SWEEPING COSTS OF JORDAN DOMAINS

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Abstract. Let D be a Jordan domain in the plane. We consider a pursuit-evasion, contamination clearing, or sensor sweep problem in which the pursuer at each point in time is modeled by a continuous curve, called the *sensor curve*. Both time and space are continuous, and the intruders are invisible to the pursuer. Given D, what is the shortest length of a sensor curve necessary to provide a sweep of domain D, so that no continuously-moving intruder in D can avoid being hit by the curve? We define this length to be the *sweeping cost* of D. We provide an analytic formula for the sweeping cost of any Jordan domain in terms of the geodesic Fréchet distance between two curves on the boundary of D with non-equal winding numbers. As a consequence, we show that the sweeping cost of any convex domain is equal to its width. We also explore the computation of the sweeping cost of Jordan domains via a geodesic pathfinding algorithm alongside the Fréchet distance algorithm for curves. Finally, we consider a natural generalization to higher dimensional Jordan domains D (homeomorphic images of n-dimensional balls) in $n \geq 2$ dimensions. Here we conjecture that the sweeping cost is realized as a hypersurface filling the "best" map $f\colon I \times S^{n-2} \to \partial D$ where f has nonzero degree.

Contents

1. Introduction

Let D be a Jordan domain, i.e. the homeomorphic image of a disk in the plane. Suppose that continuously-moving intruders wander in D. You and a friend are each given one end of a rope, and your task is to drag this rope through the domain D in such a way so that every intruder is eventually intersected or caught by the rope. What is the shortest rope length you need in order to catch every possible intruder? We refer to such a continuous rope motion as a *sweep* of D (Figure 2), and we refer to the length of the shortest such possible rope as the *sweeping cost* of D.

The problem we consider is only one example of a wide variety of interesting pursuit-evasion problems; see Section 2 for a brief introduction or [15], for example, for a survey. It is a pursuit-evasion problem in which both space and time are continuous, the pursuer is modeled at each point in time by a continuous curve, the intruder has complete information about the pursuer's location and its planned future movements, and the pursuer has no knowledge of the intruder's movements. Our problem can also be phrased as a contamination-clearing task, in which one must find the shortest rope necessary to clear domain D of a contaminant which, when otherwise unrestricted by the rope, moves at infinite speed to fill its region.

As first examples, the sweeping cost of a disk is equal to its diameter, and the sweeping cost of an ellipse is equal to the length of its minor axis. These computations follow from Theorem 5.4, in which we prove that the sweeping cost of domain D is at least as large as the shortest area-bisecting curve in D.

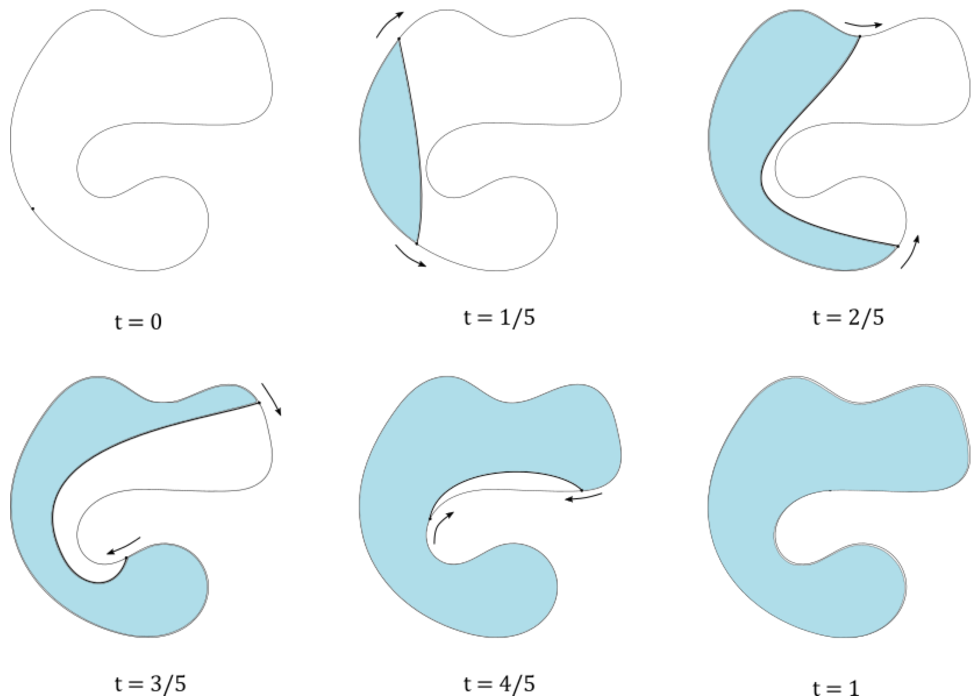

FIGURE 1. An example sweep of a domain in the plane. A time $t = 0$ the entire domain is contaminated, and at time $t = 1$ the clearing sweep is complete.

One motivation for considering a pursuer which is a rope, or a continuous curve at each point in time, is the context of mobile sensor networks. Suppose there is a large collection of disk-shaped sensors moving inside a planar domain, as considered in [19, 2]. What is the minimal number of sensors needed to clear this domain of all possible intruders? If n is the number of sensors, and $\frac{1}{n}$ is the diameter of each sensor, then as $n \to \infty$ an upper bound for the number of sensors needed is given by the sweeping cost.

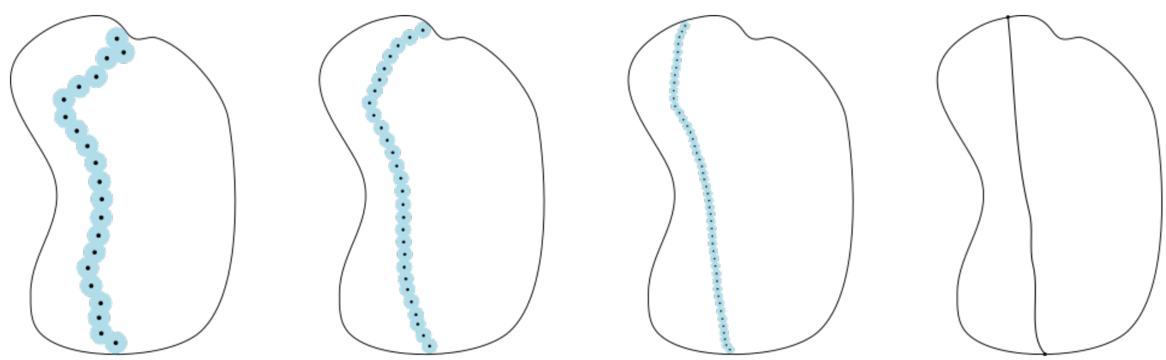

FIGURE 2. Shrinking the radius of the sensors and increasing the amount until ultimately they form a curve in the domain.

As our main result, in Theorem 7.1 we provide an analytic formula for the sweeping cost of an arbitrary planar Jordan domain D. Indeed, the sweeping cost of D is equal to the infimum geodesic Fréchet distance [20] between two curves in the boundary of D whose concatenation wraps a nontrivial number of times around the boundary. The geodesic Fréchet distance differs from the standard Fréchet distance in that the distance between two points in D is not their Euclidean distance, but instead the length of the shortest path between them in D.

Using our main result, we prove in Theorem 8.1 that the sweeping cost of a convex domain D is equal to the width of D. As a consequence, it follows that the sweeping cost of a polygonal convex domain with n vertices can be computed in time $O(n)$ and space $O(n)$ using the *rotating calipers* technique [34]. Furthermore, it follows from [30, 12] that a convex domain of unit area with the maximal possible sweeping cost—i.e. the most expensive convex domain to clear with a rope—is the equilateral triangle. Our work on sensor sweeps of planar domains also appears in [1].

With a notion of the sweeping cost of a planar Jordan domain, there become many other natural extensions of this problem. Notably, we extend this Jordan domains in dimensions $n \geq 2$. Let D now represent the homeomorphic image of an n-dimensional ball in $\mathbb{R}^d$. You are now given a $(n-1)$-dimensional hypersurface which is dragged through D in order to eventually intersect each intruder. What is the hypersurface of lowest possible $(n-1)$-dimensional volume needed in order to catch every possible intruder? This is defined as the sweeping cost for D in dimension n. Most results are generalized but without the use of the Fréchet distance as this is restricted to curves. Instead the idea of minimal hypersurfaces filling the inside of a time-varying sphere becomes useful.

An intriguing open question motivated by our work with planar domains is the following (Question 12.1). Given a Jordan domain D and two continuous injective curves α, β with image in the boundary ∂D, is the weak geodesic Fréchet distance between α and β equal to the strong geodesic Fréchet distance? The weak version of the Fréchet distance allows α and β to be repraparametrized non-injectively, whereas the strong version does not.

We review related work in Section 2, state our problem of interest in Section 3, and describe some basic properties of the sweeping cost in Section 4. In Section 5 we provide a lower bound on the sweeping cost in terms of shortest area-bisecting curves. We provide analytic formulas for the sweeping costs of Jordan and convex domains in Sections 7 and 8, and in Section 9 we deduce that the sweeping cost of a unit-area convex domain is maximized by the equilateral triangle. Following, we briefly consider the higher dimensional analog to this problem throughout Section 11. The conclusion describes related problems of interest, and the appendix contains two technical lemmas and their proofs.

2. Related work

A wide variety of pursuit-evasion problems have appeared in the mathematics, computer science, engineering, and robotics literature; see [15] for a survey. Space can modeled in a discrete fashion, for example by a graph [4, 8], or as a continuous domain in Euclidean space as we study here. Time can similarly be discrete (turn-based) or continuous, as is our case. See [3, 7, 14, 18, 28, 31] for a selection of such problems.

A further important distinction in a pursuit-evasion problem is whether information is complete (pursuers and intruders know each others' locations), incomplete (pursuers and intruders are invisible to each other), or somewhere in-between. Our problem can be considered as one in which the pursuer has no knowledge of the intruders' movements, whereas the intruders have complete knowledge of the pursuer's current position and future movements. In other words, the pursuer must catch every possible intruder. Evasion problems in which the pursuer has no information and the intruders have complete information can equivalently be cast as contamination-clearing problems, see for example [5, 19, 2]. Indeed, the contaminated region of the domain at a particular time includes all locations where an intruder could currently be located, and the uncontaminated region is necessarily free of intruders. It is the task of the pursuer to clear the entire domain of contamination, so that no possible intruders could remain undetected.

The paper [20] introduces the *geodesic width* between two polylines, a notion that is very relevant for our problem and in particular Theorem 7.1. That same paper also studies sweeps of planar domains by piecewise linear curves in which the cost of a sweep is not equal to a length, but instead to the number of vertices or joints in the curve. Related notions to the geodesic width include the isotopic Fréchet distance between two curves [13], and the minimum deformation area [35].

3. Preliminaries and notation

3.1. **Homeomorphisms.** In the field of topology, a *homeomorphism* between two spaces X and Y is considered an equivalence. This notion is helpful to study equivalences of spaces where geometrical properties such as distance and curvature are not as important. In a sense, this equivalence relation topologists use tells us about the global idea of the space we are looking at up to stretching and bending but not tearing or gluing.

Definition 3.1. Let X and Y be topological spaces. Then we say X is *homeomorphic* to Y if there exists a *homeomorphism* $f\colon X \to Y$. f is a homeomorphism if and only if f is bijective and continuous with continuous inverse.

3.2. **Jordan domains and geodesics.** Let $D \subseteq \mathbb{R}^2$ be a Jordan domain, i.e. the homeomorphic image of a closed disk in $\mathbb{R}^2$. It follows that D is compact and simply-connected, and its boundary ∂D is a topological circle. Let $d\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ denote the Euclidean metric on $\mathbb{R}^2$, and also the induced submetric on D. The distance between two subsets $X, Y \subseteq \mathbb{R}^2$ is defined as $d(X, Y) = \sup\{d(x, y) \mid x \in X \text{ and } y \in Y\}$.

We refer the reader to [11] for the basics of geodesic curves and distances. The length of a continuous path $\gamma\colon [a, b] \to D$ is defined as in [11, Defintion 2.3.1]; we denote this length by $L(\gamma)$. Curve γ is said to be *rectifiable* if $L(\gamma) < \infty$. Domain D has a *length structure* ([11, Section 2.1]) in which all continuous paths are admissible, and the length is given by the function L. The associated *geodesic metric* $d_L\colon D \times D \to \mathbb{R}$, also known as a *path-length* or *intrinsic metric*, is

$$d_L(x, y) = \inf\{L(\gamma) \mid \gamma\colon [a, b] \to D \text{ is continuous with } \gamma(a) = x,\ \gamma(b) = y\}.$$

The precise definition of a geodesic, or length-minimizing curve in D, is given in [11, Definition 2.5.27]. Since D is a Jordan domain, it follows from [10, 9] that each pair of points in D is joined by a unique shortest geodesic in D. We define the geodesic distance between two curves $\alpha, \beta\colon [a, b] \to D$ to be

$$d_L(\alpha, \beta) = \max_{t\in[a,b]} d_L(\alpha(t), \beta(t)).$$

Given a point $x \in D$, we let $B(x, \epsilon) = \{y \in D \mid d(x, y) < \epsilon\}$ denote the open ball about x in D. We denote the ϵ-offset of a set $X \subseteq D$ by $B(X, \epsilon) = \cup_{x\in X} B(x, \epsilon)$, and the closure of X in $\mathbb{R}^2$ by $\overline{X}$. Given a subset $X \subseteq D$, we define its boundary as $\partial X = \overline{X} \cap \overline{\mathbb{R}^2 \setminus X}$.

