REGULAR HOMOMORPHISMS, WITH A TWIST

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ABSTRACT. Let X/K be a variety over a field, and A/K an abelian variety. A regular homomorphism to A (in codimension i) induces, for every smooth geometrically connected pointed K-scheme (T, t_0) and every cycle class $Z \in \mathrm{CH}^i(T \times X)$, a morphism $T \to A$ of varieties over K. In this note we show that, if T admits no K-point, the data (T, Z) determines a torsor $A^{(T,Z)}$ over K under A and a K-morphism $T \to A^{(T,Z)}$. This can be used to provide an obstruction to the existence of algebraic cycles defined over K. We then connect this obstruction to some recent results of Hassett–Tschinkel and Benoist–Wittenberg on rationality of threefolds.

1. INTRODUCTION

Let K be a field with separable closure $\overline{K}$. We will use the convention that a *variety* over K is a geometrically reduced separated scheme of finite type over K. For a smooth proper variety X over K, abelian varieties can be used to understand $\mathrm{A}^i(X_{\overline{K}})$, the group of rational equivalence classes of algebraically trivial cycles on $X_{\overline{K}}$ of codimension i, via the notion of a regular homomorphism.

Recall that if A/K is an abelian variety, then a regular homomorphism over K, or Galois-equivariant regular homomorphism, is a group homomorphism $\phi\colon \mathrm{A}^i(X_{\overline{K}}) \to A(\overline{K})$ with the following property: For any smooth connected K-pointed variety (T, t_0) over K, and any cycle class $Z \in \mathrm{CH}^i(T \times X)$, the map of sets

$$T(\overline{K}) \longrightarrow \mathrm{A}^i(X_{\overline{K}}) \xrightarrow{\ \phi\ } A(\overline{K})$$

$$t \longmapsto [Z_t] - [Z_{t_0}] \longmapsto \phi([Z_t] - [Z_{t_0}])$$

is induced by a morphism $\psi_Z = \psi_{(T,t_0,Z)} : T \to A$ of schemes over K. (This concrete formulation of the functorial definition in [ACMV23] is both closer to the classical definition and better-suited to our purposes here.) An algebraic representative for X (in codimension i) is an abelian variety $\mathrm{Ab}^i_{X/K}$, and a regular homomorphism $\mathrm{A}^i(X_{\overline{K}}) \to \mathrm{Ab}^i_{X/K}(\overline{K})$, which is initial for all such maps.

At this point, the reader should keep in mind two slightly different kinds of examples of regular homomorphisms On one hand, $\mathrm{Ab}^1_{X/K}$ and $\mathrm{Ab}^{\dim X}_{X/K}$ always exist; these are, respectively, $(\mathrm{Pic}^0_{X/K})_{\mathrm{red}}$ and $\mathrm{Alb}_{X/K}$. Moreover, $\mathrm{Ab}^2_{X/K}$ exists, as well; this is due to Murre [Mur85] if K is algebraically closed and to [ACMV23, Thm. 6.1] in general. On the other hand, suppose K is a subfield of $\mathbb{C}$. By [ACMV20, Thm. A] (see also [ACMV23, Thm. 9.1]), the algebraic intermediate Jacobian $J^{2i-1}_a(X_{\mathbb{C}})$, which is by definition the image of the Abel–Jacobi map restricted to algebraically trivial cycle classes, admits a distinguished model $J^{2i-1}_{a,X/K}$ over K, and the Griffiths Abel–Jacobi map descends to a regular homomorphism $\mathrm{A}^i(X_{\overline{K}}) \to J^{2i-1}_{a,X/K}(\overline{K})$.

