer the reader to Hatcher [10] and Kozlov [12].

3 Cyclic graphs

In this section we recall finite cyclic graphs as studied in [2], and we also extend the definitions to include possibly infinite cyclic graphs. Our motivation is that if Y is an ellipse of small eccentricity and $X \subseteq Y$, then $\mathrm{VR}(X; r)$ is a cyclic graph.

The following definition generalizes [2, Definition 3.1] to the possibly infinite case.

Definition 3.1. *A directed graph with vertex set $V \subseteq S^1$ is* cyclic *if whenever there is a directed edge $v \to u$, then there are directed edges $v \to w$ and $w \to u$ for all $v \prec w \prec u \prec v$.*

In Figure 3 is pictured an example such graph.

We also generalize cyclic graphs homomorphisms ([2, Definition 3.5]) to the possibly infinite case.

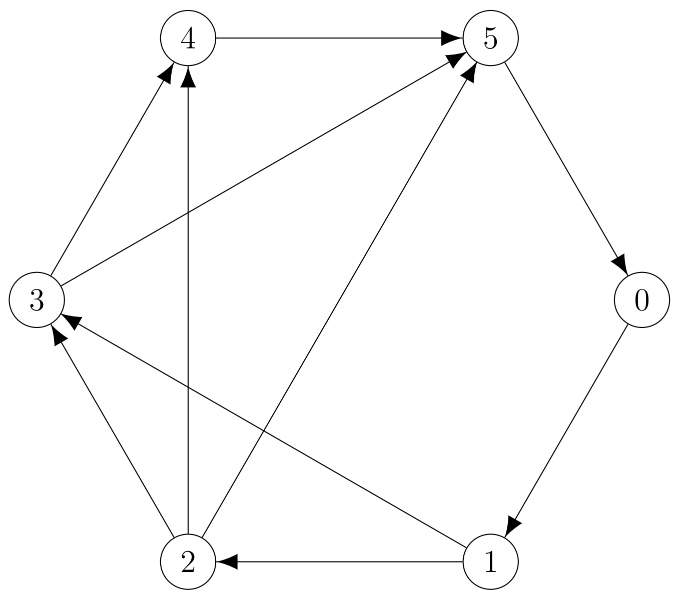

Figure 3: A cyclic graph with a dominated vertex (4).

Definition 3.2. *Let G and H be cyclic graphs. A homomorphism $f : G \to H$ of directed graphs is* cyclic *if*

- *whenever $v \prec w \prec u \prec v$ in G, then $f(v) \preceq f(w) \preceq f(u) \preceq f(v)$ in H, and*

- *f is not constant whenever G has a directed cycle of edges.*

Definition 3.3 ([2])**.** *For integers n and k with $0 \leq k < \frac{1}{2}n$, the directed graph C_n^k has vertex set $\{0, \ldots, n-1\}$ and edges $i \to (i+s) \mod n$ for all $i = 0, \ldots, n-1$ and $s = 1, \ldots, k$. Equivalently,*

$$i \to j \iff 0 < \vec{d}_n(i, j) < k$$

The graph C_9^3 is pictured in Figure 4. The graphs C_n^k will be useful in our study of Vietoris–Rips graphs. An important numerical invariant of a cyclic graph is the *winding fraction*, given for the finite case in [2, Definition 3.7].

Definition 3.4. *The* winding fraction *of a finite cyclic graph G is*

$$\mathrm{wf}(G) = \sup\left\{\frac{k}{n} \ : \ \text{there exists a cyclic homomorphism } C_n^k \to G\right\}.$$

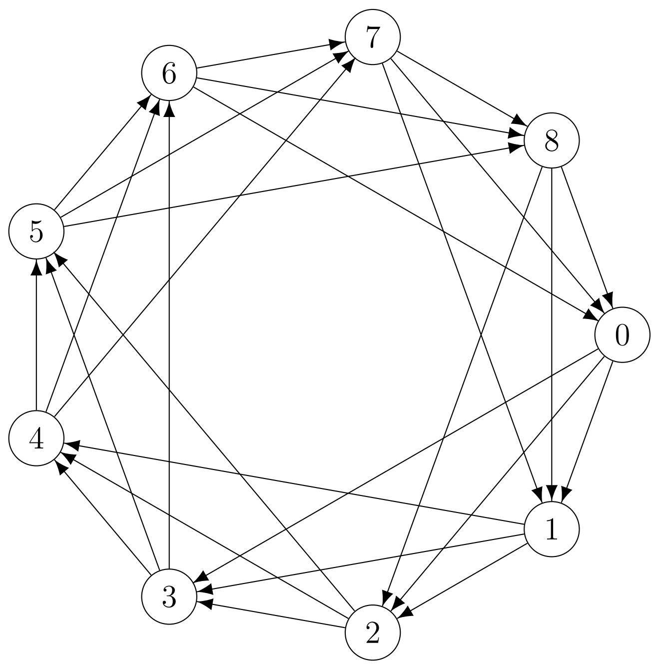

Figure 4: The cyclic graph C_9^3 which has 9 vertices, each of out-degree 3.

Note that the winding fraction is necessarily rational.

If cyclic graph G is finite, then we often label the vertices in cyclic order $v_0 \prec v_1 \prec \ldots \prec v_{n-1} \prec v_0$.

Definition 3.5 ([2]). *A vertex v_i in a finite cyclic graph G is* dominated (by v_{i+1}) *if $N^-(G, v_{i+1}) = N^-[G, v_i]$.*

In Figure 4, for example, vertex 4 is dominated by vertex 5.

