IRREDUCIBILITY OF NEWTON STRATA IN $\mathrm{GU}(1, n-1)$ SHIMURA VARIETIES

JEFFREY D. ACHTER

ABSTRACT. Let L be a quadratic imaginary field, inert at the rational prime p. Fix an integer $n \geq 3$, and let $\mathcal{M}$ be the moduli space (in characteristic p) of principally polarized abelian varieties of dimension n equipped with an action by $\mathcal{O}_L$ of signature of $(1, n-1)$. We show that each Newton stratum of $\mathcal{M}$, other than the supersingular stratum, is irreducible.

1. INTRODUCTION

For a complex abelian variety X, the isomorphism class of its p-torsion group scheme $X[p]$ and of its p-divisible group $X[p^\infty]$ depend only on the dimension of X. In contrast, in characteristic p, there are different possibilities for the corresponding isomorphism (or even isogeny) class. Each such invariant provides a stratification of a family of abelian varieties in positive characteristic.

The isogeny class of $X[p^\infty]$ is called the Newton polygon of X. The goal of the present note is to prove that the space of abelian varieties with given Newton polygon and a certain, specified endomorphism structure is irreducible.

More precisely, let L be a quadratic imaginary field, inert at the rational prime p. Fix an integer $n \geq 3$, and let $\mathcal{M}$ be the moduli space (over $\mathbb{F}_{p^2}$) of principally polarized abelian varieties of dimension n equipped with an action by $\mathcal{O}_L$ of signature $(1, n-1)$. Our main result is:

Theorem 1.1. *Let $\xi \neq \sigma$ be an admissible Newton polygon for $\mathcal{M}$ which is not supersingular. Then the corresponding stratum $\mathcal{N}^\xi$ is irreducible.*

The proof of Theorem 1.1 is modelled on, but considerably easier than, that of [4, Thm. A]. This is possible because the Newton and Ekedahl-Oort stratifications on $\mathcal{M}$ are much simpler than those of $\mathcal{A}_g$.

In the special case where $L = \mathbb{Q}(\zeta_3)$ and n is 3 or 4, $\mathcal{M}$ essentially coincides with a component of the moduli space of cyclic triple covers of the projective line. Theorem 1.1 provides a crucial base case for forthcoming work of Ozman, Pries and Weir on such covers [11], and that work was the initial impetus for the present study.

For a topological space T, let $\Pi_0(T)$ denote the set of irreducible components of T. If $T \subset \mathcal{M}$, then $\overline{T}$ is its closure in $\mathcal{M}$. The symbol k will denote an arbitrary algebraically closed field of characteristic p.

2. BACKGROUND ON $\mathcal{M}$

2.1. **Moduli spaces.** Let $\mathcal{M}$ be the moduli stack (over $\mathcal{O}_L/p \cong \mathbb{F}_{p^2}$) of principally polarized abelian varieties of dimension n with an action by $\mathcal{O}_L$ of signature $(1, n-1)$. Somewhat more precisely, $\mathcal{M}(S)$ consists of isomorphism classes of data (X, ι, λ), where $X \to S$ is an abelian variety of relative dimension n, $\iota : \mathcal{O}_L \to \mathrm{End}_S(X)$ is an embedding taking 1_L to id_X such that $\mathrm{Lie}(X)$,

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as a module over $\mathcal{O}_L \otimes \mathcal{O}_S$, has signature $(1, n-1)$; and $\lambda : X \to X^\vee$ is a principal polarization such that, if $(\dagger)$ is the induced Rosati involution on $\mathrm{End}(X)$, then for each $a \in \mathcal{O}_L$ one has $\iota(\overline{a}) = \iota(a)^{(\dagger)}$. It is standard that $\dim \mathcal{M} = 1 \cdot (n-1) = n-1$.

In fact, $\mathcal{M}$ is the moduli stack attached to the Shimura (pro-)variety constructed from a certain group G, as follows.

Let V be an n-dimensional vector space over L, equipped with a Hermitian pairing of signature $(1, n-1)$. Let $\mathrm{G}/\mathbb{Q}$ be the group of unitary similitudes of V, and let U the unitary group of V. Fix a hyperspecial subgroup $\mathbb{K}_p \subset G(\mathbb{Q}_p)$. For each sufficiently small open compact subgroup $\mathbb{K}^p \subset G(\mathbb{A}_f^p)$, there is a moduli space $\mathcal{M}_{\mathbb{K}^p} = \mathcal{M}_{\mathbb{K}_p\mathbb{K}^p}$ of abelian varieties of dimension n as above with $\mathbb{K}^p$ structure; see [7] for more details. If $\mathbb{K}^p$ is sufficiently small, then $\mathcal{M}_{\mathbb{K}^p}$ is a smooth, quasiprojective variety; and $\mathcal{M}$ may be constructed as the quotient of any $\mathcal{M}_{\mathbb{K}^p}$ by an appropriate finite group.