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Recently, *torsors* under such abelian varieties have been used to detect irrationality of geometrically rational varieties. For example, suppose X is a smooth projective variety over a subfield K of $\mathbb{C}$, and let $J = J^{2i-1}_{a,X/K}$. Given a geometrically irreducible component T of the Chow scheme of codimension-i cycles on X, Hassett and Tschinkel construct [HT21, Thm. 4.5] a torsor J^T under J; and they further show that this construction is compatible with addition in Chow, i.e., that in the Weil–Châtelet group of J one has $[J^{T_1}] + [J^{T_2}] = [J^{T_1 \times T_2}]$. If now X is a smooth projective geometrically rational threefold over an arbitrary field K, Benoist and Wittenberg construct a codimension-2 Chow scheme $\mathbf{CH}^2_{X/K}$ that represents a functor which is a certain subquotient of K-theory. Its connected component of identity $(\mathbf{CH}^2_{X/K})^\circ$ is an abelian variety – isomorphic to $\mathrm{Ab}^2_{X/K}$, if K is perfect – and so its other geometrically irreducible components are torsors under that abelian variety. Benoist and Wittenberg [BW23] and Hassett and Tschinkel [HT21] use these torsors to construct an obstruction to the rationality of the smooth complete intersection of two quadrics in $\mathbb{P}^5$. Subsequently, Frei *et. al.* studied extensions of, and limitations to, this so-called intermediate Jacobian torsor obstruction, especially for certain conic bundles over $\mathbb{P}^2$ [FJS+24, Thms. 1.4 and 1.5].

With this backdrop, we can finally explain the goal of the present note. It turns out that the various torsors constructed in [HT21] and [BW23] require neither (complex) intermediate Jacobians nor a functorial Chow scheme. Instead, they arise from an arbitrary regular homomorphism.

Theorem A. *Let X/K be a smooth proper variety, let A/K be an abelian variety, and let $\phi : \mathrm{A}^i(X_{\overline{K}}) \to A(\overline{K})$ be a regular homomorphism over K.*

(a) Let T/K be a smooth geometrically connected scheme, and let $Z \in \mathrm{CH}^i(T \times X)$. Then there exists a torsor $A^{(Z)}$ under A such that any choice of $\overline{K}$-point $t_0 \in T(\overline{K})$ induces a (K-rational) morphism $T \to A^{(Z)}$ which, after base change to $\overline{K}$, agrees with $\psi_{(Z_{\overline{K}}, t_0, T_{\overline{K}})}$.

(b) Let T_1 and T_2 be smooth geometrically connected schemes over K, and let $Z_j \in \mathrm{CH}^i(T_j \times X)$. Then there is an isomorphism of torsors

$$[A^{(Z_1)}] + [A^{(Z_2)}] = [A^{(Z_1 \boxplus Z_2)}],$$

where addition takes place in the Weil–Châtelet group $\mathrm{WC}(A/K) \cong H^1(K, A)$, and $Z_1 \boxplus Z_2 = p_{13}^* Z_1 + p_{23}^* Z_2 \in \mathrm{CH}^i(T_1 \times T_2 \times X)$.

There is a variant of this which works when the parametrizing scheme T is geometrically reducible; we work out the details of this in Theorem 2.4.

In work in progress, the authors use the ideas of §2 to construct a *big algebraic representative* for X. This is a group scheme whose connected component of identity is the (usual) algebraic representative, and which supports an action by $\mathrm{Aut}(X)$. In §3.1, we explain how the torsors of §2.1 give a conceptual framework for understanding (failures of) rationality and existence of algebraic cycles for smooth projective varieties over a field; in §3.2, we revisit the rationality of smooth complete intersections of quadrics in $\mathbb{P}^5$.

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2. Torsors unders algebraic representatives

2.1. **Torsors.** Let X/K be a smooth proper variety; we investigate codimension i cycles on $X_{\overline{K}}$. Let A/K be an abelian variety, and as in the introduction let $\phi : \mathrm{A}^i(X_{\overline{K}}) \to A(\overline{K})$ be a regular homomorphism over K.

If T_1 and T_2 are both smooth varieties over K, and if $Z_j \in \mathrm{CH}^i(T_j \times X)$ for $j = 1, 2$, we define

$$Z_1 \boxplus Z_2 = p_{13}^* Z_1 + p_{23}^* Z_2 \in \mathrm{CH}^i(T_1 \times T_2 \times X).$$

If $Z \in \mathrm{CH}^i(T \times X)$ then, following [ACMV20, §7.1], we set

$$\widetilde{Z} = Z \boxplus (-Z) \in \mathrm{A}^i(T \times T \times X) \subseteq \mathrm{CH}^i(T \times T \times X).$$

Whereas cycle classes parametrized by *K-pointed* schemes give rise to morphisms to A, in general, we obtain a map to a certain torsor under A, as follows.