3.3. **Degree of a continuous mapping.** The n-sphere S^n is defined by

$$\{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}.$$

For the planar sensor sweeps, we will use the notion of *winding number*. The winding number is the degree of a continuous map from S^1 to itself. Roughly speaking, given a continuous map $f\colon S^1 \to S^1$, the winding number $\mathrm{wn}(f)$ counts the integer number of times f wraps around the image circle. We can define this more specifically by using polar coordinates.

Definition 3.2. Let $f\colon [0, 1] \to \mathbb{R}^2$ be continuous and given by $t \mapsto (r(t), \theta(t))$ with $\theta(t) \in \mathbb{R}$. Then the *winding number* of f is given by

$$\mathrm{wn}(f) = \frac{\theta(1) - \theta(0)}{2\pi}.$$

This idea extends to a higher dimensional analog called the *degree of a continuous mapping*. In essence, if we have $f\colon S^n \to S^n$ then the degree of f, $\deg(f)$, counts the number of times the n sphere wraps around itself. We can define this rigorously using the *nth homology group* of a topological space X, $H_n(X)$. Note, $H_n(S^n) \cong \mathbb{Z}$.

Definition 3.3. Let $f\colon S^n \to S^n$, then there is a naturally induced map $f_*\colon H_n(S^n) \to H_n(S^n)$ where $f_*\colon x \mapsto \lambda x$ with $\lambda \in \mathbb{Z}$. Then the *degree of* f is given by

$$\deg(f) = \lambda.$$

3.4. **The Hausdorff and Fréchet distance.** The Hausdorff distance is used to measure the distance between sets in a given metric space. In this case, we are using the induced metric from $\mathbb{R}^n$. The Hausdorff distance is defined as follows.

Definition 3.4. Let X and Y be two non-empty subsets of a metric space (in particular we consider $\mathbb{R}^n$ with the standard Euclidean metric). Then for $x \in X$, define $d_{inf}(x, Y) = \inf\{d(x, y) | y \in Y\}$. Then the Hausdorff distance, d_H, is given by,

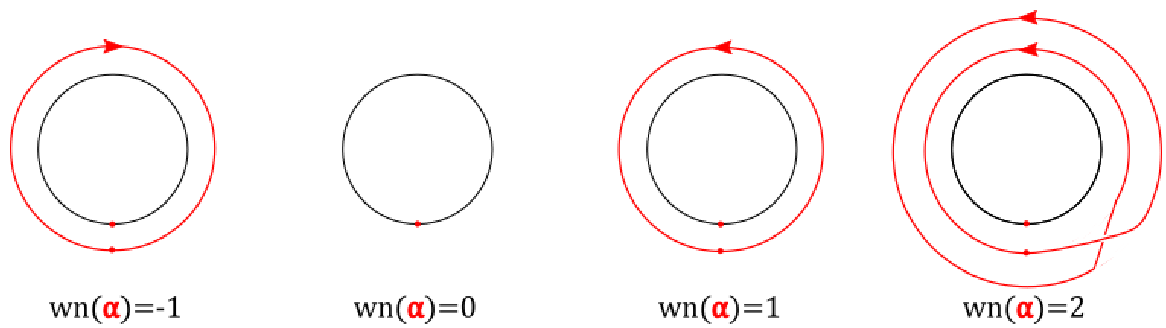

FIGURE 3. With an orientation chosen (counter clockwise is postive) and a chosen base point (shown in red), examples of the winding number can seen as depicted in the figure. Note that the red windings would be on the circle itself but are pulled away to display the winding number.

$$d_H(X,Y) = \max\{\sup_{x\in X} d_{inf}(x,Y), \sup_{y\in Y} d_{inf}(y,X)\}.$$

The *Fréchet distance* is used to develop a measure of similarity between two curves $A, B\colon [0,1] \to \mathbb{R}^n$. There are many applications in which comparing the similarity of two curves is helpful. One would be handwriting input recognition for a computer. In order to properly tell which letters a user has written, the machine must find the letters with the most obvious similarity. Other means of measurement, like the Hausdorff distance, are not necessarily good at this.

The physical idea of the Fréchet distance is that you are walking with your dog and you both follow the paths A and B and you want to know how long of a leash you need to have in order to walk the whole path. There are two notions, namely the *weak Fréchet distance* and the *strong Fréchet distance*. In the weak case, you and your dog are both allowed to back track in order to walk your paths and in the strong this is forbidden. These can provide different answers.

Definition 3.5. Let $A, B\colon [0,1] \to \mathbb{R}^n$ and let $a, b\colon [0,1] \to [0,1]$ be surjective (resp. bijective) functions. Then the *weak (resp. strong) Fréchet distance* between A and B be defined as

$$F(A,B) = \inf_{a,b} \max_{t\in[0,1]} \{d\left(A(a(t), B(b(t))\right\}.$$

It is most easy to see the difference in weak and strong cases by looking at the *Fréchet free space diagrams*. To construct these diagrams one chooses a value for ϵ which represents the leash length that you walk your dog with. The diagram is computed by sampling a discrete number of points along two curves and comparing them to this chosen ϵ.

The algorithm is fairly intuitive. Step along your first curve A and compare these distances to the first point on curve B. Then we increment to the next point along B and repeat the sampling along all of A. This is repeated until all points along B have been compared to all points along A. Points are then colored in the plot based on whether or not the distance is less than or greater than the chosen ϵ. If it is less than, then that represents a coordinate in which your leash length would allow you to stand on your curve A and your dog on curve B.

The harder portion of the algorithm to truly compute the Fréchet distance involves finding a path from the bottom left corner $A(0)$ and $B(0)$ to the top right corner $A(1)$ and $B(1)$. The idea here is to start with a very small ϵ in which no path exists and increase ϵ until a path forms. In some cases this initial path may be non-monotonic. This is the case where the weak and strong distances will differ. It is worthwhile to see some examples of this with curves in the plane and compare the weak and strong distances as well as the typical Hausdorff distance.

Example 3.6. Consider the two curves $A, B\colon [0,1] \to \mathbb{R}^2$ given by the following piecewise equations. See 4

$$A(s) = \begin{cases} (6s-1,-2) & \text{if } 0 \le s < \frac{1}{3} \\ (-6(s-1/3)+1, 3s-3) & \text{if } \frac{1}{3} \le s < \frac{2}{3} \\ (6(s-2/3)-1,-1) & \text{if } \frac{2}{3} \le s \le 1. \end{cases}$$

$$B(s) = \begin{cases} (14s-6/5, 1-s) & \text{if } 0 \le s < \frac{1}{5} \\ (-12(s-1/5)+8/5, 4/5) & \text{if } \frac{1}{5} \le s < \frac{2}{5} \\ (10(s-2/5)-4/5, 5(s-2/5)+4/5) & \text{if } \frac{2}{5} \le s \le \frac{3}{5}. \\ (-8(s-3/5)+6/5, s-6/5) & \text{if } \frac{3}{5} \le s \le \frac{4}{5}. \\ (7(s-4/5)-2/5, -5(s-4/5)+2) & \text{if } \frac{4}{5} \le s \le 1. \end{cases}$$

These curves are a single example where the Hausdorff, weak, and strong Fréchet distance are all different. In this case the Hausdorff distances was estimated by computation and yielded a result of 2.8108. The weak Fréchet distance can actually be found analytically as the points $A(0)$ and $B(0)$ are the the furthest points for the leash to stretch and this distance was $\frac{\sqrt{226}}{5} \approx 3.0067$. The strong Fréchet distance is tougher to compute, but can be seen in the diagram to be larger than the weak since the path from $s = 0$ to $s = 1$ is not monotonic when the ϵ is chosen to be 3.01.

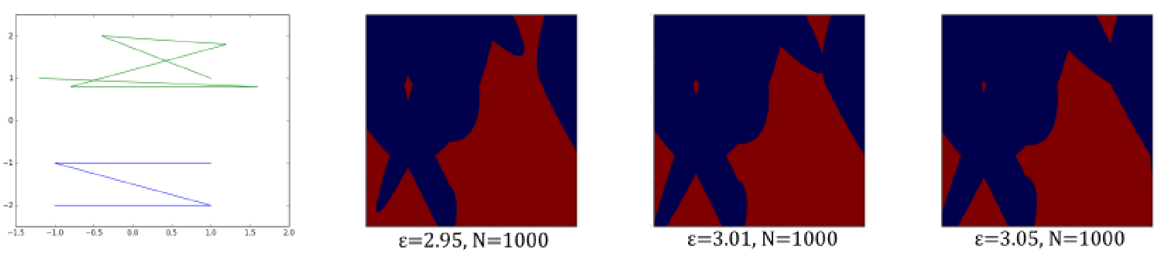

FIGURE 4. Plot of the two curves (A in green and B in blue) and three free space diagrams for various ϵ leash sizes. Blue represents two points along the curves whose distance is $\le \epsilon$ and red represents the points where it is not. Note when $\epsilon = 3.01$ a monotonic path does not exist but does when $\epsilon = 3.05$.

In this paper, we utilize this idea in sweeping Jordan domains by gluing the endpoints of these paths together forming a topological circle and we consider these curves to also be a boundary that a leash cannot pass through.

4. Sensor network sweeps

4.1. **Sensor curves.** Let $I = [0,1]$ be the unit interval. We define a sensor curve to be a time-varying rectifiable curve in D.

Definition 4.1. A *sensor curve* is a continuous map $f\colon I \times I \to D$ such that

(i) each curve $f(\cdot, t)\colon I \to D$ is rectifiable and injective for $t \in (0,1)$,
(ii) $f(I,0)$ and $f(I,1)$ are each (possibly distinct) single points in ∂D, and
(iii) $f(s,t) \in \partial D$ implies $s \in \{0,1\}$ or $t \in \{0,1\}$.

We think of the first input s as a spatial variable and of the second t as a temporal variable; in particular $f(I,t)$ is the region covered by the curve of sensors at time t. Assumption (ii) states that the images of the sensor curve at times 0 and 1 are single points, and assumption (iii) implies that (apart from times 0 and 1) only the boundary of the sensor curve intersects ∂D. We define the length of a sensor curve to be

$$L(f) = \max_{t \in I} L(f(\cdot, t)).$$

An *intruder* is a continuous path $\gamma\colon I \to D$. We say that an intruder is caught by a sensor curve f at time t if $\gamma(t) \in f(I,t)$. A path $\gamma\colon [0,t] \to D$ such that $\gamma(t') \notin f(I,t')$ for all $t' \in [0,t]$ is called an *evasion path*. Sensor curve f is a *sweep* if every continuously moving intruder $\gamma\colon I \to D$ is necessarily caught at some time t, or equivalently, if no evasion path over the full time interval I exists.[1]

The following notation will prove convenient. Fix a sensor curve f. We let $C(t) \subseteq D$ be the *contaminated region* at time t, and we let $U(t) \subseteq D$ be the *uncontaminated region* at time t. More precisely,

$$C(t) = \{x \in D \mid \exists\ \gamma\colon [0,t] \to D \text{ with } \gamma(t) = x \text{ and } \gamma(t') \notin f(I,t')\ \forall t' \in [0,t]\} \quad \text{and} \quad U(t) = D \setminus C(t).$$

Note that sensor curve f is a sweep if and only if $C(1) = \emptyset$, or equivalently $U(1) = D$.

Definition 4.2. Let $\mathcal{F}(D)$ be the set of all sensor curve sweeps of D. The *sweeping cost* of D is

$$\mathrm{SC}(D) = \inf_{f \in \mathcal{F}(D)} L(f).$$

Remark 4.3. The results of Sections 4–5 hold even if assumptions (ii) and (iii) in Definition 4.1 are removed.

4.2. **Properties.** We now prove some basic properties of sensor sweeps and the contaminated and uncontaminated regions.

Lemma 4.4. *If x and x' are in the same path-connected component of $D \setminus f(I,t)$, then $x \in U(t)$ if and only if $x' \in U(t)$.*

Proof. Suppose for a contradiction that $x \in U(t)$ but $x' \notin U(t)$. Since $x' \in C(t)$, there exists an evasion path $\gamma : [0,t] \to D$ with $\gamma(t) = x'$. Since x and x' are in the same path-connected component, there exists a path $\beta\colon I \to D \setminus f(I,t)$ with $\beta(0) = x$ and $\beta(1) = x'$.