2.2. **Newton polygons in $\mathcal{M}$.** Newton and Ekedahl-Oort stratifications on $\mathrm{GU}(1, n-1)$ Shimura varieties are well understood [2]. There are exactly $1 + \lfloor n/2 \rfloor$ ("admissible") Newton polygons which occur, and the poset of admissible Newton polygons is actually totally ordered. Let σ be the supersingular Newton polygon, so that $\sigma \preceq \xi$ for any admissible Newton polygon ξ for $\mathcal{M}$. For a Newton polygon ξ, let $\mathcal{M}^\xi$ denote the locally closed locus corresponding to abelian varieties with Newton polygon ξ. Then $\mathcal{M}^\sigma$ is pure of dimension $\lfloor \frac{n-1}{2} \rfloor$. By purity [5, 9], if $Z_\sigma \in \Pi_0(\mathcal{M}^\sigma)$ and $\sigma \preceq \xi$, then there exists some $Z_\xi \in \Pi_0(\mathcal{M}^\xi)$ such that $Z_\sigma \subseteq \overline{Z_\xi}$, the closure of Z_ξ in $\mathcal{M}$.

The Newton stratification of $\mathcal{M}$ is described in [2], as follows. Each admissible Newton polygon is determined by its smallest slope. For each integer $1 \leq j \leq \lfloor n/2 \rfloor$, there is a Newton polygon ξ_{2j}, with smallest slope

$$\lambda(2j) = \frac{1}{2} - \frac{1}{2(\lfloor n/2 \rfloor + 1 - j)};$$

then $\mathcal{M}^{\xi_{2j}}$ has codimension $\lfloor n/2 \rfloor - j$ in $\mathcal{M}$. (Admittedly, in many ways this normalization is more awkward than that of [2], in which $\mathcal{M}^{\xi_{2j}}$ is labeled $\mathcal{M}_{2(\lfloor n/2 \rfloor - j)}$; but it will be more convenient for the deformation theory below.)

Away from the supersingular locus $\mathcal{M}^\sigma$, the Newton, Ekedahl-Oort, and final stratifications coincide; a p-divisible group is determined by its $\bmod p$ truncation [2, Thm. 5.3]. This is recalled in greater detail in Section 2.3 below.

The Newton polygon and Ekedahl-Oort type of a polarized $\mathcal{O}_L$-abelian variety with prime-to-p level structure do not depend on the level structure, and we set $\mathcal{M}^\xi_{\mathbb{K}^p} = \mathcal{M}_{\mathbb{K}^p} \times_{\mathcal{M}} \mathcal{M}^\xi$.

2.3. **p-divisible groups.** In contrast to the Siegel case, it is possible to write down a finite, explicit collection of those principally quasipolarized p-divisible groups with $\mathcal{O}_L$-action which occur as $(X, \iota, \lambda)[p^\infty]$ for $(X, \iota, \lambda) \in \mathcal{M}(k)$. Following Wedhorn, we describe such p-divisible groups in terms of their covariant Dieudonné modules, as follows.

For $m \in \mathbb{N}$, let $M(m)$ be the following Dieudonné module.

- As a $W(k)$-module, $M(m)$ admits basis $\{u_1, \cdots, u_m, v_1, \cdots, v_m\}$.

- A display [10, 13] for $M(m)$ is

$$\begin{aligned} Fu_1 &= (-1)^m v_m & v_1 &= Vu_m \\ Fv_2 &= u_1 & u_2 &= Vv_1 \\ Fv_3 &= u_2 & u_3 &= Vv_2 \\ &\vdots & &\vdots \\ Fv_m &= u_{m-1} & u_m &= Vv_{m-1} \end{aligned}$$

- The two eigenspaces for the action of $\mathcal{O}_L$ on $M(m)$ are $\oplus W(k)u_i$ and $\oplus W(k)v_j$.
- The quasipolarization is given by the symplectic pairing $\langle \cdot, \cdot \rangle$ where

$$\langle u_i, v_j \rangle = (-1)^i \delta_{ij}$$
$$\langle u_i, u_j \rangle = \langle v_i, v_j \rangle = 0$$

We also define the Dieudonné module N:

- As a $W(k)$-module, N admits basis $\{u_0, v_0\}$.
- A display for N is

$$Fv_0 = -u_0 \qquad\qquad u_0 = Vv_0.$$

- The quasipolarization is given by the symplectic pairing $\langle u_0, v_0 \rangle = 1$.
- The two eigenspaces for the action of $\mathcal{O}_L$ are $W(k)u_0$ and $W(k)v_0$.