Before proceeding, recall that if B/K is an abelian variety, then $\mathrm{WC}(B/K)$, the Weil-Châtelet group of B over K, is an abelian group whose elements are isomorphism classes of torsors under B over K. (If the base field K is clear from context, we will sometimes just write $\mathrm{WC}(B)$.) It is isomorphic to the Galois cohomology group $H^1(K, B) := H^1(\mathrm{Gal}(K), B(\overline{K}))$. If $\beta : \mathrm{Gal}(K) \to B(\overline{K})$ represents a cohomology class, we may describe the corresponding torsor $B[\beta]$ by adopting the viewpoint that $B[\beta]$ "is" B equipped with a twisted action of $\mathrm{Gal}(K)$ on $B(\overline{K})$, as in [Ser02, §5.2]. For $b \in B(\overline{K})$ and $\sigma \in \mathrm{Gal}(K)$, denote the image of b under σ by b^σ; and let $b^{\widetilde{\sigma}}$ denote its image under the action of $\mathrm{Gal}(K)$ twisted by β. Then we have

$$b^{\widetilde{\sigma}} = \beta_\sigma +_{B(\overline{K})} b^\sigma. \tag{2.1}$$

Theorem 2.1. *Let X/K be a smooth proper variety, and let $\phi : \mathrm{A}^i(X_{\overline{K}}) \to A(\overline{K})$ be a regular homomorphism over K. Let T/K be a smooth geometrically connected variety, and let $Z \in \mathrm{CH}^i(T \times X)$. Let $t_0 \in T(\overline{K})$ be a $\overline{K}$-point.*

(a) There exists a torsor $A^{(T,t_0,Z)}$ over K under A and a morphism of K-varieties $T \to A^{(T,t_0,Z)}$ which, after base change to $\overline{K}$, agrees with $\psi_{Z_{\overline{K}}}$.

(b) The isomorphism class of $A^{(T,t_0,Z)}$ is independent of the choice of t_0; let $A^{(T,Z)}$ be this torsor.

(c) If T_1 and T_2 are smooth geometrically connected varieties over K, and if $Z_j \in \mathrm{CH}^i(T_j \times X)$, then there is an isomorphism of torsors

$$[A^{(T_1,Z_1)}] + [A^{(T_2,Z_2)}] = [A^{(T_1 \times T_2, Z_1 \boxplus Z_2)}],$$

where addition takes place in the Weil–Châtelet group $\mathrm{WC}(A/K) \cong H^1(K, A)$.

We will sometimes write $A^{(Z)}$ for $A^{(T,Z)}$ if the parametrizing scheme T is clear from context.

Proof. We start by verifying that the function

$$\mathrm{Gal}(K) \xrightarrow{\alpha = \alpha(T,t_0,Z)} A(\overline{K})$$

$$\sigma \longmapsto \alpha_\sigma := \phi(Z_{t_0^\sigma} - Z_{t_0})$$

is a one-cocycle, and thus its class in $H^1(K, A)$ determines a torsor $A^{(T,t_0,Z)}$ over K under A. We compute that

$$\alpha_\sigma = \phi(\widetilde{Z}_{t_0^\sigma, t_0})$$
$$(\alpha_\sigma)^\tau = \phi(\widetilde{Z}_{t_0^\sigma, t_0})^\tau$$

and, by Galois-equivariance of ϕ, we have

$$\begin{aligned}(\alpha_\sigma)^\tau &= \phi((\widetilde{Z}_{t_0^\sigma, t_0})^\tau) \\ &= \phi(\widetilde{Z}_{t_0^{\sigma\tau}, t_0^\tau})\end{aligned}$$

because Z is defined over K. We verify the cocycle condition by computing

$$\begin{aligned}(\alpha_\sigma)^\tau + \alpha_\tau &= \phi(\widetilde{Z}_{t_0^{\sigma\tau},t_0^\tau} + \widetilde{Z}_{t_0^\tau,t_0}) \\ &= \phi(\widetilde{Z}_{t_0^{\sigma\tau},t_0}) = \alpha_{\sigma\tau}.\end{aligned}$$