Note that $\beta(I)$ and $f(I,t)$ are compact, since they are each a continuous image of the compact set I. As any metric space is normal, there exist disjoint neighborhoods containing $\beta(I)$ and $f(I,t)$. Because f is uniformly continuous (it is a continuous function on a compact set), we can choose $\delta_1 > 0$ such that $f(I,[t-\delta_1,t])$ remains in this open neighborhood disjoint from $\beta(I)$, giving

$$\beta(I) \cap f(I,[t-\delta_1,t]) = \emptyset. \tag{1}$$

Since metric space D is normal, there exist disjoint neighborhoods containing $\gamma(t) = x'$ and $f(I,t)$. Since γ is continuous and f is uniformly continuous, we can choose $\delta_2 > 0$ such that $\gamma([t-\delta_2,t])$ and $f(I,[t-\delta_2,t])$ remain in these disjoint neighborhoods, giving

$$\gamma([t-\delta_2,t]) \cap f(I,[t-\delta_2,t]) = \emptyset \tag{2}$$

Let $\delta = \min\{\delta_1,\delta_2\}$. Using (1) and (2) we can define an evasion path $\gamma\colon [0,t] \to D$ with $\gamma(t) = x$. Indeed, let

$$\gamma(t') = \begin{cases} \gamma(t') & \text{if } t' \le t-\delta \\ \gamma(2t'-t+\delta) & \text{if } t-\delta < t' \le t-\frac{\delta}{2} \\ \beta(\frac{2}{\delta}(t'-t+\frac{\delta}{2})) & \text{if } t-\frac{\delta}{2} < t' \le t. \end{cases}$$

This contradicts the fact that $x \in U(t)$. □

Lemma 4.5. *For all $t \in I$, the set $U(t)$ is closed and the set $C(t)$ is open in D.*

Proof. Suppose $x \in C(t)$. Since $f(I,t)$ is closed and $x \notin f(I,t)$, there exists some $\varepsilon > 0$ such that $B(x,\varepsilon) \cap f(I,t) = \emptyset$. Note all $x' \in B(x,\varepsilon)$ are in the same path-connected component of $D \setminus f(I,t)$ as x via a straight line path. Hence Lemma 4.4 implies $B(x,\varepsilon) \subseteq C(t)$, showing $C(t)$ is open in D. It follows that $U(t) = D \setminus C(t)$ is closed in D. □

Lemma 4.6. *If $h\colon D \to h(D)$ is a homeomorphism onto its image $h(D) \subseteq \mathbb{R}^2$, then a sensor curve $f\colon I \times I \to D$ is a sweep of D if and only if sensor curve $hf\colon I \times I \to h(D)$ is a sweep of $h(D)$.*

Proof. Note that if $\gamma\colon I \to D$ is an evasion path for f, then $h\gamma\colon I \to h(D)$ is an evasion path for hf. Conversely, if $\gamma\colon I \to h(D)$ is an evasion path for hf, then $h^{-1}\gamma\colon I \to D$ is an evasion path for f. □

[1] Our definition is similar to the graph-based definition in [4, Definition 2.1].

5. A lower bound on the sweeping cost

In this section we prove that the sweeping cost of a Jordan domain is at least as large as the length of the shortest area-bisecting curve. The first lemma is a version of the intermediate value theorem with slightly relaxed hypotheses.

Lemma 5.1. *If $f\colon [a,b] \to \mathbb{R}$ is upper semi-continuous and left continuous and if $f(a) < u < f(b)$, then there exists some $c \in (a,b)$ with $f(c) = u$.*

Proof. Let S be the set of all $x \in (a,b)$ with $f(x) < u$. Then S is nonempty since $a \in S$, and S is bounded above by b. Hence by the completeness of $\mathbb{R}$, the supremum $c = \sup S$ exists. We claim that $f(c) = u$.

Let $\epsilon > 0$. Since f is left-continuous, there is some $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$ whenever $x \in (c-\delta, c]$. By the definition of supremum, there exists some $y \in (c-\delta, c]$ that is contained in S, giving $f(c) < f(y) + \epsilon < u + \epsilon$. Since this is true for all $\epsilon > 0$, it follows that $f(c) \leq u$.

It remains to show $f(c) \geq u$. Let $\epsilon > 0$. Since f is upper semi-continuous, there exists a $\delta > 0$ such that $f(c) > f(x) - \epsilon$ whenever $x \in (c-\delta, c+\delta)$. Let $y \in (c, c+\delta)$ and note that $y \notin S$, giving $f(c) > f(y) - \epsilon \geq u - \epsilon$. It follows that $f(c) \geq f(u)$. □

Lemma 5.2. *The function* $\text{area}(U(t))$ *is upper semi-continuous and left continuous.*

Proof. Let $t_0 \in I$. We will show that $\text{area}(U(t))$ is right upper semi-continuous and left continuous at t_0, which implies the function is both upper semi-continuous and left continuous.

For right upper semi-continuity, note for $t \geq t_0$ we have

$$U(t) \subseteq U(t_0) \cup f(I, [t_0, t]). \tag{3}$$

Since sensor curve $f\colon I \times I \to D$ is a continuous function on a compact domain, it is also uniformly continuous. Hence for all $\epsilon > 0$ there exists some δ such that

$$f(I, [t_0, t_0 + \delta]) \subseteq B(f(I, t_0), \epsilon). \tag{4}$$

It follows that for all $t \in [t_0, t_0 + \delta]$ we have

$$\begin{aligned} \text{area}(U(t)) - \text{area}(U(t_0)) &\leq \text{area}(f(I, [t_0, t])) && \text{by (3)} \\ &\leq \text{area}(B(f(I, t_0), \epsilon)) && \text{by (4)} \\ &\leq 2L(f(I, t_0))\epsilon + \pi\epsilon^2, \end{aligned}$$

where the last inequality is by a result of Hotelling (see for example [26, Equation (2.1)]). Hence $\text{area}(t)$ is right upper semi-continuous.

To see that $\text{area}(U(t))$ is left continuous at $t_0 \in I$, we must show that for all sequences $\{s_i\}$ with $0 \leq s_i \leq t_0$ and $\lim_i s_i = t_0$, we have $\lim_i \text{area}(U(s_i)) = \text{area}(U(t_0))$. We claim

$$U(t_0) \setminus f(I, t_0) \subseteq \liminf_i U(s_i) \subseteq \limsup_i U(s_i) \subseteq U(t_0), \tag{5}$$

where the middle containment is by definition. We now justify the first and last containment.

To prove $U(t_0) \setminus f(I, t_0) \subseteq \liminf_i U(s_i)$ it suffices to show that for any $x \in U(t_0) \setminus f(I, t_0)$ there exists an $\epsilon > 0$ such that $x \in U(t_0 - \delta)$ for all $\delta \in [0, \epsilon)$. Fix ϵ such that $x \notin f(I, t)$ for $t \in (t_0 - \epsilon, t_0]$. Suppose for a contradiction that $x \in C(t_0 - \delta)$ for some $\delta \in [0, \epsilon)$. Hence there exists an evasion path $\gamma\colon [0, t_0 - \delta] \to D$ with $\gamma(t_0 - \delta) = x$. It is possible to extend γ to an evasion path $\tilde{\gamma}\colon [0, t_0] \to D$ defined by

$$\tilde{\gamma}(t) = \begin{cases} \gamma(t) & \text{if } t \in [0, t_0 - \delta] \\ x & \text{if } t \in (t_0 - \delta, t_0]. \end{cases}$$

This contradicts the fact $x \in U(t_0)$, thus giving the first containment.

We now show $\limsup_i U(s_i) \subseteq U(t_0)$. If $x \notin U(t_0)$, then there exists an evasion path $\gamma\colon [0, t_0] \to D$ with $\gamma(t_0) = x$. Since $C(t_0)$ is open by Lemma 4.5, there exists some $\delta > 0$ such that $B(x, \delta) \subseteq C(t_0)$, and hence $B(x, \delta) \cap f(I, t_0) = \emptyset$. Since f is uniformly continuous, there is some $\epsilon_1 > 0$ sufficiently small with

$B(x,\delta) \cap f(I,[t_0-\epsilon_1,t_0]) = \emptyset$, and since γ is continuous there is some $\epsilon_2 > 0$ with $\gamma([t_0-\epsilon_2,t_0]) \subseteq B(x,\delta)$. Let $\epsilon = \min\{\epsilon_1,\epsilon_2\}$. Reparametrize γ to get a continuous curve $\tilde{\gamma}\colon [0,t_0] \to D$ with

$$\tilde{\gamma}(t) = \begin{cases} \gamma(t) & \text{if } x \in [0, t_0-\epsilon) \\ \gamma(t) \in B(x,\delta) & \text{if } t \in [t_0-\epsilon, t_0-\epsilon/2) \\ \gamma(t) = x & \text{if } t \in [t_0-\epsilon/2, t_0]. \end{cases}$$

The evasion path $\tilde{\gamma}$ shows $x \notin \limsup_i U(s_i)$, giving the third containment and finishing the proof of (5).

Set $f(I,t_0)$ has Lebesgue measure zero since curve $f(\cdot,t_0)$ is rectifiable, giving $\text{area}(U(t_0) \setminus f(I,t_0)) = \text{area}(U(t_0))$. Thus (5) implies

$$\text{area}(\limsup_i U(s_i)) = \text{area}(U(t_0)) = \text{area}(\liminf_i U(s_i)).$$

Since $\text{area}(D)$ is finite, Lemma A.1 implies $\limsup_i \text{area}(U(s_i)) \leq \text{area}(\limsup_i U(s_i))$ and $\text{area}(\liminf_i U(s_i)) \leq \liminf_i \text{area}(U(s_i))$, giving

$$\limsup_i \text{area}(U(s_i)) \leq \text{area}(U(t_0)) \leq \liminf_i \text{area}(U(s_i)).$$

Hence $\lim_i \text{area}(U(s_i)) = \text{area}(U(t_0))$ as required. □

Remark 5.3. The function $\text{area}(U(t))$ need not be right continuous. Indeed, consider a sensor curve as shown below in Figure 5, where $\text{area}(U(t_0)) > 0$, and where there is some $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$, only one point on the sensor curve at time $t_0 + \epsilon$ intersects ∂D and $\text{area}(U(t_0+\epsilon)) = 0$.

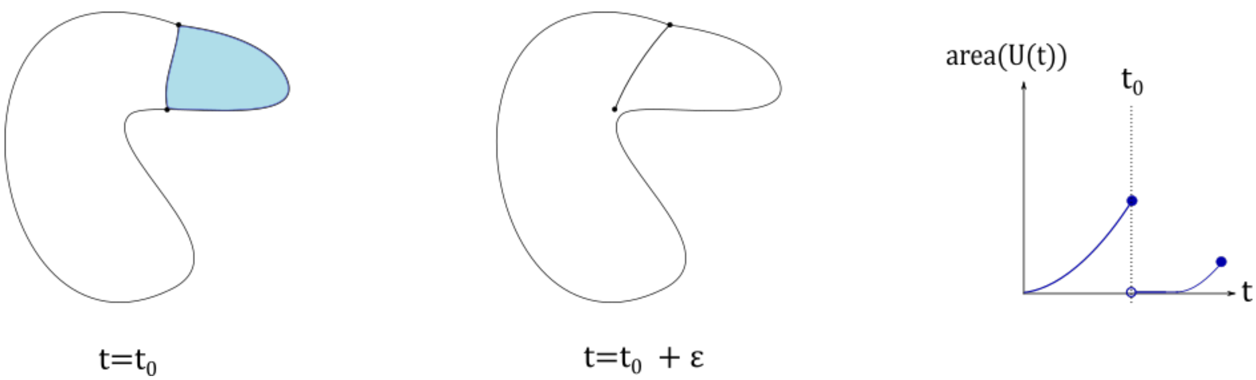

FIGURE 5. An example sensor curve where the function $\text{area}(U(t))$ is not right continuous. The shaded region is $U(t)$ and the unshaded region is $C(t)$.

As a consequence we obtain the following lower bound on the sweeping cost.

Theorem 5.4. *If D is a Jordan domain, then the sweeping cost $\text{SC}(D)$ is at least as large as the length of the shortest area-bisecting curve in D.*

Proof. Suppose that f is a sweep of D. Note that $\text{area}(U(0)) = 0$ and $\text{area}(U(1)) = \text{area}(D)$. By Lemmas 5.1 and 5.2, there exists some time $t' \in I$ with $\text{area}(U(t')) = \frac{1}{2}\text{area}(D)$. So $L(f(\cdot,t'))$ and hence $\text{SC}(D)$ is at least as large as the shortest area-bisecting curve in D. □

Example 5.5. If $D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\}$ is the unit disk, then $\text{SC}(D) = 2$.

Proof. To see $\text{SC}(D) \leq 2$, consider the sweep $f\colon [-1,1] \times I \to D$ defined by $f(s,t) = (2s\sqrt{t-t^2}, 2t-1)$ (Figure 6) which has length 2.

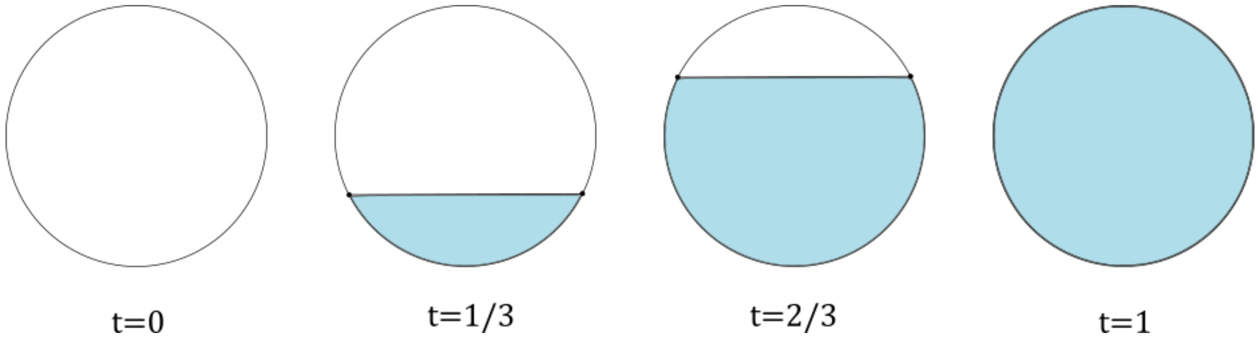

FIGURE 6. A sweep of the unit disk. The shaded region is $U(t)$ and the unshaded region is $C(t)$.