We may *dismantle* a finite cyclic graph by removing dominated vertices in rounds as follows. Let S_0 be the set of all dominated vertices in G, and in round 0 remove these vertices to obtain $G[V \setminus S_0]$. Inductively for $i > 0$, let S_i be the set of all dominated vertices in $G[V \setminus (S_0 \cup \cdots \cup S_{i-1})]$, and in round i remove these vertices to obtain $G[V \setminus (S_0 \cup \cdots \cup S_i)]$. We refer to S_i as the set of vertices *dominated in the i-th round*. By [2, Proposition 3.12] every finite cyclic graph dismantles down to an induced subgraph of the form C_n^k for some $0 \leq k < \frac{1}{2}n$.

4 Finite Cyclic Dynamical Systems

Associated to each finite cyclic graph is a finite cyclic dynamical system, as studied in [4].

Definition 4.1. *Let G be a finite cyclic graph with vertex set V. The associated* finite cyclic dynamical system *is generated by the dynamics $f: V \to V$, where $f(v)$ is defined to be the clockwise most vertex of $N^+[G, v]$.*

A vertex $v \in V$ is *periodic* if $f^i(v) = v$ for some $i \geq 1$. If v is periodic then we refer to $\{f^i(v) \mid i \geq 0\}$ as a *periodic orbit*, whose *length* ℓ is the smallest $i \geq 1$ such that $f^i(v) = v$. The *winding number* of a periodic orbit $v_1, v_2, \ldots, v_q$, denoted $\mathrm{wn}(v_1, v_2, \ldots, v_q)$ is $\sum_{i=0}^{\ell-1} \vec{d}(f^i(v), f^{i+1}(v))$, the number of times that the orbit wraps around the graph. It follows from [4, Lemma 2.3] that every periodic orbit has the same length $\ell(G)$ and the same winding number $\mathrm{wn}(G)$. We define $\mathrm{orb}(G)$ to be the number of periodic orbits.

By [2, Proposition 3.12] any cyclic graph dismantles down to an induced subgraph of the form C_n^k. Since the proof of [4, Lemma 3.4] holds in the case of arbitrary cyclic graphs, rather than solely for Vietoris–Rips graphs of the circle, we see that that the vertex set of this induced subgraph is the set of periodic vertices of f.

An important numerical invariant of a cyclic graph is the *winding fraction*, which is originally defined using homomorphisms of cyclic graphs. It follows from [2, Proposition 3.14] that the winding fraction of a cyclic graph G can be equivalently defined as $\mathrm{wf}(G) = \frac{k}{n}$, where G dismantles to C_n^k.

Thus we have $\mathrm{wf}(G) = \frac{k}{n} = \frac{\mathrm{wn}(G)}{\ell(G)}$, so the winding number of G is determined by the dynamical system f. We can describe the homotopy type of the clique complex of G via the dynamical system f as follows.

Proposition 4.2. *If G is a finite cyclic graph, then*

$$\mathrm{Cl}(G) \simeq \begin{cases} S^{2l+1} & \text{if } \frac{l}{2l+1} < \mathrm{wf}(G) < \frac{l+1}{2l+3} \text{ for some } l \in \mathbb{N}, \\ \bigvee^{\mathrm{orb}(G)-1} S^{2l} & \text{if } \mathrm{wf}(G) = \frac{l}{2l+1}. \end{cases}$$

Proof. Theorem 4.4 of [2] states

$$\mathrm{Cl}(G) \simeq \begin{cases} S^{2l+1} & \text{if } \frac{l}{2l+1} < \mathrm{wf}(G) < \frac{l+1}{2l+3} \text{ for some } l \in \mathbb{N}, \\ \bigvee^{n-2k-1} S^{2l} & \text{if } \mathrm{wf}(G) = \frac{l}{2l+1}. \end{cases}$$

Suppose G dismantles to C_n^k and

$$\frac{l}{2l+1} = \mathrm{wf}(G) = \frac{k}{n}. \tag{3}$$

Then as required we have

$$\begin{aligned} \mathrm{orb}(G) &= \gcd(n,k) \\ &= \frac{n}{2l+1} && \text{by (3) since } \gcd(l, 2l+1) = 1 \\ &= n\left(1 - \frac{2l}{2l+1}\right) \\ &= n - 2k. \end{aligned}$$

$\square$

5 Geometric lemmas for the ellipse

We next prove several geometric lemmas about ellipses. These lemmas will allow us to describe the infinite cyclic dynamical systems corresponding to Vietoris–Rips complexes of ellipses, and hence the homotopy types of Vietoris–Rips complexes of ellipses (in Section 6).

Let $d\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be the Euclidean metric. Let $Y = \{(x,y) \in \mathbb{R}^2 \mid (x/a)^2 + y^2 = 1\}$ be an ellipse of small eccentricity, meaning the length of the semi-major axis satisfies $1 < a \leq \sqrt{2}$. The threshhold $a \leq \sqrt{2}$ guarantees that all metric balls in Y are connected (Lemma 5.5). Note we equip the ellipse Y with the Euclidean metric d, even though we have been using (and [2] uses) the geodesic metric on the circle S^1.

We will often think Y as S^1, simply by fixing an orientation-preserving homeomorphism $Y \to S^1$. We write $p_1 \preceq p_2 \preceq \ldots \preceq p_n \preceq p_1$ if $p_1, p_2, \ldots, p_n \in Y$ are oriented in a clockwise fashion, allowing equality. We may also replace $p_i \preceq p_{i+1}$ with $p_i \prec p_{i+1}$ if in addition $p_i \neq p_{i+1}$. For $p, q \in Y$ with $p \neq q$ we denote the closed clockwise arc from p to q by $[p,q]_Y = \{z \in Y \mid p \preceq z \preceq q \preceq p\}$. Open and half-open arcs are defined similarly and denoted $(p,q)_Y$, $(p,q]_Y$, and $[p,q)_Y$.