We now show that $\psi := \psi_{(T_{\overline{K}},Z_{\overline{K}},t_0)} : T_{\overline{K}} \to A_{\overline{K}}$ descends to a K-rational morphism $T \to A^{(T,t_0,Z)}$. Recall that $A^{(T,Z)}$ is A equipped with a twisted action of $\mathrm{Gal}(K)$ (2.1), and consider the morphism

$$T_{\overline{K}} \xrightarrow{\psi=\psi_{(T,t_0,Z)}} A_{\overline{K}}$$

of varieties over $\overline{K}$. Identifying $A_{\overline{K}}$ with $A_{\overline{K}}^{(T,t_0,Z)}$, and in particular $A(\overline{K})$ with $A^{(T,t_0,Z)}(\overline{K})$, we have a morphism

$$T_{\overline{K}} \xrightarrow{\psi} A_{\overline{K}}^{(T,t_0,Z)}$$

which on $\overline{K}$-points is given by

$$t \longmapsto \phi(Z_t - Z_{t_0}).$$

To show that ψ descends to a morphism $T \to A^{(T,t_0,Z)}$ over K, it suffices to show that ψ is $\mathrm{Gal}(K)$-equivariant on $\overline{K}$-points. This follows from the calculation that, for $\sigma \in \mathrm{Gal}(K)$ and $t \in T(\overline{K})$, we have

$$\begin{aligned}\psi(t)^{\widetilde{\sigma}} &= \alpha_\sigma + \psi(t)^\sigma \\ &= \phi(Z_{t_0^\sigma} - Z_{t_0}) + \phi(Z_t - Z_{t_0})^\sigma \\ &= \phi(Z_{t_0^\sigma} - Z_{t_0}) + \phi(Z_{t^\sigma} - Z_{t_0^\sigma}) \\ &= \phi(Z_{t^\sigma} - Z_{t_0}) \\ &= \psi(t^\sigma).\end{aligned}$$

This proves (a). For (b), let $t_1 \in T(\overline{K})$ be any other point, with corresponding cocycle $\beta_\sigma = \phi(\widetilde{Z}_{t_1^\sigma,t_1})$. Then the difference of α and β is a coboundary, since

$$\begin{aligned}\alpha_\sigma - \beta_\sigma &= \phi(\widetilde{Z}_{t_0^\sigma,t_0} - \widetilde{Z}_{t_1^\sigma,t_1}) \\ &= \phi(\widetilde{Z}_{t_0^\sigma,t_1^\sigma} - \widetilde{Z}_{t_0,t_1}) \\ &= \phi(\widetilde{Z}_{t_0,t_1})^\sigma - \phi(\widetilde{Z}_{t_0,t_1}).\end{aligned}$$

Finally, for (c), we verify that the operator $\boxplus$ on cycles is compatible with addition in the Weil–Châtelet group of torsors over K under A. Choose $t_j \in T_j(\overline{K})$, and let $\alpha^{(j)}$ be the corresponding cocycle. Then $A^{(T_1 \times T_2, t_1 \times t_2, Z_1 \boxplus Z_2)}$ is determined by the cocycle

$$\begin{aligned}\sigma \mapsto\ & \phi((Z_1 \boxplus Z_2)_{(t_1,t_2)^\sigma} - (Z_1 \boxplus Z_2)_{(t_1,t_2)}) \\ &= \phi(Z_{1 t_1^\sigma} + Z_{2 t_2^\sigma} - Z_{1 t_1} - Z_{2 t_2}) \\ &= \alpha_\sigma^{(1)} + \alpha_\sigma^{(2)}.\end{aligned}$$

□

The isomorphism class of $A^{(Z)}$ depends only on the algebraic equivalence class of a fiber of Z:

Proposition 2.2. *For $j = 1, 2$, let T_j/K be a smooth geometrically connected variety, and let $Z_j \in \mathrm{CH}^i(T_j \times X)$. Suppose that there are $\overline{K}$-points $t_j \in T_j(\overline{K})$ such that $(Z_1)_{t_1}$ and $(Z_2)_{t_2}$ are algebraically equivalent. Then there is an isomorphism of K-torsors $A^{(T_1,Z_1)} \cong A^{(T_2,Z_2)}$.*

Proof. In fact, by hypothesis every $\overline{K}$-fiber of Z_1 is algebraically equivalent to every $\overline{K}$-fiber of Z_2. Therefore, in $\mathrm{WC}(A/K)$ we have an equality

$$[A^{(Z_1)}] - [A^{(Z_2)}] = [A^{(Z_1)}] + [A^{(-Z_2)}] = [A^{(Z_1 \boxplus (-Z_2))}] = [A].$$

$\square$

2.2. **Beyond geometric connectedness.** If the smooth parameter space T is connected but not geometrically connected, it seems unreasonable to expect that the theory of cycles will naturally induce a map from T to a (torsor under) an abelian variety (e.g., [ACMV, Cor. 1.5]). Nonetheless, we will see in this section that a cycle class $Z \in \mathrm{CH}^i(T \times X)$ naturally induces a morphism to a K-scheme which, after base change, is isomorphic to a disjoint union of copies of A.