For the reverse direction, note that the shortest area-bisecting curve in D is a diameter [21]. Hence we apply Theorem 5.4 to get $\mathrm{SC}(D) \geq 2$. □

Example 5.6. Let $a, b > 0$. If $D = \{(x,y) \in \mathbb{R}^2 \mid (x/a)^2 + (y/b)^2 \leq 1\}$ is the convex hull of an ellipse, then $\mathrm{SC}(D) = \min\{2a, 2b\}$.

Proof. To see $\mathrm{SC}(D) \leq \min\{2a, 2b\}$, construct a sweep much like in Example 5.5.

For the reverse direction, the solution to [32, Chapter X, Problem 33] states that because the ellipse has a center of symmetry, the shortest area-bisecting curve is a straight line. All area-bisecting lines pass through the center of the ellipse, and hence have length at least $\min\{2a, 2b\}$. It follows from Theorem 5.4 that $\mathrm{SC}(D) \geq \min\{2a, 2b\}$. □

6. A lemma of no progress for planar domains

In Sections 7–9 we will restrict attention to sensor curves f with boundary points $f(0,t), f(1,t) \in \partial D$ for all $t \in I$. The motivation behind this assumption is Lemma 6.2, which states that if $f(0,t) \notin \partial D$ or $f(1,t) \notin \partial D$, then the uncontaminated region at time t is as small as possible, namely $U(t) = f(I,t)$.

The following lemma is from [37]; see also its statement in [27, page 164].

Lemma 6.1 (Zoretti)**.** *If K is a bounded maximal connected subset of a plane closed set M and $\epsilon > 0$, then there exists a simple closed curve J enclosing K such that $J \cap M = \emptyset$ and $J \subseteq B(K, \epsilon)$.*

Lemma 6.2. *Let $f\colon I \times I \to D$ be a sensor curve. If $f(0,t) \notin \partial D$ or $f(1,t) \notin \partial D$ and $U(t) \neq D$, then $U(t) = f(I,t)$.*

Proof. Without loss of generality suppose $f(0,t) \notin \partial D$. It suffices to show that $D \setminus f(I,t)$ is a single path-connected component, because then Lemma 4.4 and the fact that $U(t) \neq D$ will imply $C(t) = D \setminus f(I,t)$ and hence $U(t) = f(I,t)$. Let $x, x' \in D \setminus f(I,t)$; we must find a path in $D \setminus f(I,t)$ connecting x and x'. There are two cases: when $f(1,t) \notin \partial D$, and when $f(1,t) \in \partial D$.

In the first case $f(1,t) \notin \partial D$, note that $f(I,t)$ is disjoint from ∂D. Hence by compactness there exists some $\epsilon > 0$ such that $d(f(I,t), \partial D \cup \{x, x'\}) < \epsilon$. By Lemma 6.1 (with $K = f(I,t)$ and $M = \partial D \cup \{x, x'\}$), there exists a simple closed curve J in D enclosing $f(I,t)$ but not enclosing x or x'. We may therefore connect x and x' by a path in $D \setminus f(I,t)$ consisting of three pieces: a path in D from x to J, a path in D from x' to J, and a path in J connecting these two endpoints.

In the second case $f(1,t) \in \partial D$, pick some point $y \in D \setminus (f(I,t) \cup \{x, x'\})$. By translating D in the plane we may assume that $y = \vec{0}$. Define the inversion function $i\colon \mathbb{R}^2 \setminus \{\vec{0}\} \to \mathbb{R}^2 \setminus \{\vec{0}\}$ by $i(r\cos\theta, r\sin\theta) = (\frac{1}{r}\cos\theta, \frac{1}{r}\sin\theta)$; note i^2 is the identity map. Let $\epsilon > 0$ be such that

$$d(i(f(I,t) \cup \partial D), \{i(x), i(x')\}) < \epsilon.$$

By Lemma 6.1 (with $K = i(f(I,t) \cup \partial D)$ and $M = \{i(x), i(x')\}$), there exists a simple closed curve J in $\mathbb{R}^2$ enclosing $i(f(I,t) \cup \partial D)$ but not enclosing $i(x)$ or $i(x')$. By the Jordan curve theorem, $i(x)$ and $i(x')$ are in the same (exterior) connected component E of $\mathbb{R}^2 \setminus J$. Since E is open it is also path-connected, and hence

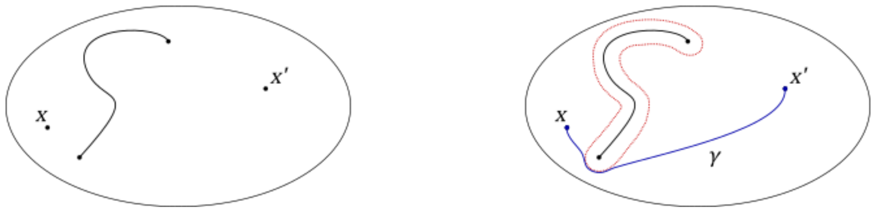

FIGURE 7. The first case in the proof of Lemma 6.2, with J in red.

we can connect $i(x)$ and $i(x')$ by a path γ in E. The path $i(\gamma)$ is therefore a path in $D \setminus f(I,t)$ connecting x and x' (Figure 8).

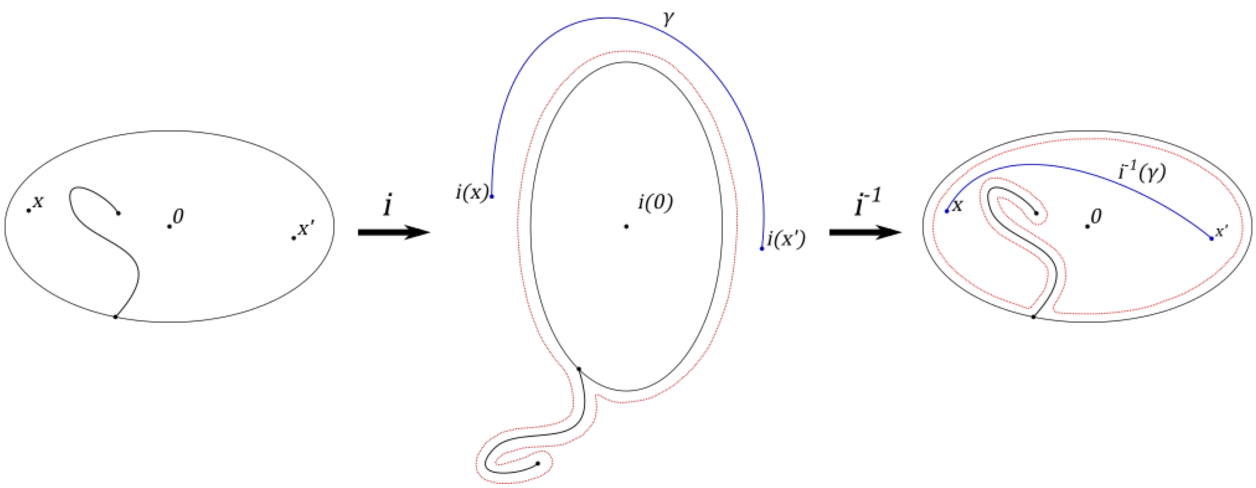

FIGURE 8. The second case in the proof of Lemma 6.2, with J and $i(J)$ drawn in red.

□

7. Sweeping cost of a planar Jordan domain

As motivated by Lemma 6.2, for the remainder of the paper we restrict attention to sensor curves satisfying $f(s,t) \in \partial D$ if and only if $s \in \{0,1\}$ or $t \in \{0,1\}$.

Given curves $\alpha, \beta \colon I \to \partial D$ with $\alpha(1) = \beta(0)$, we define the concatenated curve $\alpha \cdot \beta \colon I \to \partial D$ by

$$\alpha \cdot \beta(t) = \begin{cases} \alpha(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ \beta(2t-1) & \text{if } \frac{1}{2} < t \le 1. \end{cases}$$

We define the inverse curve $\beta^{-1} \colon I \to \partial D$ by $\beta^{-1}(t) = \beta(1-t)$.

Since ∂D is homeomorphic to the circle, given a loop $\gamma \colon I \to \partial D$ (with $\gamma(0) = \gamma(1)$) we can denote the winding number of γ, i.e. the number of times γ wraps around ∂D, by $\mathrm{wn}(\gamma)$. The winding number is positive (resp. negative) for loops that wrap around in the counterclockwise (resp. clockwise) direction. Note that if α and β are paths in ∂D with $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$, then $\alpha \cdot \beta^{-1}$ is a loop.

Our main result is an analytic formula for the sweeping cost of a Jordan domain.

Theorem 7.1. *The sweeping cost of a Jordan domain D is*

$$\mathrm{SC}(D) = \inf\{d_L(\alpha,\beta) \mid \alpha,\beta \colon I \to \partial D,\ \alpha(0) = \beta(0),\ \alpha(1) = \beta(1),\ \mathrm{wn}(\alpha \cdot \beta^{-1}) \neq 0\}. \tag{6}$$

Equation (6) is closely related to the *geodesic width* between two polylines [20], and also the *isotopic Fréchet distance* between two curves [13]. Indeed, note that the right hand side of (6) is unchanged if we replace $d_L(\alpha,\beta)$ with the geodesic Fréchet distance between α and β.

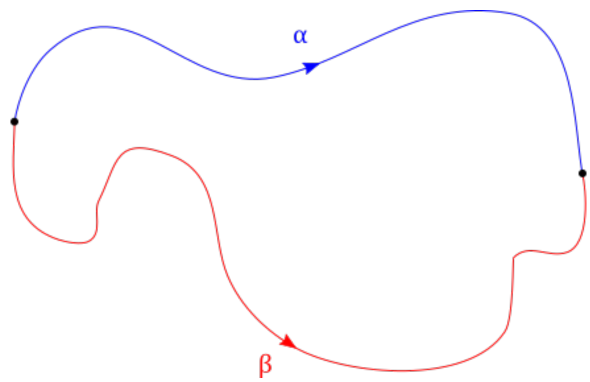

FIGURE 9. Showing how D can be constructed from two curves with endpoints $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$.

Remark 7.2. The value of the right hand side of (6) is unchanged if we replace $\mathrm{wn}(\alpha \cdot \beta^{-1}) \neq 0$ with $\mathrm{wn}(\alpha \cdot \beta^{-1}) \in \{-1, 1\}$.

Proof of Remark 7.2. Let $\alpha, \beta \colon I \to \partial D$ with $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$, and suppose $|\mathrm{wn}(\alpha \cdot \beta^{-1})| \geq 2$. Hence there exists some $0 < t < 1$ such that $\alpha(t) = \beta(t)$ and $\mathrm{wn}(\alpha|_{[0,t]} \cdot \beta|_{[0,t]}^{-1}) \in \{-1, 1\}$. The claim follows since

$$d_L(\alpha|_{[0,t]}, \beta|_{[0,t]}) \leq d_L(\alpha, \beta).$$

□

The following lemma will be used to prove the $\leq$ direction in (6).

Lemma 7.3. *Let D be a Jordan domain. Suppose $\alpha, \beta \colon I \to \partial D$ with $\alpha(0) = \beta(0)$, $\alpha(1) = \beta(1)$, and $\mathrm{wn}(\alpha \cdot \beta^{-1}) \neq 0$. If $f \colon I \times I \to D$ is any sensor curve with $f(0, t) = \alpha(t)$ and $f(1, t) = \beta(t)$, then f is a sweep of D.*

Proof. Let $\mathbb{D} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ be the unit disk. We first prove this claim in the case when $D = \mathbb{D}$.

By Lemma A.2, there exists a point $p \in \mathbb{R}^2 \setminus \mathbb{D}$ and two continuous families of curves $g_\alpha, g_\beta \colon I \times I \to \mathbb{R}^2$ such that

- $g_\alpha(0, t) = p = g_\beta(0, t)$,
- $g_\alpha(1, t) = \alpha(t)$,
- $g_\beta(1, t) = \beta(t)$, and
- $g_\alpha(s, t), g_\beta(s, t) \notin \mathbb{D}$ for $s < 1$.

Let S^1 be the circle of unit circumference, i.e. $[0, 1]$ with endpoints 0 and 1 identified. Define a continuous map $g \colon S^1 \times I \to D$ via

$$g(s,t) = \begin{cases} g_\alpha(3st, t) & \text{if } 0 \leq s < \frac{1}{3} \\ f(3s-1, t) & \text{if } \frac{1}{3} \leq s < \frac{2}{3} \\ g_\beta(3t(1-s), t) & \text{if } \frac{2}{3} \leq s \leq 1. \end{cases}$$

Note $g(\cdot, t)$ is indeed a (possibly non-simple) map from the circle since $g_\alpha(0, t) = p = g_\beta(0, t)$ for all t. Define a continuous signed distance $d^\pm \colon \mathbb{R}^2 \times I \to \mathbb{R}$ by

$$d^\pm(x,t) = \begin{cases} d(x, g(S^1, t)) & \text{if } x \in g(S^1, t) \text{ or } \mathrm{wn}(g(\cdot, t), x) = 0 \\ -d(x, g(S^1, t)) & \text{if } x \notin g(S^1, t) \text{ and } \mathrm{wn}(g(\cdot, t), x) \neq 0. \end{cases}$$

Here $\mathrm{wn}(g(\cdot, t), x)$ denotes (for $x \notin g(S^1, t)$) the winding number of the map $g(\cdot, t) \colon S^1 \to \mathbb{R}^2 \setminus \{x\} \simeq S^1$. Note that $\mathrm{wn}(g(\cdot, t), x)$ is constant on each connected component of $\mathbb{R}^2 \setminus g(S^1, t)$, and that $d^\pm$ is continuous.