With this notation in place, one has the following result of Bültel and Wedhorn:

Theorem 2.1. [2]

(a) *Suppose $1 \leq j \leq \lfloor n/2 \rfloor$. There exists an integer $r(j)$ such that, if $(X, \iota, \lambda) \in \mathcal{M}^{\xi_{2j}}(k)$, then*

$$\mathbb{D}_*((X, \iota, \lambda)[p^\infty]) \cong M(2(\lfloor n/2 \rfloor + 1 - j) \oplus N^{r(j)}.$$

(b) *There exists an open dense subspace $\mathcal{M}^{\sigma\circ} \subset \mathcal{M}^\sigma$ such that, if $(X, \iota, \lambda) \in \mathcal{M}^{\sigma\circ}(k)$, then*

$$\mathbb{D}_*((X, \iota, \lambda)[p^\infty]) \cong \begin{cases} M(n) & n \text{ odd} \\ M(n-1) \oplus N & n \text{ even} \end{cases}.$$

2.4. **Hecke operators.** An inclusion $\mathbb{K}_1^p \hookrightarrow \mathbb{K}_2^p$ of open compact subgroups of $\mathrm{G}(\mathbb{A}_f^p)$ induces a cover of Shimura varieties $\mathcal{M}_{\mathbb{K}_1^p} \to \mathcal{M}_{\mathbb{K}_2^p}$. More generally, an element $g \in \mathrm{G}(\mathbb{A}_f^p)$ induces, for each open compact $\mathbb{K}^p$, a natural morphism $\mathcal{M}_{\mathbb{K}^p} \to \mathcal{M}_{g^{-1}\mathbb{K}^p g}$.

Let $z \in \mathcal{M}_{\mathbb{K}_0^p}(k)$. Its prime-to-$p$ (unitary) Hecke orbit, $\mathcal{H}^p(z)$, is defined as follows. Consider the pro-variety $\widehat{\mathcal{M}}_{\mathbb{K}_0^p} = \varprojlim_{\mathbb{K}^p \subset \mathbb{K}_0^p} \mathcal{M}_{\mathbb{K}^p}$. Choose a lift $\hat{z}$ of z to $\widehat{\mathcal{M}}_{\mathbb{K}_0^p}$. Then $\mathcal{H}^p(z)$ is the projection to $\mathcal{M}_{\mathbb{K}_0^p}$ of $\mathrm{U}(\mathbb{A}_f^p)\hat{z}$. (One can also construct the "similitude" Hecke orbit of $\hat{z}$, by replacing the orbit $\mathrm{U}(\mathbb{A}_f^p)\hat{z}$ with $\mathrm{G}(\mathbb{A}_f^p)\hat{z}$. However, the unitary Hecke orbit is both the output of [12, Thm. 4.6] and the input to [6, Thm. 1.4], and thus better suited to the task at hand.)

3. Closures of Newton strata

Let ξ be an admissible Newton polygon for $\mathcal{M}$ such that $\xi \neq \sigma$.

Lemma 3.1. *The locus $\mathcal{M}^\xi$ is smooth.*

Proof. The isomorphism class of $(X[p^\infty], \iota[p^\infty], \lambda[p^\infty])$ for $(X, \iota, \lambda) \in \mathcal{M}^\xi(k)$ is independent of the choice of point (Theorem 2.1). By the Serre-Tate theorem, the formal neighborhoods of all points of $\mathcal{M}^\xi$ are thus isomorphic. Since $\mathcal{M}^\xi$ is by definition equipped with the reduced subscheme structure, it must be smooth. □

Lemma 3.2. *If $Z_\xi \in \Pi_0(\mathcal{M}^\xi)$, then there exists $Z_\sigma \in \Pi_0(\mathcal{M}^\sigma)$ such that $Z_\sigma \subset \overline{Z}_\xi$.*

Proof. We prove the following apparently stronger result. Suppose ν and ξ are admissible Newton polygons with $\nu \prec \xi$, and $Z_\xi \in \Pi_0(\mathcal{M}^\xi)$. We show that there exists $Z_\nu \in \Pi_0(\mathcal{N}^\sigma)$ such that $Z_\nu \subset \overline{Z}_\xi$. It suffices to prove this statement under the assumption that ν is the immediate predecessor of ξ, so that $\dim \mathcal{M}^\nu = \dim \mathcal{M}^\xi - 1$. The statement is trivially true if $\xi = \xi_{2\lfloor n/2 \rfloor}$ is the locus with positive p-rank; henceforth, we assume that ξ is strictly smaller than $\xi_{2\lfloor n/2 \rfloor}$.

