Morse Theory for Wasserstein Spaces

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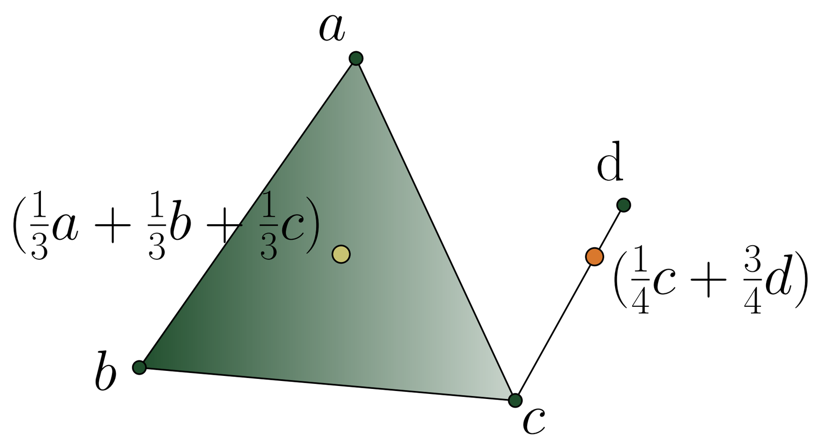
Motivation

Applied topology uses simplicial complexes to approximate a manifold based on data. This approximation is known not to always recover the homotopy type of the manifold. In this work-in-progress we investigate how to compute the homotopy type in such settings using techniques inspired by Morse theory.

Background

Points in simplices can be described with barycentric coordinates:



These can be interpreted as probability measures:

$$\sum_{i=0}^{n} \lambda_i x_i \iff \sum_{i=0}^{n} \lambda_i \delta_{x_i}$$

The set of finitely-supported probability measures on a metric space X admits a natural metric.

Definition: Let μ and ν be probability measures on X. Denote by $\Gamma(\mu, \nu)$ the set of all measures on $X \times X$ with marginals μ and ν. The *p-Wasserstein distance* is defined to be

$$d_W(\mu, \nu) = \inf_{\pi \in \Gamma} \left(\int d(x,y)^p \mathrm{d}\pi \right)^{1/p}.$$

Definition: A *metric simplicial complex* on X is a metric space (S, d_W) where S is a collection of finitely-supported probability measures on X which satisfies:

1. For all $x \in X$, the point mass δ_x is in S, and
2. If $\mu \in S$ and $\nu \ll \mu$, then $\nu \in S$.

Main Example: The Vietoris–Rips metric complex, $\mathrm{VR}^m(X; r)$, contains all finitely-supported measures, μ, such that the diameter of the support of μ is less than r.

Questions

Main Question: Given a known metric space (e.g. a compact Riemannian manifold), M, what is the homotopy type of $\mathrm{VR}^m(M; r)$ for all values of r?

Question: How is $\mathrm{VR}^m(X; r)$ related to the ordinary Vietoris–Rips simplicial complex, $\mathrm{VR}(X; r)$, with the simplicial complex topology? (Partial answer: if X is finite then $\mathrm{VR}^m(X; r) \cong \mathrm{VR}(X; r)$.)

Question: Given X and Y and some operation on metric spaces $\star$, how is $\mathrm{VR}^m(X \star Y; r)$ related to $\mathrm{VR}^m(X; r) \star \mathrm{VR}^m(Y; r)$?

Morse Theory

Classical Morse theory is based on two lemmas [6]. Given a smooth manifold, M, and a smooth function $F\colon M \to \mathbb{R}$ with no degenerate critical points, then

1. If $[a, b] \subseteq \mathbb{R}$ contains no critical values of F, then $F^{-1}(-\infty, a] \simeq F^{-1}(-\infty, b]$, and
2. If a is an index-k critical point of F, then $F^{-1}(-\infty, a+\varepsilon] \simeq F^{-1}(-\infty, a-\varepsilon] \cup D^k$ where D^k is a k-cell.

We propose to answer the questions above using a form of Morse theory for metric simplicial complexes. In particular, [4] and [5] develop a form of differential geometry for Wasserstein spaces, which should be amenable to Morse theory.

What Can Happen at Higher Scales?

The homotopy type of the Vietoris–Rips complex of S^1 is known for all r [1], and the results are surprising:

$$\mathrm{VR}_{\leq}(S^1; r) \simeq \begin{cases} S^{2\ell+1} & \frac{\ell}{2\ell+1} < r < \frac{\ell+1}{2\ell+3} \\ \bigvee^{\infty} S^{2\ell} & r = \frac{\ell}{2\ell+1} \end{cases}.$$

Conjecture: $\mathrm{VR}^m_<(S^1; r) \simeq \mathrm{VR}_<(S^1; r)$ for all r, and $\mathrm{VR}^m_{\leq}(S^1; r) \simeq \mathrm{VR}_{\leq}(S^1; r)$ except when $r = \frac{\ell}{2\ell+1}$.

Preliminary Results

Theorem: For small r, $\mathrm{VR}^m(M; r) \simeq M$.

Proof sketch: Appears in [2] and [3] for different types of M. □

Theorem: For any metric spaces X and Y, and any $r \in [0, +\infty]$, we have $\mathrm{VR}^m(X \times Y; r) \simeq \mathrm{VR}^m(X; r) \times \mathrm{VR}^m(Y; r)$ and $\mathrm{VR}^m(X \vee Y; r) \simeq \mathrm{VR}^m(X; r) \vee \mathrm{VR}^m(Y; r)$.

Proof sketch: For products, the homotopy equivalence is given by forming the product measure:

$$\left(\sum_i \lambda_i \delta_{x_i}, \sum_j \lambda_j \delta_{y_j} \right) \mapsto \sum_{i,j} \lambda_i \lambda_j \delta_{(x_i, y_j)}.$$

This has a homotopy inverse

$$\sum_{i,j} \lambda_{i,j} \delta_{(x_i, y_j)} \mapsto \left(\sum_i \sum_j \lambda_{i,j} \delta_{x_i}, \sum_j \sum_i \lambda_{i,j} \delta_{y_j} \right)$$

given by taking the marginals of a distribution. □

Additional Known Results:

- For any convex $K \subseteq \mathbb{R}^d$, $\mathrm{VR}^m(K; r)$ is contractible for all r.
- For $0 \leq r < 1/3$, $\mathrm{VR}^m(S^1; r) \simeq S^1$, and $\mathrm{VR}^m(S^1; 1/3) \simeq S^3$.
- If X is a simply-connected space of non-positive curvature, then $\mathrm{VR}^m(X; r) \simeq X$ all $r \in [0, +\infty]$.

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