Given any intruder path $\gamma \colon I \to \mathbb{D}$, the continuous function $d^\pm(\gamma(t), t) \colon I \to \mathbb{R}$ satisfies $d^\pm(\gamma(0), 0) \geq 0$ (since $f(I, 0)$ is a single point in ∂D) and $d^\pm(\gamma(1), 1) \leq 0$ (since $f(I, 1)$ is a single point in ∂D and

$\mathrm{wn}(\alpha \cdot \beta^{-1}) \neq 0)$. By the intermediate value theorem there exists some $t' \in I$ with $d^{\pm}(\gamma(t'), t') = 0$, and hence $\gamma(t') \in g(S^1, t') \cap \mathbb{D} = f(I, t')$. So γ is not an evasion path, and f is a sweep of $\mathbb{D}$.

We now handle the case when D is an arbitrary Jordan domain. By definition there exists a homeomorphism $h\colon \mathbb{D} \to D$. Note that $h^{-1}\alpha, h^{-1}\beta\colon I \to \partial\mathbb{D}$ with $h^{-1}\alpha(0) = h^{-1}\beta(0)$, $h^{-1}\alpha(1) = h^{-1}\beta(1)$, and $\mathrm{wn}(h^{-1}\alpha \cdot h^{-1}\beta^{-1}) \neq 0$. Since $h^{-1}f\colon I \times I \to \mathbb{D}$ is a sensor curve, it follows from our proof in the case of the disk that $h^{-1}f$ is a sweep of $\mathbb{D}$. Hence f is a sweep of D by Lemma 4.6. □

Proof of Theorem 7.1. Let

$$c = \inf\{d_L(\alpha, \beta) \mid \alpha, \beta\colon I \to \partial D,\ \alpha(0) = \beta(0),\ \alpha(1) = \beta(1),\ \mathrm{wn}(\alpha \cdot \beta^{-1}) \neq 0\}.$$

We first prove the $\leq$ direction of (6). Let $\epsilon > 0$ be arbitrary. By the definition of infimum there exist curves $\alpha, \beta\colon I \to \partial D$ with $\alpha(0) = \beta(0)$, $\alpha(1) = \beta(1)$, $\mathrm{wn}(\alpha \cdot \beta^{-1}) \neq 0$, and $d_L(\alpha, \beta) \leq c + \epsilon$. Define $f\colon I \times I \to D$ by letting $f(\cdot, t)\colon I \to D$ be the unique constant-speed geodesic in D between $f(0, t) = \alpha(t)$ and $f(1, t) = \beta(t)$, which exists by [10, 9]. Lemma 7.3 implies that f is a sweep, and hence we have

$$\mathrm{SC}(D) \leq L(f) = d_L(\alpha, \beta) \leq c + \epsilon.$$

Since this is true for all $\epsilon > 0$, we have $\mathrm{SC}(D) \leq c$.

For the $\geq$ direction of (6), suppose that f is a sensor curve with $L(f) < c$. For notational convenience, define $\alpha, \beta\colon I \to \partial D$ by $\alpha(t) = f(0, t)$ and $\beta(t) = f(1, t)$. Then necessarily $\mathrm{wn}(\alpha \cdot \beta^{-1}) = 0$, and furthermore

$$\alpha(t) = \beta(t) \quad \text{implies} \quad \mathrm{wn}(\alpha|_{[0,t]} \cdot \beta|_{[0,t]}^{-1}) = 0, \tag{7}$$

since otherwise we'd have

$$L(f) \geq d_L(\alpha, \beta) \geq d_L(\alpha|_{[0,t]}, \beta|_{[0,t]}) \geq c,$$

a contradiction. We will show that f is not a sweep of D by showing the existence of an evasion path $\gamma\colon I \to D$ whose image furthermore lives in ∂D.

Indeed, consider the 1-dimensional evasion problem in ∂D where the region covered by the sensors at time t is $\{\alpha(t), \beta(t)\}$. In this 1-dimensional problem, it is clear that the uncontaminated region in ∂D is either (i) a single point $\alpha(t) = \beta(t)$, (ii) a closed interval in ∂D with endpoints $\alpha(t)$ and $\beta(t)$, or (iii) all of ∂D. Equation (7), however, rules out the possibility of (iii). It follows that the contaminated region in ∂D is always a nonempty open interval in ∂D with continuously varying endpoints $\alpha(t)$ and $\beta(t)$. Therefore we can define an evasion path $\gamma\colon I \to \partial D$, for example by letting $\gamma(t)$ be the midpoint of the open interval of the uncontaminated region in ∂D. This evasion path γ is also an evasion path for our original 2-dimensional problem in D, as $\gamma\colon I \to \partial D \subseteq D$ satisfies $\gamma(t) \notin f(I, t)$ for all t. This gives the $\geq$ direction of (6). □

Question 7.4. Does Theorem 7.1 hold even if assumption (iii) in Definition 4.1 is removed, i.e. if the interior of a sensor curve is also allowed to touch ∂D?

8. Sweeping cost of a planar convex domain

Given a convex Jordan domain $D \subseteq \mathbb{R}^2$, its *width* $w(D)$ is defined as

$$w(D) = \min_{\|v\|=1} \max_{x \in \mathbb{R}^2} L(D \cap \{x + tv \mid t \in \mathbb{R}\}),$$

where v is a unit direction vector in $\mathbb{R}^2$. Alternatively, the width $w(D)$ is the smallest distance between two parallel supporting lines on opposite sides of D.

Theorem 8.1. *If D is a convex Jordan domain, then* $\mathrm{SC}(D) = w(D)$.

Proof. We first show $\mathrm{SC}(D) \leq w(D)$. By Theorem 7.1, it suffices to show

$$\inf\{d_L(\alpha, \beta) \mid \alpha, \beta\colon I \to \partial D,\ \alpha(0) = \beta(0),\ \alpha(1) = \beta(1),\ \mathrm{wn}(\alpha \cdot \beta^{-1}) \neq 0\} \leq w(D).$$

Let v some direction vector realizing the width, i.e. $w(D) = \max_{x \in \mathbb{R}^2} L(D \cap \{x + tv \mid t \in \mathbb{R}\})$. Consider sweeping through all lines in $\mathbb{R}^2$ parallel to v; the intersection of these lines with ∂D traces out two continuous curves $\alpha, \beta\colon I \to \partial D$ with $\alpha(0) = \beta(0)$, $\alpha(1) = \beta(1)$, and $\mathrm{wn}(\alpha \cdot \beta^{-1}) = \pm 1$. We have $d_L(\alpha, \beta) \leq w(D)$, giving $\mathrm{SC}(D) \leq w(D)$.

To finish the proof, we need some background on planar convex domains. A point $x \in \partial D$ is *smooth* if it has a unique supporting hyperplane, and otherwise x is a *vertex* containing a range of angles $[\theta_1, \theta_2] \subseteq S^1$

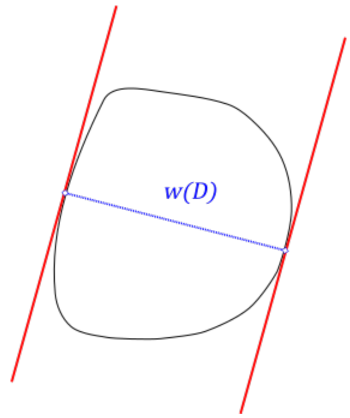

FIGURE 10. Visual representation of the width of a convex domain. Parallel supporting lines shown in red.

(with $\theta_1 \neq \theta_2$) which are the outward normal directions of supporting hyperplanes of D at x. Away from the vertices, the unique supporting hyperplane of $x \in \partial D$ varies continuously with x. By [6, Proposition 11.6.2], the set of vertices of the closed convex domain D is countable.

We now show $\mathrm{SC}(D) \geq w(D)$. Given $\epsilon > 0$, let α and β be ϵ-close to realizing the infimum in (6), meaning $\mathrm{SC}(D) + \epsilon \geq d_L(\alpha, \beta)$. For notational convenience we assume that $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$ are not vertices of ∂D (our same proof technique works regardless). Let $T = \{t_1, t_2, t_3, \ldots\} \subseteq I$ be a countable subset such that $t \in T$ if and only if either $\alpha(t)$ or $\beta(t)$ is a vertex of ∂D. Let $t_i = 0$, and if $|T|$ is finite, then let $t_{|T|+1} = 1$. For $i = 1, 2, \ldots, |T|$, choose weights $w_i > 0$ such that $\sum_i w_i = w < \infty$; this is possible since T is countable. Let $p_1 \colon \partial D \times S^1 \to \partial D$ and $p_2 \colon \partial D \times S^1 \to S^1$ be the projection maps. It is possible to define a *continuous* map $g_\alpha \colon [0, 1 + w] \to \partial D \times S^1$ satisfying the following properties.

- Each $g_\alpha(s)$ is equal to a point $(\alpha(t), v) \in \partial D \times S^1$ with $t \in I$ such that v is the outward normal vector to a supporting hyperplane of D at $\alpha(t)$.
- If $t_i \leq t \leq t_{i+1}$, then $\alpha(t) = p_1 g_\alpha(t + \sum_{i=1}^{i} w_i)$.
- For all $0 \leq s \leq w_i$, we have $p_1 g_\alpha(s + t_i + \sum_{i=1}^{i-1} w_i) = \alpha(t_i)$.
- As s varies from 0 to w_i, angle $p_2 g_\alpha(s + t_i + \sum_{i=1}^{i-1} w_i)$ varies over the range of supporting hyperplanes of D at $\alpha(t_i)$ (which may be a single angle if $\alpha(t_i)$ is not a vertex of ∂D).

Define $g_\beta \colon [0, 1 + w] \to \partial D \times S^1$ similarly (with α replaced everywhere by β). Note that $g_\alpha(0) = g_\beta(0)$, that $g_\alpha(1 + w) = g_\beta(1 + w)$, and that $p_2 g_\alpha$ and $p_2 g_\beta$ wrap in opposite directions around S^1. Hence for some $s \in [0, 1 + w]$ the supporting hyperplanes corresponding to $g_\alpha(s)$ and $g_\beta(s)$ will be parallel and on opposite sides of D. It follows that

$$\mathrm{SC}(D) + \epsilon \geq d_L(\alpha, \beta) \geq d(p_1 g_\alpha(s), p_1 g_\beta(s)) \geq w(D).$$

Since this is true for all $\epsilon > 0$, we have $\mathrm{SC}(D) \geq w(D)$. □

The paper [25] shows that for $D \subseteq \mathbb{R}^2$ a convex polygonal domain with n vertices, the width and hence the sweeping cost of D can be computed in time $O(n)$ and space $O(n)$ using the *rotating calipers* technique.

9. EXTREMAL SHAPES IN THE PLANE

Which convex shape of unit area has the largest sweeping cost? The papers [12, Theorem 4.3] and [30] state that if D is a bounded planar convex domain, then $\mathrm{area}(D) \geq w(D)^2/\sqrt{3}$, where equality is achieved if D is an equilateral triangle. The next corollary follows immediately from Theorem 8.1.

Corollary 9.1. *Let D be a convex Jordan domain. Then*

$$\mathrm{area}(D) \geq \frac{\mathrm{SC}(D)^2}{\sqrt{3}},$$

where equality is achieved if D is an equilateral triangle. Hence the equilateral triangle has the maximal sweeping cost over all planar convex domains of the same area.

The next example shows that there is no extremal shape for non-convex Jordan domains.

Example 9.2. A (non-convex) domain D of unit area may have arbitrarily large sweeping cost.

Proof. Consider a deformation of an equilateral triangle with unit side lengths where we deform each edge towards the center of the triangle (Figure 11). Note that as the sweeping cost converges to $\frac{1}{\sqrt{3}}$ (the distance from the center to a vertex) from above, the area of the shape tends zero. Rescaling each shape in this deformation to have area one shows that a non-convex domain of unit area may have arbitrarily large sweeping cost.

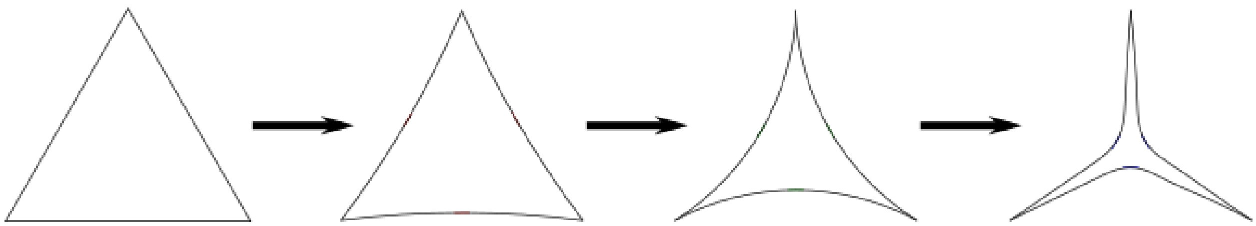

FIGURE 11. We deform each edge of the triangle towards the center of the triangle, producing a three-pronged shape. As the sweeping costs of the shapes converge to a fixed constant, the areas converge to zero.

□

10. Computational approximation of the geodesic Fréchet distance

How does one efficiently estimate geodesic Fréchet distances with computer algorithms? This problem has been addressed by Cook and Wenk in [17]. Cook and Wenk study the Fréchet distance inside of a polygon and also allows for the two curves to be along the boundary of the polygon. However, their method involves more advanced means of computation than we study here.

To get results quickly, we constructed a simpler algorithm which breaks the boundary of a polygon into two curves which are then discretized. The geodesic Fréchet distance is then computed for the two polygonal boundary curves.

