Persistence and Simplicial Metric Thickenings

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Family and friends
Applied topology/Topological data analysis

- Roughly 25 year history
- Topological techniques for applied problems
- Heavy emphasis on mathematical theory
My work...

- revolves around the theory of one-parameter persistent homology
- builds on previous research on geometric constructions used in persistent homology
- solves some preexisting problems and provides tools for future research
Simplicial complexes
Vietoris–Rips

“Connect points that are close”

\[ \text{VR}_\leq (X; r) = \{ \sigma \subseteq X \mid \sigma \text{ finite, diam } \sigma \leq r \} \]
One-parameter filtration: “spaces evolving over time”

\[ X_{s \leq t} : X_s \rightarrow X_t \]

Apply \( H_n \) to get the **persistent homology module** \( H_n(X) \)
The “output” of persistent homology

The *barcode* of $H_n(X)$ records the lifetimes of homological features:

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Barcodes exist under reasonable conditions and are “stable” to perturbations.
Vietoris–Rips persistent homology
Topology of simplicial complexes

- Standard coherent / colimit topology
- Classical metric topology (not used here)
- Metric thickening topology

All agree for finite simplicial complexes but can differ for infinite
Simplicial metric thickenings


If $K$ is a simplicial complex with vertex set a metric space $(X, d)$,

$$K^m = \left\{ \sum_{i=1}^{n} \lambda_i \delta_{x_i} \mid \lambda_i \geq 0 \text{ for all } i, \sum_{i=1}^{n} \lambda_i = 1, [x_1, \ldots, x_n] \in K \right\},$$

equipped with the 1-Wasserstein metric

- Isometric embedding $X \hookrightarrow K^m$

- Vietoris–Rips metric thickenings: $\text{VR}^m(X; r)$
Vietoris–Rips properties

Theorem (Hausmann / Adamaszek, Adams, and Frick)

For a closed Riemannian manifold $M$, $\text{VR}(M; r) \simeq \text{VR}^m(M; r) \simeq M$ for small enough $r$.

Theorem (Equivalence of Vietoris–Rips persistent homology)

If $X$ is a totally bounded metric space, then $H_n(\text{VR}(X))$ and $H_n(\text{VR}^m(X))$ have identical barcodes up to open vs. closed endpoints.

Theorem (Gillespie)

The natural map $\text{VR}_<(X; r) \rightarrow \text{VR}^m_<(X; r)$ is a weak homotopy equivalence.
Vietoris–Rips properties

“Close filtrations produce close barcodes”

Theorem (Stability of Vietoris–Rips persistent homology)

If $X$ and $Y$ are totally bounded metric spaces, then

\[
d_B\left(\bar{\text{bar}}(H_n(\text{VR}(X))), \bar{\text{bar}}(H_n(\text{VR}(Y)))\right) \leq 2d_{GH}(X, Y)
\]

and

\[
d_B\left(\bar{\text{bar}}(H_n(\text{VR}^m(X))), \bar{\text{bar}}(H_n(\text{VR}^m(Y)))\right) \leq 2d_{GH}(X, Y)
\]

Applies even in the case of spaces with infinitely many points.
Homotopy types of \( \text{VR}(S^1; r) \): Adamaszek and Adams, 2015.

Matching homotopy types of the metric thickenings \( \text{VR}_m^\leq(S^1; r) \): Vietoris-Rips Metric Thickenings of the Circle, Journal of Applied and Computational Topology, 2023

**Theorem**

If \( r \in \left[ \frac{2k\pi}{2k+1}, \frac{(2k+2)\pi}{2k+3} \right) \), then \( \text{VR}_m^\leq(S^1; r) \cong S^{2k+1} \).
Implications for persistence

- Interpretation of persistent homology in practice
- New techniques in persistent homology

Footprints of Geodesics in Persistent Homology – Virk
What measures are possible?
Method outline

- Gather clusters
- Identify every measure with an odd polygonal measure while preserving homotopy type
Technical properties

- Need properties of homotopies of simplicial metric thickenings
- Support homotopies
- Homotopy extension property

Extend classical ideas:

*Algebraic Topology* – Hatcher
The homotopy types

- Result: a CW complex
- Induction to find homotopy types

$D^0 \quad S^1 \quad D^2 \quad S^3$
Barcodes of $\text{VR}_{\leq}^m(S^1)$
“Connect points that are far”

- \( \text{AVR}(X; r) = \{ \sigma \subseteq X \mid \sigma \text{ finite, spread } \sigma \geq r \} \)
- Contravariant filtration
- \( \text{AVR}^m(X; r) \): metric thickening topology
- Connections/applications to graph coloring
An $n$-coloring of $G$ is the same as a homomorphism $G \to K_n$

Replace $K_n$ with $\text{AVR}(S^1; \frac{2\pi}{n})$

Circular chromatic number: $\lceil \chi_C \rceil = \chi$
Again, collapse to regular polygonal measures to get a cell complex

Now we get $n$-gons for both even and odd $n$
First step: gluing in “diameters” produces Möbius strip $M$

Degree two map $S^1 \to M \simeq S^1$
Rely on degrees of attaching maps
Third step uses homotopy group $\pi_n(S^n \vee S^n)$
Fourth step introduces a manifold and applies Mayer–Vietoris
Homotopy types

**Theorem**

If \( r \in \left( \frac{2\pi}{2k+1}, \frac{2\pi}{2k-1} \right] \), then \( \text{AVR}^m(S^1; r) \cong S^{2k-1} \).

Degree two maps \( \Rightarrow \) persistent homology depends on characteristic of field