Let $L \subseteq \overline{K}$ be a finite separable extension of K. (Note that $\overline{K}$ is also a separable closure of L.) Then $\phi : \mathrm{A}^i(X_{\overline{K}}) \to A(\overline{K})$ induces a regular homomorphism $\mathrm{A}^i((X_L)_{\overline{K}}) \to A_L(\overline{K})$ over L ([ACMV23, Lemma 2.2]).

Now suppose $\tau \in \mathrm{Gal}(K)$; let $L^\tau := \tau(L)$. If W is a scheme over L, set $W^\tau := \tau^* W$; it is naturally a scheme over L^τ.

Let B/L be an abelian variety, and let $\beta : \mathrm{Gal}(L) \to B(\overline{L})$ be a one-cocycle. Let β^τ be the one-cocycle $\mathrm{Gal}(L^\tau) \to B^\tau(\overline{L})$ which makes the following diagram commute:

$$\begin{array}{ccccccc} \sigma & \qquad & \mathrm{Gal}(L) & \xrightarrow{\beta} & B(\overline{K}) & \qquad & P \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \sigma' := \tau\sigma\tau^{-1} & & \mathrm{Gal}(L^\tau) & \xrightarrow{\beta^\tau} & B^\tau(\overline{K}) & & P^\tau \end{array}$$

Direct calculation shows that

$$(\beta^\tau)_{\sigma'} = (\beta_\sigma)^\tau;$$

analyzing the induced actions of $\mathrm{Gal}(L^\tau)$ on $B^\tau(\overline{K})$ then shows that there is an isomorphism

$$B[\beta^\tau] \cong B[\beta]^\tau \tag{2.2}$$

of torsors over L^τ under B^τ.

Lemma 2.3. *Suppose that T/L is a smooth geometrically connected scheme, and let $Z \in \mathrm{CH}^i(T \times_L (X_L))$. Let $t_0 \in T(\overline{K})$ be a $\overline{K}$-point. Then*

(a) $A^{(T^\tau, t_0^\tau, Z^\tau)} \cong (A^{(T,t_0,Z)})^\tau$ *and*
(b) $\psi_{(T^\tau, t_0^\tau, Z^\tau)} = (\psi_{T,t_0,Z})^\tau$.

Proof. The one-cocycle corresponding to the pulled-back data $(T^\tau, t_0^\tau, Z^\tau)$ is

$$\begin{aligned} \alpha(T^\tau, t_0^\tau, Z^\tau)_{\sigma'} &= \phi((Z^\tau)_{t_0^{\tau\sigma'}} - (Z^\tau)_{t_0^\tau}) \\ &= \phi((Z^\tau)_{t_0^{\tau\tau^{-1}\sigma\tau}} - (Z_{t_0})^\tau) \\ &= \phi((Z^\tau)_{t_0^{\sigma\tau}} - (Z_{t_0})^\tau) \\ &= \phi((Z_{t_0^\sigma} - Z_{t_0})^\tau) \\ &= \phi(Z_{t_0^\sigma} - Z_{t_0})^\tau \\ &= (\alpha(T, t_0, Z)_\sigma)^\tau. \end{aligned}$$

Therefore $\alpha(T^\tau, t_0^\tau, Z^\tau) = \alpha(T, t_0, Z)^\tau$, and (a) follows from (2.2).

For (b), it suffices to verify the commutativity of the diagram

$$\begin{array}{ccc} T & \xrightarrow{\psi_{(T,t_0,Z)}} & A \\ \downarrow{\tau^*} & & \downarrow{\tau^*} \\ T^\tau & \xrightarrow{\psi_{(T^\tau,t_0^\tau,Z^\tau)}} & A \end{array}$$

on $\overline{K}$-points. Since

$$\tau^*\psi_Z(t) = \tau^*\phi(Z_t - Z_{t_0}) = \phi(Z_t - Z_{t_0})^\tau$$

while

$$\psi_{Z^\tau}(\tau^* t) = \psi_{Z^\tau}(t^\tau) = \phi(Z^\tau_{t^\tau} - Z^\tau_{t_0^\tau}),$$

the claim (again) follows from the $\mathrm{Gal}(K)$-equivariance of ϕ. ☐

If W is a scheme, let $\Pi_0(W)$ denote its set of irreducible components.