Definition 5.1. *Define the continuous function $h\colon Y \to Y$ which maps a point $p \in Y$ to the unique point in the intersection of $Y \setminus \{p\}$ with the normal line to Y at p.*

Lemma 5.2. *Given a point $p \in Y$ with $p \notin \{(\pm a, 0), (0, \pm 1)\}$, there is exactly one point q in the diametrically opposite quadrant (on the other side of both axes) satisfying $h(q) = p$. In particular $h\colon Y \to Y$ is surjective.*

Proof. Without loss of generality, let $p = (x,y) \in Y$ be in the bottom left quadrant, meaning $x < 0$ and $y < 0$. We must find some $q = (a\cos t, \sin t)$ with $0 < t < \frac{\pi}{2}$ and $h(q) = p$. The direction of the vector perpendicular to the ellipse at q is $(1, a\tan t)^T$, and the direction of the vector from p to q is $(a\cos t - x, \sin t - y)^T$. Thus $h(q) = p$ whenever

$$c\begin{pmatrix} a\cos t - x \\ \sin t - y \end{pmatrix} = \begin{pmatrix} 1 \\ a\tan t \end{pmatrix}$$

for some $c \in \mathbb{R}$, or alternatively, when

$$g(t) := (a^2 - 1)\sin t - ax\tan t + y = 0.$$

The derivative $g'(t) = (a^2-1)\cos t - ax\sec^2 t$ is strictly positive on $0 < t < \frac{\pi}{2}$ since $a > 1$ and $x < 0$. Therefore $g(t)$ is strictly increasing on $0 < t < \frac{\pi}{2}$, and so there is at most one q with $h(q) = p$. To see that there is at least one such q, note that g is continuous with $g(0) = y < 0$ and $\lim_{t\to\frac{\pi}{2}^-} g(t) = \infty$.

To see that $h\colon Y \to Y$ is surjective, note $h((\pm a, 0)) = (\mp a, 0)$ and $h((0, \pm 1)) = (0, \mp 1)$. $\square$

Lemma 5.3. *Ellipse $Y = \{(x, y) \in \mathbb{R}^2 \mid (x/a)^2 + y^2 = 1\}$ with $a > 1$ is an ellipse of small eccentricity (meaning $a \le \sqrt{2}$) if and only if for each point $q \in Y$ not on an axis, $h(q)$ is in the quadrant diametrically opposite from q.*

Proof. We restrict attention, without loss of generality, to the case where $q = (a \cos t, \sin t)$ with $0 < t < \pi/2$. Let L be the line perpendicular to Y at q. The slope of L is $a \tan t$, so the intersection of L with the x-axis is the point $(0, y_0)$ where

$$y_0 = \sin t - (a \cos t)(a \tan t) = (1 - a^2) \sin t.$$

For $1 < a \le \sqrt{2}$ we have $-1 < y_0 < 0$, and hence the $h(q)$ must lie in the lower left quadrant. For $a > \sqrt{2}$ we see that for t sufficiently close to $\pi/2$, we'll have $y_0 < -1$, meaning that point $h(q)$ will be in the lower right quadrant and the property does not hold. $\square$

Lemma 5.4. *The function h is bijective if and only if Y is an ellipse of small eccentricity.*

Proof. If Y is an ellipse of small eccentricity, then h is bijective by Lemmas 5.2 and 5.3. If Y is not an ellipse of small eccentricity, then by Lemmas 5.3 and 5.2 there is some $p \in Y$ with points q in an adjacent quadrant and q' in the opposite quadrant satisfying $h(q) = p = h(q')$. Hence h is not injective. $\square$

For $p \in Y$, define the function $d_p\colon Y \to \mathbb{R}$ by $d_p(q) = d(p, q)$.

Lemma 5.5. *For Y an ellipse of small eccentricity and $p \in Y$, the only critical points of $d_p\colon Y \to \mathbb{R}$ are a global minimum at p and a global maximum at $h^{-1}(p)$. It follows that nonempty metric balls in Y are either contractible or all of Y. Furthermore, d_p is continuous and strictly increasing on $(p, h^{-1}(p))_Y$.*

Proof. Clearly p is the global minimum of d_p. For any critical point $q \ne p$ of d_p it must be the case that the circle of radius $d(p, q)$ centered at p is tangent to Y at q, and hence $h(q) = p$. By Lemma 5.4, h is bijective, and since Y is compact it follows that $h^{-1}(p)$ is the global maximum of d_p and that there are no other critical points besides p and $h^{-1}(p)$. $\square$

Let Y be an ellipse of small eccentricity. If $p \in Y$ and $0 \leq r \leq d_p(h^{-1}(p))$, then we define $g_r(p) \in Y$ to be the unique solution q to $d_p(q) = r$ on $[p, h^{-1}(p)]_Y$. Uniqueness and existence are guaranteed by the fact that d_p is continuous and strictly increasing on the interval. It is not hard to see that $g_r \colon Y \to Y$ is bijective and of degree (i.e. winding number) one. It follows that

Lemma 5.6. *For Y an ellipse of small eccentricity and $p \in Y$, the function $(0, 2) \to Y$ defined by $r \mapsto g_r(p)$ is continuous.*