The unique geodesic is found by using a pathfinding algorithm seen in reference [33]. The pathfinder works by taking a a source point and target point in the domain and finding the shortest path that connects the points. If the target point is connected to the source by a straight line then no further calculations are done. Otherwise, points connected to the source point by a straight line are considered as next possible points in the path to the target. This continues for each of the possible points connected while keeping track of the total distance traveled. Finally, Dijkstra's algorithm is used to find the minimal path length.

Example 10.1. Consider a simple non-convex domain shaped like an L. It is given by connecting each vertex to the next in the list via a straight line, and connecting the last with the first via a straight line

$$(0,0),(5,0),(5,1),(1,1),(1,5),(0,5)$$

11. Sweeps of higher dimensional Jordan domains

While the planar case is an easier setting in which to explore this type of contamination clearing problem, we also consider extensions of this problem into dimensions $n > 2$. Specifically, the $n = 3$ case still has validity for real world searching problems, but higher $n \geq 3$ dimension cases inspire interesting mathematics.

We will generalize to higher dimensional sweeps by avoiding the use of two curves parameterizing the boundary of our domain and instead considering maps from spheres into ∂D. In the two dimensional case, we chose to build the boundaries with curves α and β but we could have instead considered $\alpha(t)$ and $\beta(t)$ as maps from S^0 since S^0 is the disjoint union of two points.

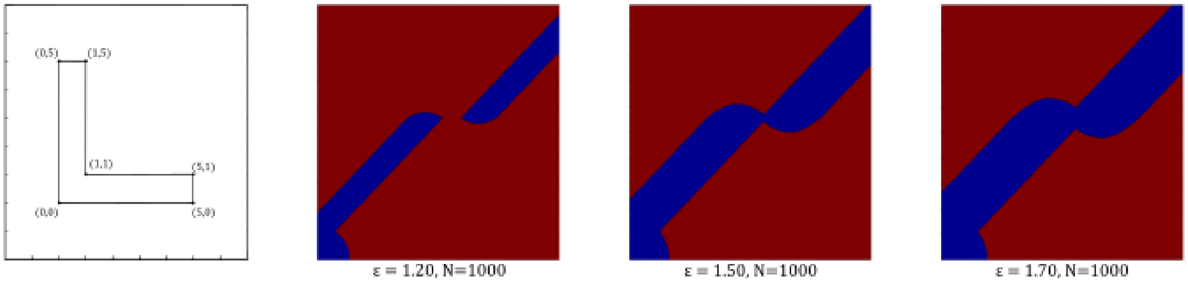

FIGURE 12. Plot of the L shape with α and *beta* beginning at $(0,5)$ and ending at $(5,0)$. Next are three free space plots with varying ϵ leash size. Blue represents two points on the boundary where the leash is $\leq \epsilon$ and red where it is not. This shape has a sweeping cost of $\sqrt{2}$.

Higher dimensional preliminaries and notation.

11.1. **Jordan curve theorem and the natural generalization.** The Jordan curve theorem in the plane states the following. Let $\varphi\colon [0,1] \to \mathbb{R}^2$ be a curve with $\varphi(0) = \varphi(1)$ and φ is injective $\forall t \in [0,1)$. Then this curve divides the plane into two connected components, namely the bounded portion contained in the interior to the curve and the unbounded portion which is exterior to the curve.

This can be stated in a slightly different way by letting $\varphi\colon S^1 \to \mathbb{R}^2$ where φ is injective. This is beneficial as it allows for the higher dimensional analog where $\varphi\colon S^{n-1} \to \mathbb{R}^n$. With φ injective, this will split $\mathbb{R}^n$ into the interior bounded of the $(n-1)$-sphere and the exterior unbounded region outside the $(n-1)$-sphere.

Theorem 11.1 (Jordan–Brouwer separation theorem)**.** *Any compact, connected hypersurface X in $\mathbb{R}^n$ will divide $\mathbb{R}^n$ into two connected regions; the "outside" D_0 and the "inside" D_1. Furthermore, $\bar{D}_1$ (closure of D_1) is itself a compact manifold with boundary $\partial \bar{D}_1 = X$.*

Proof. See reference [23, Chapter 2, §5]. □

11.2. **Ball-shaped domains and geodesics.** Let $D \subseteq \mathbb{R}^n$ be the homeomorphic image of a closed n-dimensional ball. It follows that D is contractible, and its boundary ∂D is a topological $(n-1)$-dimensional sphere. Let $d\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ denote the Euclidean metric on $\mathbb{R}^n$, and also the induced submetric on D. The distance between two subsets $X, Y \subseteq \mathbb{R}^n$ is defined as $d(X,Y) = \sup\{d(x,y) \mid x \in X \text{ and } y \in Y\}$.

Let $B^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ be the closed n-dimensional ball, and let $I = B^1 = [0,1]$ be the unit interval. The boundary $S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$ is the $(n-1)$-dimensional unit sphere. Given a point $x \in D$, we let $B(x,\epsilon) = \{y \in D \mid d(x,y) \leq \epsilon\}$ denote the closed ball about x in D. We denote the ϵ-offset of a set $X \subseteq D$ by $B(X,\epsilon) = \cup_{x \in X} B(x,\epsilon)$.

Denote the closure of X in $\mathbb{R}^n$ by $\overline{X}$. Given a subset $X \subseteq D$, we define its boundary as $\partial X = \overline{X} \cap \overline{\mathbb{R}^n \setminus X}$.

Given a k-dimensional manifold $M \subseteq \mathbb{R}^n$ (with $k \leq n$), we let $\mathrm{vol}_k(M)$ denote its k-dimensional volume, as defined for example in [29, Section 22].

11.3. **Sensor hypersurfaces.** We define a sensor hypersurface to be a time-varying $(n-1)$-dimensional hypersurface in D.

Definition 11.2. A *sensor hypersurface* is a continuous map $f\colon B^{n-1} \times I \to D$ such that

- (i) Each hypersurface $f(B^{n-1}, t)$ is a C^1 $(n-1)$-dimensional submanifold in $\mathbb{R}^n$ with $\mathrm{vol}_{n-1}(f(B^{n-1}, t)) < \infty$.
- (ii) for $t \notin \{0,1\}$, $f(\cdot, t)\colon B^{n-1} \to D$ is a homeomorphism onto its image,
- (iii) $f(B^{n-1}, 0)$ and $f(B^{n-1}, 1)$ are each (possibly distinct) single points in ∂D, and
- (iv) $f(s,t) \in \partial D$ implies $s \in \partial B^{n-1}$ or $t \in \{0,1\}$.

Again, we think of the first input s as a spatial variable and of the second t as a temporal variable; in particular $f(B^{n-1}, t)$ is the region covered by the hypersurface of sensors at time t. Assumption (ii) states that the images of the sensor hypersurface at times 0 and 1 are single points, and assumption (iii) implies that (apart from times 0 and 1) only the boundary of the sensor hypersurface intersects ∂D.

We define the volume of a sensor hypersurface to be

$$\mathrm{vol}_{n-1}(f) = \max_{t \in I} \ \mathrm{vol}_{n-1}(f(\cdot, t)).$$

A sensor hypersurface f is a *sweep* if every continuously moving intruder $\gamma\colon I \to D$ is necessarily intersected at some time t, or equivalently, if no evasion path over the full time interval I exists. Using previous notation, we can say that a sensor hypersurface f is a sweep if and only if $C(1) = \emptyset$, or equivalently $U(1) = D$.

Definition 11.3. Let $\mathcal{F}(D)$ be the set of all sensor hypersurface sweeps of D. The *sweeping cost* of D is

$$\mathrm{SC}(D) = \inf_{f \in \mathcal{F}(D)} \ \mathrm{vol}_{n-1}(f).$$

11.4. **Properties of sensor hypersurface sweeps.** We now generalize some of the basic properties of sensor sweeps from the plane utilizing the contaminated and uncontaminated regions.

Lemma 11.4. *If x and x' are in the same path-connected component of $D \setminus f(B^{n-1}, t)$, then $x \in U(t)$ if and only if $x' \in U(t)$.*

Proof. Suppose for a contradiction that $x \in U(t)$ but $x' \notin U(t)$. Since $x' \in C(t)$, there exists an evasion path $\gamma : [0, t] \to D$ with $\gamma(t) = x'$. Since x and x' are in the same path-connected component, there exists a path $\beta\colon I \to D \setminus f(B^{n-1}, t)$ with $\beta(0) = x$ and $\beta(1) = x'$.

Note that $\beta(I)$ and $f(B^{n-1}, t)$ are compact, since they are each a continuous image of a compact set. As any metric space is normal, there exist disjoint neighborhoods containing $\beta(I)$ and $f(B^{n-1}, t)$. Because f is uniformly continuous (it is a continuous function on a compact set), we can choose $\delta_1 > 0$ such that $f(B^{n-1}, [t - \delta_1, t])$ remains in this open neighborhood disjoint from $\beta(I)$, giving

$$\beta(I) \cap f(B^{n-1}, [t - \delta_1, t]) = \emptyset. \tag{8}$$

Since metric space D is normal, there exist disjoint neighborhoods containing $\gamma(t) = x'$ and $f(B^{n-1}, t)$. Since γ is continuous and f is uniformly continuous, we can choose $\delta_2 > 0$ such that $\gamma([t - \delta_2, t])$ and $f(B^{n-1}, [t - \delta_2, t])$ remain in these disjoint neighborhoods, giving

$$\gamma([t - \delta_2, t]) \cap f(B^{n-1}, [t - \delta_2, t]) = \emptyset \tag{9}$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Using (8) and (9) we can define an evasion path $\gamma\colon [0, t] \to D$ with $\gamma(t) = x$. Indeed, let

$$\gamma(t') = \begin{cases} \gamma(t') & \text{if } t' \leq t - \delta \\ \gamma(2t' - t + \delta) & \text{if } t - \delta < t' \leq t - \frac{\delta}{2} \\ \beta(\frac{2}{\delta}(t' - t + \frac{\delta}{2})) & \text{if } t - \frac{\delta}{2} < t' \leq t. \end{cases}$$

This contradicts the fact that $x \in U(t)$. ☐

Lemma 11.5. *For all $t \in I$, the set $U(t)$ is closed and the set $C(t)$ is open in D.*

Proof. Suppose $x \in C(t)$. Since $f(B^{n-1}, t)$ is closed and $x \notin f(B^{n-1}, t)$, there exists some $\varepsilon > 0$ such that $B(x, \varepsilon) \cap f(B^{n-1}, t) = \emptyset$. Note all $x' \in B(x, \varepsilon)$ are in the same path-connected component of $D \setminus f(B^{n-1}, t)$ as x via a straight line path. Hence Lemma 11.4 implies $B(x, \varepsilon) \subseteq C(t)$, showing $C(t)$ is open in D. It follows that $U(t) = D \setminus C(t)$ is closed in D. ☐

Lemma 11.6. *If $h\colon D \to h(D)$ is a homeomorphism onto its image $h(D) \subseteq \mathbb{R}^n$, then a sensor hypersurface $f\colon B^{n-1} \times I \to D$ is a sweep of D if and only if sensor hypersurface $hf\colon B^{n-1} \times I \to h(D)$ is a sweep of $h(D)$.*

Proof. Note that if $\gamma\colon I \to D$ is an evasion path for f, then $h\gamma\colon I \to h(D)$ is an evasion path for hf. Conversely, if $\gamma\colon I \to h(D)$ is an evasion path for hf, then $h^{-1}\gamma\colon I \to D$ is an evasion path for f. ☐

11.5. **A lower bound on the sweeping cost in higher dimension.** In this section we prove that the sweeping cost of a ball-shaped domain is at least as large as the length of the smallest volume-bisecting hypersurface. The first lemma is a version of the intermediate value theorem with slightly relaxed hypotheses.

Conjecture 11.7. The function $\mathrm{vol}_n(U(t))$ is upper semi-continuous and left continuous.

Potential proof outline. Let $t_0 \in I$. We conjecture that $\mathrm{vol}_n(U(t))$ is right upper semi-continuous and left continuous at t_0, which implies the function is both upper semi-continuous and left continuous.

For right upper semi-continuity, note for $t \geq t_0$ we have

$$U(t) \subseteq U(t_0) \cup f(B^{n-1}, [t_0, t]). \tag{10}$$

Since the sensor hypersurface $f\colon B^{n-1} \times I \to D$ is a continuous function on a compact domain, it is also uniformly continuous. Hence for all $\epsilon > 0$ there exists some δ such that

$$f(B^{n-1}, [t_0, t_0 + \delta]) \subseteq B(f(B^{n-1}, t_0), \epsilon). \tag{11}$$

It follows that for all $t \in [t_0, t_0 + \delta]$ we have

$$\begin{aligned} \mathrm{vol}_n(U(t)) - \mathrm{vol}_n(U(t_0)) &\leq \mathrm{vol}_n(f(B^{n-1}, [t_0, t])) && \text{by (10)} \\ &\leq \mathrm{vol}_n(B(f(B^{n-1}, t_0), \epsilon)) && \text{by (11)}. \end{aligned}$$

By [36, Equation (2)] or [22, Equation (1.1)], the function $p(\epsilon) = \mathrm{vol}_n(B(f(B^{n-1}, t_0), \epsilon))$ is a polynomial of degree at most n. Note that Weyl's formula only applies to manifolds without boundary. However, $\partial f(\cdot, t)$ is a manifold without boundary and thus Weyl's formula applies there. Thus we hope to find a statement bounding the ϵ offset of the interior of $f(\cdot, t)$ we would have a polynomial with degree n. However, we have not confirmed this idea regarding the interior. Since $f(B^{n-1}, t)$ is a C^1 $(n-1)$-dimensional submanifold, it follows that $\mathrm{vol}_n(f(B^{n-1}, t_0)) = 0$. Hence the constant term of p is zero, meaning $p(0) = 0$. Since polynomials are continuous, it follows that $\mathrm{vol}_n(U(t))$ is right upper semi-continuous.