It is is slightly more convenient to work with fine moduli schemes. Let $\mathbb{K}^p \subset G(\mathbb{A}_f^p)$ be an open compact subgroup which is small enough that $\mathcal{M}_{\mathbb{K}^p}$ is a smooth, quasiprojective variety. Let $W_\xi \in \Pi_0(Z_\xi \times_\mathcal{M} \mathcal{M}_{\mathbb{K}^p})$ be an irreducible component of $\mathcal{M}^\xi_{\mathbb{K}^p}$ lying over Z_ξ. It suffices to show that the closure of W_ξ in $\mathcal{M}_{\mathbb{K}^p}$ contains an irreducible component of $\mathcal{M}^\nu_{\mathbb{K}^p}$.

Let $\widetilde{\mathcal{M}}_{\mathbb{K}^p}$ be a toroidal compactification of $\mathcal{M}_{\mathbb{K}^p}$ (e.g., [8, 6.4.1.1]). It is a smooth, projective variety. Let $\widetilde{W}_\xi$ be the closure of W_ξ in $\widetilde{\mathcal{M}}_{\mathbb{K}^p}$, and let $\partial W_\xi = \widetilde{W}_\xi \smallsetminus W_\xi$. Newton strata (other than the supersingular stratum) coincide with Ekedahl-Oort strata, and the latter are known to be affine (e.g., [9]). Because W_ξ is positive dimensional, ∂W_ξ is nonempty. The first slope of ξ is positive, while the boundary of $\widetilde{\mathcal{M}}_{\mathbb{K}^p}$ parametrizes semiabelian varieties with nontrivial toric part. Consequently, $\partial W_\xi \cap (\widetilde{\mathcal{M}}_{\mathbb{K}^p} \smallsetminus \mathcal{M}_{\mathbb{K}^p})$ is empty, and $\partial W_\xi \subset \mathcal{M}_{\mathbb{K}^p}$. Again by purity ([9]), $\dim \partial W_\xi = \dim W_\xi - 1$. By semicontinuity of Newton polygons, there is an a priori containment $\partial W_\xi \subseteq \cup_{\tau \prec \xi} \mathcal{M}^\tau_{\mathbb{K}^p}$. The result now follows from dimension counts: $\mathcal{M}^\nu_{\mathbb{K}^p}$ is pure of dimension $\dim W_\xi - 1$, while if $\tau \prec \nu$ then $\dim \mathcal{M}^\tau_{\mathbb{K}^p} < \dim W_\nu = \dim \partial W_\xi$. □

Conversely,

Lemma 3.3. *If $Z_\sigma \in \Pi_0(\mathcal{M}^\sigma)$, then there is a unique $Z_\xi \in \Pi_0(\mathcal{M}^\xi)$ such that $Z_\sigma \subset \overline{Z}_\xi$.*

Proof. The existence of such a Z_ξ follows from purity and dimension-counting. If there were two such components, then they would intersect along $Z_\sigma \cap \mathcal{M}^{\sigma\circ}$, which would contradict the smoothness shown in Lemma 4.1. □

4. Local calculations

Lemma 4.1. *Let ξ be an admissible Newton polygon which is not supersingular, and suppose $z \in \mathcal{M}^{\sigma\circ}(k)$. Then $\overline{\mathcal{M}}^\xi$ is smooth at z.*

Proof. This follows directly from the explicit calculation (Lemmas 4.3 and 4.8) of the Newton stratification on the formal neighborhood $\mathcal{M}^{/z}$ of z. □

The necessary calculations are somewhat sensitive to the parity of n. We first work out the details when n is odd, and then indicate the changes necessary to accommodate even n.

4.1. **The case of n odd.** Throughout this section, assume that n is odd.

4.1.1. *Explicit deformations.* Suppose $z=(X,\iota,\lambda)\in\mathcal{M}^{\sigma\circ}(k)$. Our goal is to understand the Newton stratification on the formal neighborhood $\mathcal{M}^{/z}=\operatorname{Spf}\widetilde{R}$ of z in $\mathcal{M}$. This will be accomplished using (covariant) Dieudonné theory. Suppose $z=(X,\iota,\lambda)\in\mathcal{M}^{\sigma\circ}(k)$. Then the Dieudonné module $\mathbb{D}_*(X[p^\infty])$ is isomorphic to $M:=M(n)$ (Theorem 2.1).