Proof. Function $d_p \colon (p, h^{-1}(p))_Y \to \mathbb{R}$ is continuous since $(p, h^{-1}(p))_Y$ is the image of the continuous curve in the plane. Hence given $\epsilon > 0$, there exists $\delta > 0$ such that $d(q, g_r(p)) < \delta$ implies $|d_p(q) - r| < \epsilon$. Choose $q^- \in (p, g_r(p))_Y \cap B_{\min(\delta,\epsilon)}(g_r(p))$, and $q^+ \in (g_r(p), h^{-1}(p))_Y \cap B_{\min(\delta,\epsilon)}(g_r(p))$; these intersections are nonempty since $(p, h^{-1}(p))_Y$ is a continuous curve through $g_r(p)$. By construction, $d(g_r(p), q^-), d(g_r(p), q^+) < \epsilon$. Since d_p is strictly increasing, as we move along Y from p to $h^{-1}(p)$, by Lemma 5.5, we have $d_p(q^-) < r < d_p(q^+)$. Thus for $|r - r'| < \min(r - d_p(q^-), d_p(q^+) - r)$, we see that $g_{r'}(p) \in (q^-, q^+)_Y$ and hence $d(g_{r'}(p), g_r(p)) < \epsilon$. $\square$

Lemma 5.7. *For Y an ellipse of small eccentricity and $0 \leq r < 2$, the function $g_r \colon Y \to Y$ is continuous.*

Proof. Let $\epsilon > 0$ and $p \in Y$. By Lemma 5.6, there exists some $\delta > 0$ such that $|r - r'| \leq \delta$ implies $d(g_r(p), g_{r'}(p)) < \epsilon$; choose such a δ with $\delta < \epsilon$. By the triangle inequality, we see that if p' satisfies $d(p', p) < \delta$, then $d(p', g_{r-\delta}(p)) < r < d(p', g_{r+\delta}(p))$. Hence, by continuity of d, there is some point q' in the interval $(g_{r-\delta}(p), g_{r+\delta}(p))_Y \subseteq B_\epsilon(g_r(p))$ satisfying $d(p', q') = r$, and thus $g_r \colon Y \to Y$ is continuous. $\square$

For $X \subseteq Y$ and $0 \leq r < 2$, we orient the edges of $\mathrm{VR}_\leq(X; r)$ (resp. $\mathrm{VR}_<(X; r)$) in a clockwise fashion by specifying that $p \to p'$ is a directed edge if $p' \in (p, g_r(p)]_Y$ (resp. $p' \in (p, g_r(p))_Y$).

Lemma 5.8. *Let Y be an ellipse of small eccentricity, let $X \subseteq Y$, and let $0 \leq r < 2$. Then $\mathrm{VR}_\leq(X; r)$ and $\mathrm{VR}_<(X; r)$ are cyclic graphs.*

Proof. We can think of the vertex set X as a subset of S^1, simply by fixing an orientation-preserving homeomorphism $Y \to S^1$. If $p \to p'$ is a directed edge of $\mathrm{VR}_\leq(X; r)$, then $p' \in (p, g_r(p)]_Y$. If $p'' \in X$ satisfies $p \prec p'' \prec p' \prec p$,

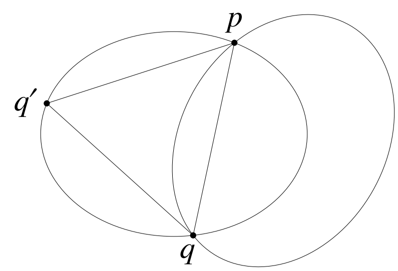

Figure 5: Proof of Lemma 5.9

then we have $p'' \in (p, g_r(p)]_Y$ by definition and $p' \in (p'', g_r(p''))]_Y$ since g_r is cyclic. Hence we have directed edges $p \to p''$ and $p'' \to p'$ in $\mathrm{VR}_{\leq}(X;r)$, and therefore $\mathrm{VR}_{\leq}(X;r)$ is a cyclic graph. The proof that $\mathrm{VR}_{<}(X;r)$ is a cyclic graph is identical after replacing each half-open interval with an open interval. $\square$

Lemma 5.9. *Let Y be an ellipse of small eccentricity and let $p \in Y$. Then there exists a unique inscribed equilateral triangle in Y containing y as one of its vertices.*

Proof. Rotate Y by $60°$ clockwise about y. Clearly there must be at least some intersection point between the two ellipses other than y. This intersection point corresponds to two points p and q on the original ellipse. By construction, we see that $\overline{yp} = \overline{yq}$ and that $\angle pyq = 60°$, so we must have an equilateral triangle.

For uniqueness, suppose for a contradiction that $\{p, g_r(p), g_r^2(p)\}$ and $\{p, g_{r'}(p), g_{r'}^2(p)\}$ are two distinct inscribed equilateral triangles in Y with side lengths $r < r'$ (we will learn later in Theorem 5.12 that necessarily $r' < 2$, but this isn't needed now). Necessarily we have $g_r(p), g_{r'}(p) \in (p, h^{-1}(p))_Y$ and $g_r^2(p), g_{r'}^2(p) \in (h^{-1}(p), p)_Y$. Note $g_r(p) \in (p, g_{r'}(p))_Y$ implies, since g_r is a bijection of degree one,

$$g_r^2(p) \in (g_r(p), g_r(g_{r'}(p)))_Y \subseteq (g_r(p), g_{r'}^2(p))_Y.$$

This gives $p \prec h^{-1}(p) \prec g_r^2(p) \prec g_{r'}^2(p) \prec p$. Hence $r = d(g_r^2(p), p) > d(g_{r'}^2(p), p) = r'$, a contradiction. $\square$

Remark 5.10. *The proof of Lemma 5.9 relies on $a \leq \sqrt{2}$ and Lemmas 5.4 and 5.5. Simulations show that Lemma 5.9 is true also for $a \leq \sqrt{\frac{7}{3}}$.*

For Y an ellipse of small eccentricity, by Lemma 5.9 we may define a continuous function $s\colon Y \to \mathbb{R}$ which maps each $y \in Y$ to the side length of the unique inscribed equilateral triangle containing this vertex.