Proposition 2.4. *Suppose T/K is a smooth connected scheme, and let $Z \in \mathrm{CH}^i(T \times X)$. Let $t_0 \in T(\overline{K})$ be a $\overline{K}$-point, and let $L \subseteq \overline{K}$ be a finite separable extension of K such that each irreducible component of T_L is geometrically connected. Then*

(a) There is a canonical L/K descent datum on

$$\coprod_{V \in \Pi_0(T_L)} A^{(V, t_0|_V, Z|_V)}; \tag{2.3}$$

let $A^{(T,t_0,Z)}$ be the corresponding model over K.

(b) The L-morphism

$$T_L = \coprod_{V \in \Pi_0(T_L)} V \xrightarrow{\psi_{(V,t_0|_V,Z|_V)}} \coprod_{V \in \Pi_0(T_L)} A^{(V,t_0|_V,Z|_V)} \tag{2.4}$$

descends to a K-morphism

$$\psi_{(T,t_0,Z)} : T \longrightarrow A^{(T,t_0,Z)}.$$

Proof. Since T is irreducible, the $\overline{K}$-point t_0 of T pulls back to a $\overline{K}$-point on each component of T_L. Fix some irreducible component W of T_L, and let $w_0 = t_0|_W$ be the pullback of t_0 to W. Similarly, let $Y \in \mathrm{CH}^i(W \times X_L)$ be the restriction of Z to $W \times X_L$.

The Galois group $\mathrm{Gal}(K)$ acts transitively on the set of irreducible components of T_L. Let H be the stabilizer of W in $\mathrm{Gal}(K)$, and let $R \subseteq \mathrm{Gal}(K)$ be a system of representatives for the set of cosets $H\backslash \mathrm{Gal}(K)$. Then

$$\Pi_0(T_L) = \{W^\tau : \tau \in R\},$$

and by Theorem 2.3(a), the object of (2.3) becomes

$$\coprod_{\tau \in R} A^{(W^\tau, w_0^\tau, Y^\tau)} \cong \coprod_{\tau \in R} (A^{(W,w_0,Y)})^\tau,$$

which descends to K. Call the resulting scheme $A^{(T,t_0,Z)}$.

Similarly, let $\psi = \psi_{(W,w_0,Y)}$. Thanks to Theorem 2.3(b), the morphism (2.4) is actually

$$\textstyle\coprod_{\tau \in R} W^\tau \xrightarrow{\psi^\tau} (A^{(W,w_0,Y)})^\tau,$$

which visibly descends to K. ☐

2.3. **Symmetric products.** If C/K is a smooth projective curve, then there is a well-known Abel map $\mathrm{Sym}^{(d)} C \to \mathrm{Pic}^d_{C/K}$. We will see (Theorem 2.6) that this is a special case of the following observation.

Proposition 2.5. *Let T/K be a smooth geometrically connected scheme, and suppose $Z \in \mathrm{CH}^i(T \times X)$. Let $Z^{\boxplus d} = Z \boxplus Z \boxplus \cdots \boxplus Z \in \mathrm{CH}^i(T^{\times d} \times X)$. Then*

- *(a) the isomorphism class of the torsor $A^{(T^{\times d}, t_0^{\times d}, Z^{\boxplus d})}$ is equal to the d-fold sum $d[A^{(T,t_0,Z)}] = [A^{(T,t_0,Z)}] + \cdots + [A^{(T,t_0,Z)}]$; and*
- *(b) $\psi_{Z^{\boxplus d}}$ factors through the symmetric product $\mathrm{Sym}^{(d)}(T)$. In other words, there is a diagram of K-schemes*

$$\begin{array}{ccc} T^{\times d} = T \times \cdots \times T & \xrightarrow{\psi_{Z^{\boxplus d}}} & A^{(T^{\times d}, t_0^{\times d}, Z^{\boxplus d})} \\ & {\scriptstyle s_d} \searrow \quad \nearrow {\scriptstyle \psi_Z^{(d)}} & \\ & \mathrm{Sym}^{(d)}(T) & \end{array}$$

