Metric Thickening, Orbitopes, and Borsuk–Ulam Theorems

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Summary

Metric thickenings of a metric space capture local geometric properties of the space. We use the combinatorial and geometric structure of convex bodies in Euclidean space to give geometric proofs of the homotopy type of certain metric thickenings of the circle. Consequently, we discover interconnections between the geometry of circle actions on Euclidean space, the structure of zeros of trigonometric polynomials, and theorems of Borsuk–Ulam type.

Definitions

- Fix a metric space $(X, d)$. Given a simplicial complex $K$ with vertex set $X$, the metric thickening of $K$ is the metric space $K^m = ([K], d_W)$, where $d_W$ denotes the 1-Wasserstein metric and $[K] = \left\{ \sum_{i=0}^{n} x_i \mid n \in \mathbb{N}, \lambda_i \geq 0, \sum_i \lambda_i = 1, \{x_0, \ldots, x_n\} \in K \right\}$ [1].
- Let $VR(X; r)$ and $CH(X; r)$ denote the Vietoris–Rips and Čech simplicial complexes of a metric space $X$ at scale $r$, respectively. Let $VR^m(X; r)$ and $CH^m(X; r)$ denote the metric thickenings of these complexes.
- An orbitope is a convex hull of an orbit of a compact group acting linearly on a vector space [2].
- The Barvinok–Novik orbitope (see [3]) is $B_k = \text{conv}(SM_k(S^3))$, where $SM_k : S^1 \to \mathbb{R}^k$ by $SM_k(t) = (\cos(t), \sin(t), \cos(3t), \sin(3t), \ldots, \cos((2k-1)t), \sin((2k-1)t))$.

Main Theorem ([4])

Equip $S^3$ with the geodesic metric (of total circumference 1). Then,

$VR^m(S^3; r) \simeq S^{3k-1}$ if $r = \frac{1}{2k+1}$

Homotopy equivalences are $p \circ SM_k$ and $\iota$ in the following diagram:

$VR^m(S^3; r) \xymatrix@1{\ar[r]^{SM_k} & \mathbb{R}^k \ar[r]^{\iota} & \theta \ar[r]^{p} & VR^m(S^3; r)}$

Here, the domain of $SM_k$ has been linearly extended to $VR^m(S^1; r)$, $p$ denotes the radial projection, and $\iota$ denotes the inclusion $\iota : \mathbb{R}^k \to VR^m(S^3; r)$.

Intuition: Simplices contributing to the homotopy type of $VR^m(S^3; r)$ are contained in $\partial B_1$. Consequently, $p \circ SM_k$ reduces the dimension of $VR^m(S^3; r)$ while maintaining the correct topology. In fact, $B_1$ is simplicial, meaning its faces are simplices, and if $(SM_k(t_0), \ldots, SM_k(t_n))$ is a simplex in $\partial B_1$, then $(t_0, \ldots, t_n)$ belongs to the 2-skeleton of $VR^m(S^3; r)$ [3].

Conjecture

 Equip $S^3$ with the geodesic metric (of total circumference 1). Then,

$VR^m(S^3; r) \simeq S^{3k-1}$ if $r = \frac{1}{2k+1}$

Desired proof (outline). For $\frac{1}{2k+1} \leq r < \frac{1}{2k+1}^+$, there exist homotopy equivalences $p \circ SM_k$ and $\iota$ in the following diagram:

$VR^m(S^3; r) \xymatrix@1{\ar[r]^{SM_k} & \mathbb{R}^k \setminus \{0\} \ar[r]^{\iota} & \theta \ar[r]^{p} & VR^m(S^3; r)}$

where $p$ denotes the radial projection, $\iota$ denotes the inclusion, and $\partial B_1 \cong S^{3k-1}$.

- Continuity depends crucially on the topology of the metric thickening. In fact, $X \to [VR(X; r)]$ is not continuous if $VR(X; r)$ is not locally finite, whereas $X \to VR^m(X; r)$ is always continuous.
- Exact structure of $\partial B_{2k}$ is unknown for $k > 2$, and showing $\iota \circ p \circ SM_k \simeq i_{VR^m(S^3; r)}$ is difficult for $r > \frac{1}{2}$. 

Consequences

- Theorem. If $f : S^1 \to \mathbb{R}^{2k+1}$ is continuous, there exists a subset $\{x_1, \ldots, x_n\} \subseteq S^1$ of diameter at most $\frac{2k+1}{r}$ and with $m \leq 2k+1$ such that $\sum_{i=0}^{m} \lambda_i f(x_i) = \sum_{i=0}^{m} \lambda_i f(-x_i)$, for some choice of convex coefficients $\lambda_i$ [4]. This result is sharp.

- Theorem. Let $r_n$ denote the diameter of an inscribed regular $(n + 1)$-simplex in $S^n$. If $f : S^n \to \mathbb{R}^{n+2}$ is continuous, there is a subset $\{x_1, \ldots, x_n\} \subseteq S^n$ of diameter at most $r_n$ such that $\sum_{i=0}^{n} \lambda_i f(x_i) = \sum_{i=0}^{n} \lambda_i f(-x_i)$, for some choice of convex coefficients $\lambda_i$ [4].

Theorem. Given a subset $X \subseteq S^1$ of diameter less than $\frac{1}{2k+1}$, there exists a raked homogeneous trigonometric polynomial of degree $2k - 1$ that is positive on all of the points in $X$ [4]. This result is sharp.

Future Work

- If the orbitopes $B_{2k}$ are simplicial for all $k$, it would follow that the $(2k - 1)$-dimensional homology, cohomology, and homotopy groups of $VR^m(S^3; r)$ are nontrivial for $\frac{1}{2k+1} \leq r < \frac{1}{2k+1}^+$.

References


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