Lemma 5.11. *For Y an ellipse of small eccentricity, the function $s\colon Y \to \mathbb{R}$ is continuous.*

Proof. By the fact that d_p is strictly increasing along this interval, we see that $g_r(p)$ is strictly increasing in r, in the sense that, for $r < r' < d(p, h^{-1}(p)) < 2$, we have

$$p \prec g_r(p) \prec g_{r'}(p) \prec h^{-1}(p) \prec p.$$

Therefore,

$$g_{r'}(p) \prec g_{r'}(g_r(p)) \prec g_{r'}^2(p) \prec g_{r'}(h^{-1}(p)) \prec g_{r'}(p)$$

and thus for $|r - r^*|$ sufficiently small,

$$g_r(p) \prec g_r^2(p) \prec g_{r'}^2(p) \prec g_r(h^{-1}(p)) \prec g_r(p)$$

and also

$$g_r^2(p) \prec g_r^3(p) \prec g_{r'}^3(p) \prec g_r^2(h^{-1}(p)) \prec g_r^2(p).$$

Let $\epsilon > 0$, $p, p^* \in Y$, and $r = s(p)$. Without loss of generality suppose that $p \prec p^* \prec g_r(p) \prec p$. Choose r^+, r^-, such that

$$r - \epsilon < r^- < r < r^+ < r + \epsilon$$

and

$$g_r^2(p) \prec g_{r^-}^3(p) \prec p \prec g_{r^+}^3(p) \prec g_r^2(h^{-1}(p)) \prec g_r^2(p).$$

Then by the continuity and monotonicity of g_r^3, if $d(p, p^*)$ is sufficiently small,

$$g_r^2(p) \prec g_{r^-}^3(p^*) \prec p^* \prec g_{r^+}^3(p^*) \prec g_r^2(h^{-1}(p)) \prec g_r^2(p).$$

By continuity of g_r^3, there exists r^* such that $r^- < r^* < r^+$ and $g_{r^*}^3(p^*) = p^*$. Then, $r^* = s(p^*)$ and $|r - r^*| < \epsilon$ so s is continuous.

$\square$

One can verify that

$$\Delta_{(a,0)} = \Big\{(a,0), \Big(-\frac{3a-a^3}{a^2+3}, \frac{2\sqrt{3}a}{a^2+3}\Big), \Big(-\frac{3a-a^3}{a^2+3}, -\frac{2\sqrt{3}a}{a^2+3}\Big)\Big\}$$
$$\Delta_{(-a,0)} = \Big\{(-a,0), \Big(\frac{3a-a^3}{a^2+3}, \frac{2\sqrt{3}a}{a^2+3}\Big), \Big(\frac{3a-a^3}{a^2+3}, -\frac{2\sqrt{3}a}{a^2+3}\Big)\Big\}$$

are inscribed equilateral triangles of side length $r_1 = s(\pm a, 0) = \frac{4\sqrt{3}a}{a^2+3}$, and that

$$\Delta_{(0,1)} = \Big\{(0,1), \Big(\frac{2\sqrt{3}a^2}{3a^2+1}, -\frac{3a^2-1}{3a^2+1}\Big), \Big(-\frac{2\sqrt{3}a^2}{3a^2+1}, -\frac{3a^2-1}{3a^2+1}\Big)\Big\}$$
$$\Delta_{(0,-1)} = \Big\{(0,-1), \Big(\frac{2\sqrt{3}a^2}{3a^2+1}, \frac{3a^2-1}{3a^2+1}\Big), \Big(-\frac{2\sqrt{3}a^2}{3a^2+1}, \frac{3a^2-1}{3a^2+1}\Big)\Big\}$$

are inscribed equilateral triangles of side length $r_2 = s(0, \pm 1) = \frac{4\sqrt{3}a^2}{3a^2+1}$. Furthermore we have $r_1 < r_2 < 2$.

Theorem 5.12. *For Y an ellipse of small eccentricity, the six vertices in the two inscribed equilateral triangles $\Delta_{(\pm a,0)}$ are global minima of $s\colon Y \to \mathbb{R}$, and the six vertices in the two inscribed equilateral triangles $\Delta_{(0,\pm 1)}$ are global maxima. There are no other local extrema of s.*

Proof. We first show that given any $r > 0$, there are at most twelve points $p \in Y$ satisfying $s(p) = r$. Let $(x_1, y_1), (x_2, y_2) \in Y$ be two points in the ellipse at distance r apart. We consider all points $(x, y) \in \mathbb{R}^2$ at distance r from each of (x_1, y_1) and (x_2, y_2). The triangle $\{(x_1, y_1), (x_2, y_2), (x, y)\}$ is then an inscribed equilateral triangle in Y if and only if $(x, y) \in Y$.