To see that $\mathrm{vol}_n(U(t))$ is left continuous at $t_0 \in I$, we must show that for all sequences $\{s_i\}$ with $0 \leq s_i \leq t_0$ and $\lim_i s_i = t_0$, we have $\lim_i \mathrm{vol}_n(U(s_i)) = \mathrm{vol}_n(U(t_0))$. We claim

$$U(t_0) \setminus f(B^{n-1}, t_0) \subseteq \liminf_i U(s_i) \subseteq \limsup_i U(s_i) \subseteq U(t_0), \tag{12}$$

where the middle containment is by definition. We now justify the first and last containment.

To prove $U(t_0) \setminus f(B^{n-1}, t_0) \subseteq \liminf_i U(s_i)$ it suffices to show that for any $x \in U(t_0) \setminus f(B^{n-1}, t_0)$ there exists an $\epsilon > 0$ such that $x \in U(t_0 - \delta)$ for all $\delta \in [0, \epsilon)$. Fix ϵ such that $x \notin f(B^{n-1}, t)$ for $t \in (t_0 - \epsilon, t_0]$. Suppose for a contradiction that $x \in C(t_0 - \delta)$ for some $\delta \in [0, \epsilon)$. Hence there exists an evasion path $\gamma\colon [0, t_0 - \delta] \to D$ with $\gamma(t_0 - \delta) = x$. It is possible to extend γ to an evasion path $\tilde{\gamma}\colon [0, t_0] \to D$ defined by

$$\tilde{\gamma}(t) = \begin{cases} \gamma(t) & \text{if } t \in [0, t_0 - \delta] \\ x & \text{if } t \in (t_0 - \delta, t_0]. \end{cases}$$

This contradicts the fact $x \in U(t_0)$, thus giving the first containment.

We now show $\limsup_i U(s_i) \subseteq U(t_0)$. If $x \notin U(t_0)$, then there exists an evasion path $\gamma\colon [0, t_0] \to D$ with $\gamma(t_0) = x$. Since $C(t_0)$ is open by Lemma 11.5, there exists some $\delta > 0$ such that $B(x, \delta) \subseteq C(t_0)$, and hence $B(x, \delta) \cap f(B^{n-1}, t_0) = \emptyset$. Since f is uniformly continuous, there is some $\epsilon_1 > 0$ sufficiently small with $B(x, \delta) \cap f(B^{n-1}, [t_0 - \epsilon_1, t_0]) = \emptyset$, and since γ is continuous there is some $\epsilon_2 > 0$ with $\gamma([t_0 - \epsilon_2, t_0]) \subseteq B(x, \delta)$. Let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. Reparametrize γ to get a continuous curve $\tilde{\gamma}\colon [0, t_0] \to D$ with

$$\tilde{\gamma}(t) = \begin{cases} \gamma(t) & \text{if } x \in [0, t_0 - \epsilon) \\ \gamma(t) \in B(x, \delta) & \text{if } t \in [t_0 - \epsilon, t_0 - \epsilon/2) \\ \gamma(t) = x & \text{if } t \in [t_0 - \epsilon/2, t_0]. \end{cases}$$

The evasion path $\tilde{\gamma}$ shows $x \notin \limsup_i U(s_i)$, giving the third containment and finishing the proof of (12).

Since $\mathrm{vol}_n(f(B^{n-1}, t_0)) = 0$, we have $\mathrm{vol}_n(U(t_0) \setminus f(B^{n-1}, t_0)) = \mathrm{vol}_n(U(t_0))$. Thus (12) implies

$$\mathrm{vol}_n(\limsup_i U(s_i)) = \mathrm{vol}_n(U(t_0)) = \mathrm{vol}_n(\liminf_i U(s_i)).$$

Since $\mathrm{vol}_n(D)$ is finite, Lemma A.1 implies $\limsup_i \mathrm{vol}_n(U(s_i)) \leq \mathrm{vol}_n(\limsup_i U(s_i))$ and $\mathrm{vol}_n(\liminf_i U(s_i)) \leq \liminf_i \mathrm{vol}_n(U(s_i))$, giving

$$\limsup_i \mathrm{vol}_n(U(s_i)) \leq \mathrm{vol}_n(U(t_0)) \leq \liminf_i \mathrm{vol}_n(U(s_i)).$$

Hence $\lim_i \mathrm{vol}_n(U(s_i)) = \mathrm{vol}_n(U(t_0))$ as required. ☐

Remark 11.8. The function $\mathrm{vol}_n(U(t))$ need not be right continuous by 5.3.

As a consequence we obtain the following lower bound on the sweeping cost.

Conjecture 11.9. If B^n is a ball-shaped domain, then the sweeping cost $\mathrm{SC}(D)$ is at least as large as the length of the smallest volume-bisecting hypersurface in D.

Potential proof outline. Suppose that f is a sweep of D. Note that $\mathrm{vol}_n(U(0)) = 0$ and $\mathrm{vol}_n(U(1)) = \mathrm{vol}_n(D)$. By Lemmas **??** and Conjecture 11.7, there exists some time $t' \in I$ with $\mathrm{vol}_n(U(t')) = \frac{1}{2}\mathrm{vol}_n(D)$. So $\mathrm{vol}_{n-1}(f(\cdot, t'))$ and hence $\mathrm{SC}(D)$ is at least as large as the smallest volume-bisecting hypersurface in D. ☐

Example 11.10. If $B^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ is the unit n-ball, then $\mathrm{SC}(B^n) = \mathrm{vol}_{n-1}(B^{n-1})$.

Proof. To see $\mathrm{SC}(B^n) \leq \mathrm{vol}_{n-1}(B^{n-1})$, consider the sweep $f\colon B^{n-1} \times I \to D$ defined by $f(s,t) = (2s\sqrt{t-t^2}, 2t-1)$ (Figure 13) which has volume $\mathrm{vol}_{n-1}(B^{n-1})$.

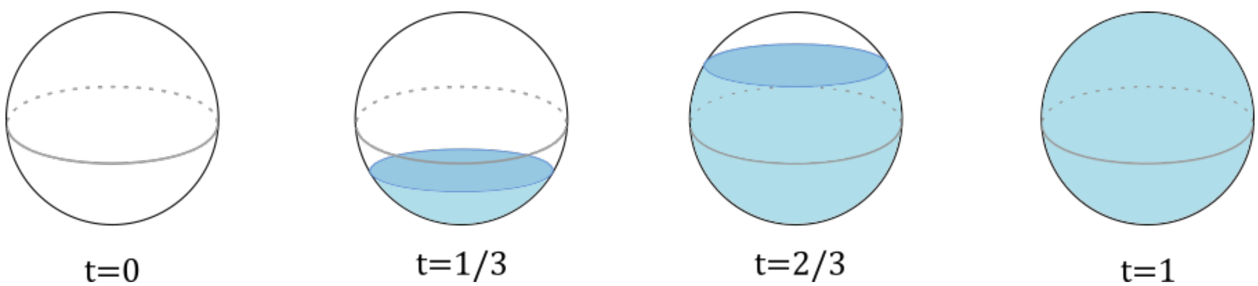

Figure 13. A sweep of the unit 3 ball. The shaded region is $U(t)$ and the unshaded region is $C(t)$.

For the reverse direction, note that the smallest volume-bisecting hypersurface in B^n is a cross-sectional ball of dimension $n-1$ through the equator. Indeed, this is stated in the Bisection Lemma of [24] Hence we apply Conjecture 11.9 to get $\mathrm{SC}(B^n) \geq \mathrm{vol}_{n-1}(B^{n-1})$. ☐

11.6. **Generalization of the lemma of no progress.** In Sections 11.7–11.8 we will restrict attention to sensor hypersurfaces f with boundary points $f(s,t) \in \partial D$ for all $s \in \partial B^{n-1}$. The motivation behind this assumption is Lemma 11.12, which states that if $f(s,t) \notin \partial D$ for some $s \in \partial B^{n-1}$, then the uncontaminated region at time t is as small as possible, namely $U(t) = f(B^{n-1}, t)$.

Conjecture 11.11. If K is a bounded maximal connected subset of a closed set $M \subseteq \mathbb{R}^n$ and $\epsilon > 0$, then there exists the homeomorphic image J of an $(n-1)$-dimensional sphere enclosing K such that $J \cap M = \emptyset$ and $J \subseteq B(K, \epsilon)$.

Potential proof outline. For arbitrary n we must generalize the statement from reference [37] and reference [27, page 164] which only provide a result in $n = 2$. ☐

Conjecture 11.12. Let $f\colon B^{n-1} \times I \to D$ be a sensor hypersurface. If $f(s,t) \notin \partial D$ for some $s \in \partial B^{n-1}$ and $U(t) \neq D$, then $U(t) = f(B^{n-1}, t)$.

Potential proof outline. Suppose $f(s,t) \notin \partial D$ for some $s \in \partial B^{n-1}$. It suffices to show that $D \setminus f(B^{n-1}, t)$ is a single path-connected component, because then Lemma 11.4 and the fact that $U(t) \neq D$ will imply $C(t) = D \setminus f(B^{n-1}, t)$ and hence $U(t) = f(B^{n-1}, t)$. Let $x, x' \in D \setminus f(B^{n-1}, t)$; we must find a path in $D \setminus f(B^{n-1}, t)$ connecting x and x'. There are two cases: when $f(\partial B^{n-1}, t)$ is disjoint from ∂D, and when it is not.

In the first case note that $f(B^{n-1},t)$ is disjoint from ∂D. Hence by compactness there exists some $\epsilon > 0$ such that $d(f(B^{n-1},t), \partial D \cup \{x,x'\}) < \epsilon$. By Conjecture 11.11 (with $K = f(B^{n-1},t)$ and $M = \partial D \cup \{x,x'\}$), there exists some $(n-1)$-dimensional sphere J in D enclosing $f(B^{n-1},t)$ but not enclosing x or x'. We may therefore connect x and x' by a path in $D \setminus f(B^{n-1},t)$ consisting of three pieces: a path in D from x to J, a path in D from x' to J, and a path in J connecting these two endpoints.

In the second case there is at least one point $f(s',t) \in \partial D$. Pick some point $y \in D \setminus (f(B^{n-1},t) \cup \{x,x'\})$. By translating D in the $\mathbb{R}^n$ we may assume that $y = \vec{0}$. Define the inversion function $i\colon \mathbb{R}^n \setminus \{\vec{0}\} \to \mathbb{R}^n \setminus \{\vec{0}\}$ by $i(u) = u/\|u\|$; note i^2 is the identity map. Let $\epsilon > 0$ be such that

$$d(i(f(B^{n-1},t) \cup \partial D), \{i(x), i(x')\}) < \epsilon.$$

By Conjecture 11.11 (with $K = i(f(B^{n-1},t) \cup \partial D)$ and $M = \{i(x), i(x')\}$), there exists a simple closed curve J in $\mathbb{R}^n$ enclosing $i(f(B^{n-1},t) \cup \partial D)$ but not enclosing $i(x)$ or $i(x')$. By the Jordan–Brouwer separation theorem, $i(x)$ and $i(x')$ are in the same (exterior) connected component E of $\mathbb{R}^n \setminus J$. Since E is open it is also path-connected, and hence we can connect $i(x)$ and $i(x')$ by a path γ in E. The path $i(\gamma)$ is therefore a path in $D \setminus f(B^{n-1},t)$ connecting x and x' (Figure 8).

□

11.7. **Sweeping cost of a ball-shaped domain.** As motivated by Lemma 6.2, for the remainder of the paper we restrict attention to sensor hypersurfaces satisfying $f(s,t) \in \partial D$ if and only if $s \in \partial B^{n-1}$ or $t \in \{0,1\}$. In this case, let $n \geq 2$

Let $\alpha\colon S^{n-2} \times I \to \partial D$ be a time-varying map with $\alpha(S^{n-2},0)$ and $\alpha(S^{n-2},1)$ each (possibly distinct) single points in ∂D. Define the equivalence relation $\sim$ on $S^{n-2} \times I$ by $(x,0) \sim (x',0)$ and $(x,1) \sim (x',1)$ for all $x, x' \in S^{n-2}$. Note $(S^{n-2} \times I)/\sim$ is homeomorphic to S^{n-1}, and hence α induces a map $\tilde{\alpha}\colon S^{n-1} \to \partial D$. Since ∂D is homeomorphic to S^{n-1}, we denote the degree of this map $\tilde{\alpha}$ between spheres as $\deg(\alpha)$. Given such an α, let its *filling volume* be

$$\mathrm{fvol}(\alpha) = \max_{t \in I} \mathrm{vol}_{n-1}(M_{\alpha(S^{n-2},t)}),$$

where $M_{\alpha(S^{n-2},t)}$ is the minimal surface with boundary $\alpha(S^{n-2},t)$. Our main conjecture is an analytic formula for the sweeping cost of a ball-shaped domain.

The idea behind $M_{\alpha(S^{n-2},t)}$ and fvolα is to provide the extension of the geodesic Fréchet distance for a planar domain in a natural way. In the case where $n = 2$ we recover the exact definition. In higher dimensions the leash is replaced with a minimal surface and we want to find the smallest such minimal surface over all times of the sweep. The minimal surface "fills the sphere" we map to ∂D in the very same way the geodesic "fills between the two points" mapped to ∂D when D is a planar domain.