Deformations of $X[p^\infty]$ are parametrized by $\operatorname{Hom}(VM/pM,M/VM)$; those which preserve the $\mathcal{O}_L$-structure are classified by $\operatorname{Hom}_{\mathcal{O}_L\otimes k}(VM/pM,M/VM)$ (e.g., [1]). The display we have chosen gives coordinates on VM/pM and M/VM:

$$\begin{aligned} VM/pM &= k\{v_1,u_2,u_3,\cdots,u_n\}\\ M/VM &= k\{u_1,v_2,v_3,\cdots,v_n\}\end{aligned}$$

Consequently,

$$\begin{aligned}\operatorname{Hom}_{\mathcal{O}_L\otimes k}(VM/pM,M/VM) &= k\{v_1^*v_2,v_1^*v_3,\cdots,v_1^*v_n,u_2^*u_1,u_3^*u_1,\cdots,u_n^*u_1\}\\ &\subset \operatorname{Hom}_k(VM/pM,M/VM)=(VM/pM)^*\otimes(M/VM),\end{aligned}$$

and the universal equicharacteristic deformation ring of $(X[p^\infty],\iota[p^\infty])$ is

$$\widetilde{R}'=k[\![t(v_1v_2),t(v_1v_3),\cdots,t(v_1v_n),t(u_2u_1),\cdots,t(u_nu_1)]\!].$$

For $t(xy)\in\widetilde{R}'$, let $\underline{t}(xy)$ be its Teichmuller lift to $W(\widetilde{R}')$. Then $\widetilde{X}[p^\infty]$ is displayed over $\widetilde{R}'$ by

$$\begin{aligned}
\widetilde{F}u_1 &= -v_n & v_1 &= \widetilde{V}(u_n+\underline{t}(u_nu_1)u_1)\\
\widetilde{F}v_2 &= u_1 & u_2 &= \widetilde{V}(v_1+\sum_{2\le j\le n}\underline{t}(v_1v_j)v_j)\\
\widetilde{F}v_3 &= u_2+\underline{t}(u_2u_1)u_1 & u_3 &= \widetilde{V}v_2\\
&\vdots & &\vdots\\
\widetilde{F}v_n &= u_{n-1}+\underline{t}(u_{n-1}u_1)u_1 & u_n &= \widetilde{V}v_{n-1}
\end{aligned}$$

The pairing $\langle\cdot,\cdot\rangle$ extends to $\widetilde{M}=M\otimes_{W(k)}W(\widetilde{R}')$ by linearity. We would like to identify the largest quotient $\widetilde{R}$ of $\widetilde{R}'$ to which $\langle\cdot,\cdot\rangle$ extends as a pairing of Dieudonné modules; for then $\mathcal{M}^{/z}\cong\operatorname{Spf}\widetilde{R}$.

The quasipolarization extends to a ring R if and only if, for each $x,y\in\widetilde{M}_R:=\widetilde{M}\otimes_{W(\widetilde{R}')}W(R)$, one has

$$\langle\widetilde{F}x,y\rangle=\langle x,\widetilde{V}\rangle^\sigma.$$

Suppose $3\le j\le n$, and let $(x,y)=(v_j,v_1+\sum_{2\le k\le n}\underline{t}(v_1v_k)v_k)$. Then

$$\begin{aligned}\langle\widetilde{F}x,y\rangle &= \langle u_{j-1}+\underline{t}(u_{j-1}u_1)u_1,v_1+\sum_{2\le k\le n}\underline{t}(v_1v_k)v_k\rangle\\ &= \underline{t}(v_1v_{j-1})\langle u_{j-1},v_{j-1}\rangle+\underline{t}(u_{j-1}u_1)\langle u_1,v_1\rangle\\ &= (-1)^{j-1}\underline{t}(v_1v_{j-1})-\underline{t}(u_{j-1}u_1),\end{aligned}$$

while

$$\begin{aligned}\langle x,\widetilde{V}y\rangle &= \langle v_j,u_2\rangle\\ &= 0.\end{aligned}$$

Consequently, if $\widetilde{M}_R$ is quasipolarized by $\langle\cdot,\cdot\rangle$, then for each $2\le k\le n-1$, the image of $(-1)^kt(v_1v_k)-t(u_ku_1)$ in R is zero.