Consider the following system of polynomial equations in $x_1, y_1, x_2, y_2, x, y, a, r \in \mathbb{R}$:

$$\begin{aligned}
(x_1/a)^2 + y_1^2 - 1 &= 0 \\
(x_2/a)^2 + y_2^2 - 1 &= 0 \\
(x/a)^2 + y^2 - 1 &= 0 \\
(x_1 - x_2)^2 + (y_1 - y_2)^2 - r^2 &= 0 \\
(x - x_1)^2 + (y - y_1)^2 - r^2 &= 0 \\
(x - x_2)^2 + (y - y_2)^2 - r^2 &= 0.
\end{aligned} \tag{13}$$

The first three equations encode the requirement that $(x_1, y_1), (x_2, y_2), (x, y) \in Y$, and the last three equations encode the requirement that $\{(x_1, y_1), (x_2, y_2), (x, y)\}$ is an equilateral triangle. One can use elimination theory to eliminate the variables x_1, y_1, x_2, y_2, y from (13), producing a polynomial $p(x, a, r)$ such that any solution to (13) is also a root of p. We use the mathematics software system Sage to compute

$$
\begin{aligned}
p(x,a,r) = {} & 12288a^{12}r^4 - 6912a^{12}r^6 - 10752a^{10}r^6 + 13568a^8r^6 + 1296a^{12}r^8 \\
& + 4032a^{10}r^8 - 2208a^8r^8 - 6720a^6r^8 + 3600a^4r^8 - 81a^{12}r^{10} \\
& - 378a^{10}r^{10} - 63a^8r^{10} + 1044a^6r^{10} - 63a^4r^{10} - 378a^2r^{10} - 81r^{10} \\
& - 36864a^{10}r^4x^2 + 36864a^8r^4x^2 + 13824a^{10}r^6x^2 - 4608a^8r^6x^2 \\
& + 16896a^6r^6x^2 - 26112a^4r^6x^2 - 1296a^{10}r^8x^2 - 432a^8r^8x^2 \\
& - 4512a^6r^8x^2 + 4512a^4r^8x^2 + 432a^2r^8x^2 + 1296r^8x^2 + 36864a^8r^4x^4 \\
& - 73728a^6r^4x^4 + 36864a^4r^4x^4 - 6912a^8r^6x^4 + 15360a^6r^6x^4 \\
& - 16896a^4r^6x^4 + 15360a^2r^6x^4 - 6912r^6x^4 - 12288a^6r^4x^6 \\
& + 36864a^4r^4x^6 - 36864a^2r^4x^6 + 12288r^4x^6.
\end{aligned}
$$

Note p is a polynomial in x of degree 6. It follows that given any a and r, a solution to the system of equations (13) can contain at most 6 distinct values for x, and hence at most 12 distinct points $p = (x, y) \in Y$ satisfying $s(p) = r$.

We next show that the extrema of $s\colon Y \to \mathbb{R}$ are as claimed. Symmetry of s about the horizontal and vertical axes, along with the fact that the cardinality of each set $s^{-1}(r)$ is finite[2], gives that the points $(\pm a, 0), (0, \pm 1)$ are local extrema of s. Since $s(p) = s(g_{s(p)}(p)) = s(g^2_{s(p)}(p))$, it follows that all twelve vertices in the triangles $\Delta_{(\pm a,0)}, \Delta_{(0,\pm 1)}$ are local extrema. Since the six vertices of $\Delta_{(\pm a,0)}$ are interleaved with the six vertices of $\Delta_{(0,\pm 1)}$, it follows from the intermediate value theorem that for all $r_1 < r < r_2$ we have $|s^{-1}(r)| \geq 12$. Since $|s^{-1}(r)| \leq 12$ for all $r > 0$, it follows that s is either monotonically increasing or monotonically decreasing between any two adjacent vertices of the triangles $\Delta_{(\pm a,0)}$ and $\Delta_{(0,\pm 1)}$. Hence the vertices of $\Delta_{(\pm a,0)}$ are global minima, the vertices of $\Delta_{(0,\pm 1)}$ are global maxima, and there are no other local extrema of s. $\square$

6 Vietoris–Rips complexes of ellipses of small eccentricity

We restrict attention in this section to Y an ellipse of small eccentricity.

[2] This rules out infinite oscillations such as in the topologists' sine curve

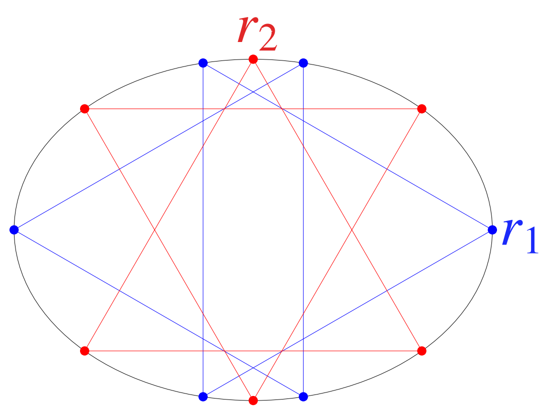

Figure 6: The blue equilateral triangles have side length r_1 and the red equilateral triangles have side length r_2, where $r_1 < r_2$. Theorem 5.12 says that r_1 and r_2 are the smallest and largest values of $s\colon Y \to \mathbb{R}$.

For finite $X \subseteq Y$ it follows from Lemma 5.8 that $\mathbf{VR}(X;r)$ is homotopy equivalent to an odd sphere or a wedge of even spheres of the same dimension.