Conjecture 11.13. The sweeping cost of a ball-shaped domain D is

(13) $\mathrm{SC}(D) = \inf\{\mathrm{fvol}(\alpha) \mid \alpha\colon S^{n-2} \times I \to \partial D,\ \alpha(S^{n-2},0) = x \in \partial D,\ \alpha(S^{n-2},1) = x' \in \partial D,\ \deg(\alpha) \neq 0\}.$

Question 11.14. Is the value of the right hand side of (13) is unchanged if we replace $\deg(\alpha) \neq 0$ with $\deg(\alpha) = 1$?

The following lemma will be used to give a potential proof outline for the $\leq$ direction in (13).

Conjecture 11.15. Let D be a ball-shaped domain. Suppose $\alpha\colon S^{n-2} \times I \to \partial D$ is such that $\alpha(S^{n-2},0)$ and $\alpha(S^{n-2},1)$ are each single points, and $\deg(\alpha) \neq 0$. If $f\colon B^{n-1} \times I \to D$ is any sensor hypersurface with $f(S^{n-2},t) = \alpha(t)$, then f is a sweep of D.

Potential proof outline. Let $\mathbb{D} = \{x \in \mathbb{R}^{n+1} \mid \|x\| \leq 1\}$ be the unit disk. We first give a potential proof outline for this claim in the case when $D = \mathbb{D}$.

By Conjecture A.3, there exists a point $p \in \mathbb{R}^2 \setminus \mathbb{D}$ and a map $g_\alpha\colon B^n \times I \to \mathbb{R}^{n+1}$ such that

- $g_\alpha(\vec{0},t) = p$ (where $\vec{0} \in B^n$ is the center of the ball),
- For $s \in S^{n-1}$ we have $g_\alpha(s,t) = \alpha(s,t)$, and
- $g_\alpha(s,t) \notin \mathbb{D}$ for $s \notin S^n$.

Note that $S^n = D^n \cup_{S^{n-1}} D^n$; we denote the two hemispheres in this gluing by $S^n = D^n_- \cup_{S^{n-1}} D^n_+$. Hence we can define a continuous map $g\colon S^n \times I \to D$ via

$$g(s,t) = \begin{cases} g_\alpha(s,t) & \text{if } s \in D^n_- \\ f(s,t) & \text{if } s \in D^n_+. \end{cases}$$

Define a continuous signed distance $d^\pm\colon \mathbb{R}^{n+1} \times I \to \mathbb{R}$ by

$$d^\pm(x,t) = \begin{cases} d(x, g(S^n,t)) & \text{if } x \in g(S^n,t) \text{ or } \deg(g(\cdot,t),x) = 0 \\ -d(x, g(S^n,t)) & \text{if } x \notin g(S^n,t) \text{ and } \deg(g(\cdot,t),x) \neq 0. \end{cases}$$

Here $\deg(g(\cdot,t),x)$ denotes (for $x \notin g(S^n,t)$) the winding number of the map $g(\cdot,t)\colon S^n \to \mathbb{R}^{n+1} \setminus \{x\} \simeq S^n$. Note that $\deg(g(\cdot,t),x)$ is constant on each connected component of $\mathbb{R}^{n+1} \backslash g(S^n,t)$, and that $d^\pm$ is continuous.

Given any intruder path $\gamma\colon I \to \mathbb{D}$, the continuous function $d^\pm(\gamma(t),t)\colon I \to \mathbb{R}$ satisfies $d^\pm(\gamma(0),0) \geq 0$ (since $f(B^n,0)$ is a single point in ∂D) and $d^\pm(\gamma(1),1) \leq 0$ (since $f(B^n,1)$ is a single point in ∂D and $\deg(\alpha) \neq 0$). By the intermediate value theorem there exists some $t' \in I$ with $d^\pm(\gamma(t'),t') = 0$, and hence $\gamma(t') \in g(S^n,t') \cap \mathbb{D} = f(B^n,t')$. So γ is not an evasion path, and f is a sweep of $\mathbb{D}$.

We now handle the case when D is an arbitrary ball-shaped domain. By definition there exists a homeomorphism $h\colon \mathbb{D} \to D$. Note that $h^{-1}\alpha\colon S^{n-1} \times I \to \partial\mathbb{D}$ satisfies the property that $h^{-1}\alpha(S^{n-1},0)$ and $h^{-1}\alpha(S^{n-1},1)$ are each single points, and $\deg(h^{-1}\alpha) \neq 0$. Since $h^{-1}f\colon B^n \times I \to \mathbb{D}$ is a sensor hypersurface, it follows from our proof in the case of the disk that $h^{-1}f$ is a sweep of $\mathbb{D}$. Hence f is a sweep of D by Lemma 4.6. □

Potential proof of $\leq$ direction of conjecture 11.13. Let

$$c = \inf\{\mathrm{fvol}(\alpha) \mid \alpha\colon S^{n-1} \times I \to \partial D,\ \alpha(S^{n-1},0) = x \in \partial D,\ \alpha(S^{n-1},1) = x' \in \partial D,\ \deg(\alpha) \neq 0\}.$$

We first prove the $\leq$ direction of (6). Let $\epsilon > 0$ be arbitrary. By the definition of infimum there exists a map $\alpha\colon S^{n-1} \times I \to \partial D$ with $\alpha(S^{n-1},0) = x \in \partial D$, $\alpha(S^{n-1},1) = x' \in \partial D$, $\deg(\alpha) \neq 0$, and $\mathrm{fvol}(\alpha) \leq c + \epsilon$. Define $f\colon B^n \times I \to D$ by letting $f(\cdot,t)\colon I \to D$ be the unique minimal surface in D with $\partial f(0,t) = \alpha(t)$. We must show that there is indeed a unique minimal surface and that they vary continuously. Lemma 7.3 implies that f is a sweep, and hence we would have

$$\mathrm{SC}(D) \leq \mathrm{vol}_n(f) = \mathrm{fvol}(\alpha) \leq c + \epsilon.$$

Since this is true for all $\epsilon > 0$, we would have $\mathrm{SC}(D) \leq c$. □

11.8. **Sweeping cost of a higher dimensional convex domain.** For a unit vector $v \in \mathbb{R}^n$, let $H_v = \{u \in \mathbb{R}^n \mid u \cdot v = 0\}$ be the hyperplane orthogonal to v. Let $x + H_v$ denote the set of all vectors of the form $x + u$ with $u \in H_v$.

Definition 11.16. Given a convex ball-shaped domain $D \subseteq \mathbb{R}^{n+1}$, the *$n$-dimensional width* $w_n(D)$ is given by

$$w_n(D) = \min_{\|v\|=1} \max_{x \in \mathbb{R}^{n+1}} \mathrm{vol}_n(D \cap (x + H_v)),$$

where v is a unit direction vector in R^{n+1}.

Conjecture 11.17. If D is a convex ball-shaped domain, then $\mathrm{SC}(D) = w_n(D)$.

12. Related open questions

Another question would be sweeping multiply connected domains in the plane. In other words, consider a punctured Jordan domain which can be viewed as a domain with obstacles which the rope must avoid. Here the use of multiple sensor curves may be necessary in order to achieve the smallest possible sweeping cost. In a simply connected planar domain, it could also be that the sweeping cost when using multiple curves is lower than when restricted to a single curve if we consider the cost to be the sum of the individual curve lengths. Another question motivated by computation of the Fréchet distance is whether knowing the pairwise geodesic distances for all points along two curves allows one to recover the shape of the two curves exactly? Finally, we open up the idea what the possible extremal shapes for $n \geq 2$.

In studying the planar case specifically, we utilized the notion of the *geodesic Fréchet distance*. However, there are two subtypes of this metric. First is the *weak* case where the parameterizations of α and β need not be injective and the second *strong* case in which they are required to be.

Question 12.1. For any Jordan domain D in the plane and injective curves $\alpha, \beta\colon I \to \partial D$, is the weak geodesic Fréchet distance between α and β is equal to their strong geodesic Fréchet geodesic distance? Figure 4 is not a counterexample since we are not using the geodesic distance in a domain D with $\alpha, \beta\colon I \to \partial D$.

There are simple counterexamples to Question 12.1 when α and β are not injective, or when they do not map to ∂D. Closely related is following question: is the value of (6) unchanged if we require α and β to be injective?

Computation of the *geodesic Fréchet distance* for polygons split into two curves brought up another interesting question.

Question 12.2. Given the pairwise distance between the points on two curves, can the two curves be reconstructed exactly?

Through the research related to sweeping generalized Jordan domains, many other questions have arisen.

Question 12.3. Does Theorem 11.13 hold even if assumption (iii) in Definition 11.2 is removed, i.e. if the interior of a sensor hypersurface is also allowed to touch ∂D?

Question 12.4. For example, in a compact multiply connected planar domain, is it possible to establish the sweeping cost in a natural way? We conjecture that this will take multiple curves and the sweeping cost will be the sum of the individual lengths of each curve.

Question 12.5. In a higher dimensional case, what would be the sweeping cost of the interior of a hollow torus? What if more than one sweeping surface is allowed? In this case, is the minimal sweeping cost achieved two copies of a disk with area equal to that of $2\pi r^2$ where r is the minor radius.

Question 12.6. What are the convex unit area shapes for $n \geq 2$ with largest sweeping cost? For $n = 3$ is it a tetrahedron with equilateral triangles as faces? Does this same idea continue for larger n? Do higher-dimensional non-convex shapes with unit area have unbounded sweeping cost like in the planar case?

13. Conclusion

Given a Jordan domain D in the plane, we show that the sweeping cost of D is at least as large as the shortest area-bisecting curve in D, and we give a formula for the sweeping cost in terms of the geodesic Fréchet distance between two curves on the boundary of D with non-equal winding numbers. We show that the sweeping cost of any convex domain is equal to its width. Therefore, the sweeping cost of a polygonal convex domain with n vertices can be computed in time and space $O(n)$, and a convex domain of unit area with maximal sweeping cost is the equilateral triangle.

Most of this theory behaves well when extended to dimensions $n \geq 2$. Indeed, our work can be thought of as one higher-dimensional analogue of the geodesic Fréchet distance.

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Appendix A. Additional lemmas and proofs

Lemma A.1. *If (X, μ) is a measure space and U_i is a sequence of measurable sets in X, then*

(1) $\mu(\liminf_i U_i) \leq \liminf_i \mu(U_i)$, *and*
(2) $\mu(\limsup_i U_i) \geq \limsup_i \mu(U_i)$ *if* $\mu(X) < \infty$.

Proof. Recall $\limsup_i U_i = \cap_{i=1}^{\infty}(\cup_{j=i}^{\infty} U_i)$. Since the $\cup_{j=i}^{\infty} U_i$ are a decreasing sequence of sets, and since $\mu(X) < \infty$, [16, Proposition 1.2.3] implies $\mu(\limsup_i U_i) = \lim_i \mu(\cup_{j=i}^{\infty} U_i)$. Since $U_i \subseteq \cup_{j=i}^{\infty} U_j$, we have $\mu(U_i) \leq \mu(\cup_{j=i}^{\infty} U_j)$, and hence

$$\mu(\limsup_i U_i) = \lim_i \mu(\cup_{j=i}^{\infty} U_i) \geq \limsup_i \mu(U_i),$$

giving (2). The proof of (1) is similar except that the finiteness assumption is unnecessary. □

Lemma A.2. *Let* $\mathbb{D} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ *be the unit disk. Given any point* $p \in \mathbb{R}^2 \setminus \mathbb{D}$ *and any curve* $\alpha\colon I \to \partial\mathbb{D}$*, there exists a continuous function* $g\colon I \times I \to \mathbb{R}^2$ *such that*

- $g(0, t) = p$ *for all* $t \in I$*,*
- $g(1, t) = \alpha(t)$ *for all* $t \in I$*, and*
- $g(s, t) \notin \mathbb{D}$ *for* $s < 1$*.*

Proof. Given $r \geq 0$, let $\alpha_r(t)\colon I \to \mathbb{R}^2$ be defined by $\alpha_r(t) = (1+r)\alpha(t)$. Fix some $\epsilon > 0$. Pick a single curve $\gamma\colon I \to \mathbb{R}^2 \setminus \mathbb{D}$ with $\gamma(0) = p$ and $\gamma(1) = \alpha_\epsilon(0)$; this is possible since $\mathbb{R}^2 \setminus \mathbb{D}$ is connected by the Jordan curve theorem. We define the function $g\colon I \times I \to \mathbb{R}^2$ as follows:

$$g(s, t) = \begin{cases} \gamma(3s) & \text{if } 0 \leq s < \frac{1}{3} \\ \alpha_\epsilon((3s-1)t) & \text{if } \frac{1}{3} \leq s < \frac{2}{3} \\ \alpha_{(3-3s)\epsilon}(t) & \text{if } \frac{2}{3} \leq s \leq 1. \end{cases}$$

Note that g is continuous and satisfies all of the required conditions. □

Conjecture A.3. Let B^n be the unit n-ball. Given any point $p \in \mathbb{R}^n \setminus B^n$ and any curve $\alpha\colon I \to \partial B^n$, there exists a continuous function $g\colon B^{n-1} \times I \to \mathbb{R}^n$ such that

- $g(0, t) = p$ for all $t \in I$,
- $g(1, t) = \alpha(t)$ for all $t \in I$, and
- $g(s, t) \notin B^n$ for $s < 1$.