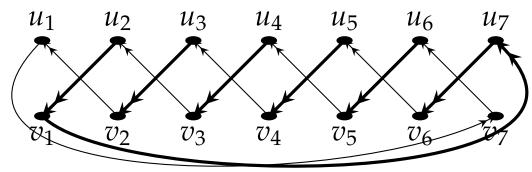

FIGURE 1. The graph Γ for $n = 7$.

Similarly, by considering $(x, y) = (v_1, v_2 + \sum \underline{t}(v_2 v_j) v_j)$, we see that the image of $t(v_1 v_n)$ in such an R must be zero.

The quotient $\widetilde{R}$ of $\widetilde{R}'$ by these relations is a smooth, local ring of dimension $n-1$, and thus we identify $\widetilde{R}$ with

$$\widetilde{R} = k[\![s_2, \cdots, s_n]\!],$$

where s_j is the image of $t(u_j u_1)$ in $\widetilde{R}$. We record these calculations as follows.

Lemma 4.2. *The formal neighborhood $\mathcal{M}^{/z}$ of z is isomorphic to $\widetilde{R} = \operatorname{Spf} k[\![s_2, \cdots, s_n]\!]$. Over $\widetilde{R}$, the Dieudonné module $\widetilde{M} = \widetilde{M}_{\widetilde{R}}$ of the universal deformation of $(X[p^\infty], \iota[p^\infty], \lambda[p^\infty])$ is displayed by*

$$\begin{aligned}
\widetilde{F}u_1 &= -v_n & v_1 &= \widetilde{V}(u_n - \underline{s}_n u_1)\\
\widetilde{F}v_2 &= u_1 & u_2 &= \widetilde{V}(v_1 + \sum_{2 \le j \le n} (-1)^j \underline{s}_j v_j)\\
\widetilde{F}v_3 &= u_2 + \underline{s}_2 u_1 & u_3 &= \widetilde{V} v_2\\
&\vdots & &\vdots\\
\widetilde{F}v_n &= u_{n-1} + \underline{s}_{n-1} u_1 & u_n &= \widetilde{V} v_{n-1}
\end{aligned}$$

4.1.2. *Newton strata in local coordinates.* In this choice of coordinates, it is easy to calculate the Newton stratification on $\mathcal{M}^{/z}$. For $1 \le j \le \lfloor n/2 \rfloor$, let

$$\mathcal{M}^{/z}_{<j} := \mathcal{M}^{/z} \cap (\mathcal{M}^{\sigma} \cup \bigcup_{1 \le i < j} \mathcal{M}^{\xi_{2i}})$$

be the locus in $\mathcal{M}^{/z}$ parametrizing those deformations whose first slope is strictly larger than $\lambda(2j)$.

Lemma 4.3. *Suppose* $1 \le j \le \lfloor n/2 \rfloor$. *Then*

$$\mathcal{M}^{/z}_{<j} = \operatorname{Spf} \frac{k[\![s_2, \cdots, s_n]\!]}{(s_{2j}, s_{2(j+1)}, \cdots, s_{2\lfloor n/2 \rfloor})} \subset \mathcal{M}^{/z} = \operatorname{Spf} k[\![s_2, \cdots, s_n]\!].$$

Before proceeding with the proof, we construct a graph to encode part of the structure of (a deformation of) M. Initially, construct a graph Γ as follows (see Figure 4.1.2). With a slight abuse of notation, let the vertex set be $\{u_1, \cdots, u_n, v_1, \cdots, v_n\}$. For $2 \le i \le n$, draw a (light) gray arrow from v_i to u_{i-1}, to encode the fact that $Fv_i = u_{i-1}$. Similarly, draw a gray arrow from u_1 to v_n.

Also, for each $2 \le i \le n$, draw a black arrow from u_i to v_{i-1}, to encode the fact that $Fu_i = pv_{i-1}$. Similarly, draw a black arrow from v_1 to u_n.

Note that Γ is a (colored) cycle. In fact, starting from vertex u_1, one successively visits

$$\{u_1, v_n, u_{n-1}, v_{n-2}, u_{n-3}, \cdots, v_1, u_n, v_{n-1}, u_{n-2}, \cdots, v_2, u_1\}.$$

Now let S be an integral domain equipped with a surjection $\phi : k[\![s_2, \cdots, s_n]\!] \to S$, and let K be the field of fractions of S. Construct a graph Γ_S by (possibly) augmenting the edge set of Γ, as follows.

For each $2 \le i \le n-1$, if $\phi(s_i) \neq 0$, then add a gray edge from v_j to u_1. (For the sake of completeness, if $\phi(s_n) \neq 0$, then add a black edge from v_1 to u_1. For each $2 \le i \le n-1$, if $\phi(s_i) \neq 0$, then add a black edge from u_2 to v_i. These additional black edges will not affect the final calculation.)