The following notation will prove convenient. Given a point $p \in Y$, let $p' = g_{s(p)}(p)$ and let $p'' = g^2_{s(p)}(p)$. By Theorem 5.12 and the intermediate value theorem, there are cyclically ordered points

$$z_0 \prec z_1 \prec z_2 \prec z_3 \prec z_0' \prec z_1' \prec z_2' \prec z_3' \prec z_0'' \prec z_1'' \prec z_2'' \prec z_3'' \prec z_0$$

in Y such that

- $s(z_0) = s(z_1) = s(z_2) = s(z_3) = r$.
- $I_0 = (z_0, z_1)_Y \cup (z_0', z_1')_Y \cup (z_0'', z_1'')_Y$ and $I_2 = (z_2, z_3)_Y \cup (z_2', z_3')_Y \cup (z_2'', z_3'')_Y$ are each sets of three equivalence classes of fast points for dynamics $g_r : Y \to Y$, meaning that $s(p) < r$ for all points p in these intervals.
- $I_1 = (z_1, z_2)_Y \cup (z_1', z_2')_Y \cup (z_1'', z_2'')_Y$ and $I_3 = (z_3, z_0')_Y \cup (z_3', z_0'')_Y \cup (z_3'', z_0)_Y$ are each sets of three equivalence classes of slow points for dynamics $g_r : Y \to Y$, meaning that $s(p) > r$ for all points p in these intervals.

Theorem 6.1. *Let Y be an ellipse of small eccentricity. For any sufficiently dense finite sample $X \subseteq Y$, the graph $\mathrm{VR}(X;r)$ for $r_1 < r < r_2$ has at least* *two periodic orbits of length three, and hence* $\mathbf{VR}(X;r) \simeq \bigvee^i S^2$ *for some* $i \geq 1$.

Proof. We cyclically order the elements of X as $p_0 \prec p_1 \prec \ldots \prec p_{n-1} \prec p_0$, where all arithmetic operations on vertex indices are performed modulo n.

Recall from Definition 4.1 that associated to the cyclic graph $\mathrm{VR}(X;r)$ we have a map $f : X \to X$. Since f^m converges pointwise to g_r^m as the density increases, for X sufficiently dense each interval

$$(z_0, z_2)_Y, (z_2, z_0')_Y, (z_0', z_2')_Y, (z_2', z_0'')_Y, (z_0'', z_2'')_Y, (z_2'', z_0)_Y \tag{1}$$

contains points $p_i \in (z_0, z_1)_Y$, and $p_j \in (z_1, z_2)_Y$ such that

$$p_i \prec f^3(p_i) \preceq f^3(p_j) \prec p_j \prec p_i. \tag{2}$$

Now, suppose for a contradiction that there is no point $x \in [p_i, p_j]_Y \cap X$ with $x = f^3(x)$. Since f^3 weakly preserves the cyclic ordering we have

$$p_i \prec f^3(p_i) \preceq f^3(p_{i+1}) \preceq f^3(p_j) \prec p_j \prec p_i,$$

and since $p_{i+1} \neq f^3(p_{i+1})$ this gives

$$p_{i+1} \prec f^3(p_{i+1}) \preceq f^3(p_j) \prec p_j \prec p_{i+1}.$$

Note this is (2) with i replaced by $i+1$. Iterating this process gives

$$p_{j-1} \prec f^3(p_{j-1}) \preceq f^3(p_j) \prec p_j \prec p_{j-1},$$

contradicting the fact that $(p_{j-1}, p_j)_Y \cap X = \emptyset$. Hence there must be some $x \in [p_i, p_j]_Y \cap X$ with $x = f^3(x)$. Repeating this argument with each interval in (1) gives 6 periodic points, and hence at least two periodic orbits of length three.

The statement about the homotopy type of $\mathbf{VR}(X;r)$ follows from Proposition 4.2. □

Theorem 6.2. *Let Y be an ellipse of small eccentricity. For any $r_1 < r < r_2$, $\epsilon > 0$, and $m \geq 2$, there exists an ϵ-dense finite subset $X \subseteq Y$ so that the graph $\mathrm{VR}(X;r)$ has exactly m periodic orbits of length three, and hence* $\mathbf{VR}(X;r) \simeq \bigvee^{m-1} S^2$.

Proof. We will prove the theorem by showing that for any $n, n' \geq 1$ with $n + n' = m$, one can produce an ϵ-dense subset X of $Y \setminus \{z_i, z_i', z_i'' \mid i = 0, 1, 2, 3\}$ such that $I_0 \cap X$ contains n periodic orbits for $\mathrm{VR}(X; r)$, $I_2 \cap X$ contains n' periodic orbits for $\mathrm{VR}(X; r)$, and $I_1 \cap X$ and $I_3 \cap X$ contain no periodic orbits for f_r.

Pick a point $p_1 \in (z_0, z_1)_Y$ such that $d(p_1, z_1), d(p_1', z_1'), d(p_1'', z_1'') < \epsilon$. Add the points p_1, p_1', p_1'' to X; these points will form our first periodic orbit in $I_0 \cap X$. To see this, note that since p_1 is a fast point, $s(p_1) < r$, so:

$$p_1' = g_{s(p)}(p_1) \prec g_r(p_1) \prec g_r(z_1) = z_1' \prec p_1'$$

So long as we don't add any points in $(p_1', g_r(p_1)]_Y$, $(p_1'', g_r(p_1')]_Y$, or $(p_1, g_r(p_1'')]_Y$, we have that $f_r(p_1) = p_1'$, $f_r(p_1') = p_1''$, and $f_r(p_1'') = p_1$, and hence a periodic orbit of length 3. Next pick a point p_2 satisfying $p_2 \in (g_r(p_1''), z_1)_Y$, $p_2' \in (g_r(p_1), z_1')_Y$, and $p_2'' \in (g_r(p_1'), z_1'')_Y$. Add the points p_2, p_2', p_2'' to X to form our second periodic orbit. Iterating, for $2 \leq i \leq n$ we pick a point p_i satisfying $p_i \in (g_r(p_{i-1}''), z_1)_Y$, $p_i' \in (g_r(p_{i-1}), z_1')_Y$, and $p_i'' \in (g_r(p_{i-1}'), z_1'')_Y$, and we add the points p_i, p_i', p_i'' to X to form our i-th periodic orbit. This creates n periodic orbits in $I_0 \cap X$.