Let C be a cycle or path in Γ_S. The length of C is the number of edges in C, while the weight of C is the number of black edges in C. Define the slope of C to be

$$\lambda(C) = \frac{\text{weight}(C)}{\text{length}(C)}.$$

Note that for the trivial deformation, corresponding to Γ itself, we have $\lambda(\Gamma) = \frac{n}{2n} = \frac{1}{2}$.

Lemma 4.4. *If $C \subset \Gamma_S$ is a cycle through u_1, then the smallest slope of the Newton polygon is at most $\lambda(C)$.*

Proof. It is harmless, and convenient, to replace K by its perfection. Suppose there is a cycle C of length b and weight a; let $\widetilde{N}_K = W(K)\{u_2, \cdots, u_n, v_1, \cdots, v_n\}$. Then $F^b u_1 \in p^a W(K)\{u_1\} + \widetilde{N}_K$ but $F^b u_1 \notin \widetilde{N}_K$, and $\widetilde{M}_K / \widetilde{N}_K$ is an F-σ^a-crystal of slope at most a/b. Therefore, the smallest slope of $\widetilde{M}_K$ is at most a/b. □

Remark 4.5. Let $B(K) = \operatorname{Frac} W(K)$; then the $B(K)[F]$-span of u_1 in $\widetilde{M}_K \otimes B(K)$ is all of $\widetilde{M}_K \otimes B(K)$. Therefore, one can in fact show that the smallest slope of $\widetilde{M}_K$ is

$$\min_{C \subset \Gamma \text{ a cycle through } u_1} \lambda(C).$$

Lemma 4.6. *If $\phi(s_{2i}) \neq 0$, then there is a cycle in Γ_S of length $n+1-2j$ and weight $\frac{n-1}{2} - j$.*

Proof. In Γ, the unique path P from u_1 to v_{2j+1} has length $n+1-(2j+1) = n-2j$ and weight $\frac{n-(2j+1)}{2} = \frac{n-1}{2} - j$. If $\phi(s_{2j}) \neq 0$, then in Γ_S there is a cycle, obtained by concatenating u_1 to P, of length length$(P)+1$ and weight weight(P). □

Lemma 4.7. *If the smallest slope of $\widetilde{M}_K$ is greater than $\lambda(2j)$, then*

$$\phi(s_{2j}) = \phi(s_{2(j+1)}) = \cdots = \phi(2\lfloor n/2 \rfloor) = 0.$$

Proof. The contrapositive follows immediately from Lemmas 4.6 and 4.4; if there is some $i \ge j$ with $\phi(s_{2i}) \neq 0$, then the smallest slope of $\widetilde{M}_K$ is at most $\lambda(2i)$. □

Proof of Lemma 4.3. By Lemma 4.7, the sought-for neighborhood $\mathcal{N}_{<j}^{/z}$ is the formal spectrum of a quotient of $R_{<j} := k[\![s_1, \cdots, s_n]\!]/(s_{2j}, s_{2(j+1)}, \cdots, s_{2\lfloor n/2 \rfloor})$. We thus have $\mathcal{N}_{<j}^{/z} \hookrightarrow \operatorname{Spf} R_{<j} \hookrightarrow \mathcal{M}^{/z}$. Since both $\mathcal{M}_{<j}^{/z}$ and $\operatorname{Spf} R_{<j}$ have codimension $\lfloor n/2 \rfloor - j + 1$, the result follows. □

4.2. **The case of n even.** We now indicate the changes which must be made in order to perform the calculations of Section 4.1 in the case where n is even.

Suppose $z = (X, \iota, \lambda) \in \mathcal{M}^{\sigma\circ}(k)$. The quasipolarized Dieudonné module M of $X[p^\infty]$, as a p-divisible group with $\mathcal{O}_L$-action, is $M(n-1) \oplus N$ (Theorem 2.1). A calculation exactly like that in Section 4.1.1 shows $\mathcal{M}^{/z} \cong \operatorname{Spf} \widetilde{R} = \operatorname{Spf} k[[s_0, s_2, \cdots, s_{n-1}]]$; the corresponding deformation $\widetilde{M}$ of M is displayed by

$$\begin{aligned}
\widetilde{F}v_0 &= -u_0 - \underline{s}_0 u_1 & u_0 &= \widetilde{V}v_0 \\
\widetilde{F}u_1 &= -v_{n-1} & v_1 &= \widetilde{V}(u_{n-1} - \underline{s}_{n-1}u_1) \\
\widetilde{F}v_2 &= u_1 & u_2 &= \widetilde{V}(v_1 + \sum_{2\le j\le n-1} (-1)^j \underline{s}_j v_j) \\
\widetilde{F}v_3 &= u_2 + \underline{s}_2 u_1 & u_3 &= \widetilde{V}v_2 \\
&\vdots & &\vdots \\
\widetilde{F}v_{n-1} &= u_{n-2} + \underline{s}_{n-2}u_1 & u_{n-1} &= \widetilde{V}v_{n-2}
\end{aligned}$$