We now create ϵ-density in I_0. First note that by the continuity of $g_{s(p)}(p)$ in p, that for $p \in (z_0, z_1)_Y$, and $q \in (z_0, p)_Y$ sufficiently close to p,

$$d(q, p), d(q', p'), d(q'', p'') < \epsilon$$

Furthermore, for any point $p \in (z_0, z_1)_Y$, since p is fast, $d(p', g_r(p)) > 0$. Hence, by continuity of g_r, we see that for $q \in (z_0, p)_Y$ sufficiently close to p:

$$q' \prec p' \prec g_r(q) \prec g_r(p) \prec q'$$

Combining these facts: for any $p \in (z_0, z_1)_Y$, and $q \in (z_0, p)_Y$ sufficiently close to p, we have both of the above sets of conditions. Let $u \in (z_0, z_1)_Y$ satisfy $d(z_0, u) < \epsilon/4$. By compactness of $[u, p_1]_Y$, we have that there exists $\delta' > 0$ such that for all $p \in [u, p_1]_Y$, and $q \in (z_0, p)_Y$, with $d(p, q) < \delta'$, the above conditions hold. Let $\delta = \min(\delta', \epsilon)/2$. Add $q_1 = g_\delta^{-1}(p_1)$, q_1', q_1'' to X. Note that this doesn't create any periodic orbits. To see this, simply observe that $p_1' \prec g_r(q_1') \prec g_r(p_1) \prec p_2' \prec p_1'$, and hence $f_r(q_1) = p_1'$. Inductively define $q_n = g_\delta^{-1}(q_{n-1})$, for $n > 1$. Iteratively add q_n, q_n', q_n'' to X, so long as $q_n \in [u, p_1]_Y$. Notice similarly that this does not create any additional periodic orbits. Since $\delta > 0$, this process will terminate with some q_n. Notice that $d(q_n, z_0) \leq d(q_n, u) + d(u, z) < \frac{3}{4}\epsilon$, so we have ϵ-density in I_0.

Create n' periodic orbits in an ϵ-dense subset of I_2 via an analogous procedure.

Finally, we make X ϵ-dense by adding otherwise arbitrary points to I_1 and I_3. These new points cannot create any new periodic orbits for f_r because I_1 and I_3 are sets of slow points for $g_r : Y \to Y$. This concludes the construction of an ϵ-dense subset X so that $\mathrm{VR}(X; r)$ has exactly m periodic orbits.

The statement about the homotopy type of $\mathbf{VR}(X; r)$ follows from Proposition 4.2. $\square$

In an upcoming paper [5] [Note to the committee: The preprint will be upon arXiv by the time I submit the final draft of my thesis, and will be cited here] we generalize the cyclic dynamic approach presented herein to handle the infinite case, with the following results:

Theorem 6.3. *Let $1 < a \leq \sqrt{2}$, $r_1 = \frac{4\sqrt{3}a}{a^2+3}$, and $r_2 = \frac{4\sqrt{3}a^2}{3a^2+1}$. Then*

$$\mathbf{VR}_{<}(Y; r) \simeq \begin{cases} S^1 & \text{for } 0 < r \leq r_1 \\ S^2 & \text{for } r_1 < r \leq r_2 \end{cases}$$

$$\text{and} \quad \mathbf{VR}_{\leq}(Y; r) \simeq \begin{cases} S^1 & \text{for } 0 < r < r_1 \\ S^2 & \text{for } r = r_1 \text{ or } r_2 \\ \bigvee^5 S^2 & \text{for } r_1 < r < r_2. \end{cases}$$

Furthermore,

- *For $0 < r < r' \leq r_1$ or $r_1 < r < r' \leq r_2$, inclusion $\mathbf{VR}_{<}(Y; r) \hookrightarrow \mathbf{VR}_{<}(Y; r')$ is a homotopy equivalence.*
- *For $0 < r < r' < r_1$, inclusion $\mathbf{VR}_{\leq}(Y; r) \hookrightarrow \mathbf{VR}_{\leq}(Y; r')$ is a homotopy equivalence.*
- *For $r_1 \leq r < r' \leq r_2$, inclusion $\mathbf{VR}_{\leq}(Y; r) \hookrightarrow \mathbf{VR}_{\leq}(Y; r')$ induces a rank 1 map on 2-dimensional homology $H_2(\ ; \mathbb{F})$ for any field $\mathbb{F}$.*

Corollary 6.3. *The 1-dimensional persistent homology of $\mathbf{VR}_{<}(Y; r)$ (resp. $\mathbf{VR}_{\leq}(S^1; r)$) consists of a single interval $(0, r_1]$ (resp. $[0, r_1)$).*

The 2-dimensional persistent homology of $\mathbf{VR}_{<}(Y; r)$ consists of a single interval $(r_1, r_2]$. The 2-dimensional persistent homology of $\mathbf{VR}_{\leq}(S^1; r)$ consists of the interval $[r_1, r_2]$, as well as a point $[r, r]$ on the diagonal with multiplicity four for every $r_1 < r < r_2$.

7 Future work

In the future, we hope to expand on this work two primary ways. First we wish to expand to the case of $r_2 < r < 2$, so as to more fully understand the homotopy types of small eccentricity ellipses. Secondly, we hope to study the case of ellipses of larger eccentricity $a > \sqrt{2}$. In this case, we are no longer guaranteed cyclic graphs, and must therefore resort to alternative techniques to study homotopy type.

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