Construction and analysis of graphs Γ and Γ_S, for quotients S of $\widetilde{R}$, shows that Lemma 4.3 holds for even n, too:

Lemma 4.8. *Suppose* $1 \le j \le n/2$. *Then*

$$\mathcal{M}^{/z} \cap (\cup_{1\le i\le j} \mathcal{M}^{\xi_{2i}}) = \operatorname{Spf} \frac{k[[s_0, s_2, s_3, \cdots, s_n]]}{(s_{2j}, s_{2(j+1)}, \cdots, s_n)}.$$

5. Hecke orbits for the supersingular locus

Lemma 5.1. *Let $\mathbb{K}^p \subset G(\mathbb{A}_f^p)$ be a compact open subgroup. The $\mathrm{U}(\mathbb{A}_f^p)$-Hecke operators act transitively on $\Pi_0(\mathcal{M}^\sigma_{\mathbb{K}^p})$.*

Proof. Let $(X, \iota, \lambda) = \overline{\eta}$ be a geometric generic point of $\mathcal{M}^\sigma_{\mathbb{K}^p}$. The central leaf $\mathcal{C}([\overline{\eta}])$, which in a general PEL Shimura variety context parametrizes those $(Y, \jmath, \mu)$ with $(Y[p^\infty], \jmath[p^\infty], \mu[p^\infty]) \cong (X[p^\infty], \iota[p^\infty], \lambda[p^\infty])$, in this case coincides with (the union of a choice of geometric point over each generic point of) $\mathcal{M}^{\sigma\circ}$. There is an a priori inclusion $\mathcal{H}^p(\overline{\eta}) \subseteq \mathcal{C}([\overline{\eta}])$. Since $\overline{\eta}$ is basic and G, the reductive group defining $\mathcal{M}$, has simply connected derived group, the prime-to-p Hecke orbit of $\overline{\eta}$ coincides with the central leaf $\mathcal{C}([X[p^\infty], \iota[p^\infty], \lambda[p^\infty]])$ [12, Thm. 4.6(1) and Rem. 4.7(3)]. □

6. Irreducibility of Newton strata

Proof of Theorem 1.1. Chai and Oort identify nine steps in their proof of [4, Thm. 3.1], which is the analogue for $\mathcal{A}_g$ of Theorem 1.1. We proceed here in a similar fashion. Fix an open compact subgroup $\mathbb{K}^p \subset \mathrm{G}(\mathbb{A}_f^p)$; it suffices to prove that $\mathcal{M}^\xi_{\mathbb{K}^p}$ is irreducible.

Steps 1-6: By Lemma 3.3, there is a well-defined map of sets

$$\Pi_0(\mathcal{M}^\sigma) \longrightarrow \Pi_0(\mathcal{M}^\xi).$$

It is surjective, by Lemma 3.2. From it, we deduce the existence of a surjective

$$\Pi_0(\mathcal{M}^\sigma_{\mathbb{K}^p}) \longrightarrow \Pi_0(\mathcal{M}_{\mathbb{K}^p}),$$

visibly $\mathrm{U}(\mathbb{A}_f^p)$-equivariant.

Steps 7-8: By Lemma 5.1, the action of $\mathrm{U}(\mathbb{A}_f^p)$ on $\Pi_0(\mathcal{M}^\sigma_{\mathbb{K}^p})$ is transitive.

Step 9: Taken together, this shows that $\mathrm{U}(\mathbb{A}_f^p)$ acts transitively on $\Pi_0(\mathcal{M}^{\zeta}_{\mathbb{K}^p})$. By [6, Thm. 1.4], which is the PEL analogue of [3], $\mathcal{M}^{\xi}_{\mathbb{K}^p}$ is connected. Since $\mathcal{M}^{\xi}_{\mathbb{K}^p}$ is also smooth (Lemma 3.1), it is irreducible.

$\square$

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E-mail address: `j.achter@colostate.edu`

DEPARTMENT OF MATHEMATICS, COLORADO STATE UNIVERSITY, FORT COLLINS, CO 80523

URL: `http://www.math.colostate.edu/~achter`