Proof. Part (a) is a special case of Theorem 2.1(c). For part (b), after base change to $\overline{K}$ it is clear that $\psi^{(d)}_{\overline{K}}$ factors through some morphism $\psi^{(d)} : \mathrm{Sym}^{(d)}(X)_{\overline{K}} \to A_{\overline{K}}$. Since s_d is surjective on $\overline{K}$-points, and since s_d and $\psi_{Z^{\boxplus d}}$ are $\mathrm{Gal}(K)$-equivariant on $\overline{K}$-points, it follows that $\psi^{(d)}$ is, too, and thus descends to K. □

As an example, now suppose that X/K has dimension n. Then X admits an algebraic representative in codimension n, namely, the *Albanese variety* $\mathrm{Alb}_{X/K} = \mathrm{Ab}^n_{X/K}$. The diagonal cycle $\Delta_X \in \mathrm{CH}^n(X \times X)$ determines a torsor $\mathrm{Alb}^{(X,\Delta_X)}_{X/K}$ over K under $\mathrm{Alb}_{X/K}$, which is usually denoted $\mathrm{Alb}^{(1)}_{X/K}$, and the theory of regular homomorphisms gives a canonical morphism

$$\psi_{\Delta_X} : X \longrightarrow \mathrm{Alb}^{(1)}_{X/K} := \mathrm{Alb}^{(X,\Delta_X)}_{X/K}.$$

Let $\Delta_X^{(d)}$ denote the d-fold sum $\Delta_X \boxplus \ldots \boxplus \Delta_X$. Then the isomorphism class of $\mathrm{Alb}^{(d)}_{X/K} := \mathrm{Alb}^{\Delta_X^{(d)}}_{X/K}$ is the same as the d-fold sum $[\mathrm{Alb}^{(1)}_{X/K}] + \cdots + [\mathrm{Alb}^{(1)}_{X/K}]$ (Theorem 2.1), and in this special case Proposition 2.5 is the assertion:

Corollary 2.6. *The morphism $\psi_{\Delta_X^{(d)}}$ factors through the symmetric product; there is a diagram of K-varieties*

$$\begin{array}{ccc} X \times \cdots \times X & \xrightarrow{\psi_{\Delta_X^{(d)}}} & \mathrm{Alb}^{(d)}_{X/K} \\ & {\scriptstyle s_d} \searrow \quad \nearrow {\scriptstyle \psi^{(d)}} & \\ & \mathrm{Sym}^{(d)}(X) & \end{array} \tag{2.5}$$

If X is a curve, then ψ_{Δ_X} is the Abel map, $\mathrm{Alb}^{(1)}_{X/K} \cong \mathrm{Pic}^1_{X/K}$, and more generally $\mathrm{Alb}^{(d)}_{X/K} \cong \mathrm{Pic}^d_{X/K}$. Theorem 2.6 also appears as (the existence part of) [HT21, Cor. 3.2]; in contrast to the argument given there, which relies on the universal property of Albanese varieties, our proof is a simple consequence of the theory of regular homomorphisms.

3. Application to rationality and cycles

3.1. **A framework for an obstruction to the existence of cycles defined over K.** It is well-known that for a smooth projective curve C over a field K, the schemes $\mathrm{Pic}^d_{C/K}$ provide obstructions to the existence of 0-cycles of degree d. More precisely, consider the Abel map

$$\alpha^{(d)} := \psi^{(d)} : C^{(d)} \longrightarrow \mathrm{Pic}^d_{C/K}$$

reviewed in §2.3. After base change to the separable closure, this is given by $D \mapsto \mathcal{O}_{C_{\overline{K}}}(D)$. The scheme $\mathrm{Pic}^d_{C/K}$ is a torsor under $\mathrm{Pic}^0_{C/K}$, and its class $[\mathrm{Pic}^d_{C/K}]$ in the Weil–Châtelet group $\mathrm{WC}(\mathrm{Pic}^0_{C/K})$ provides an obstruction to the existence of an effective 0-cycle of degree d on C, defined over K. Indeed, the existence of such a cycle implies that $\mathrm{Pic}^d_{C/K}$ admits a K-point (namely, the image under $\alpha^{(d)}$ of the K-point in $\mathrm{Sym}^{(d)} C$ corresponding to the 0-cycle), which in turn implies that $[\mathrm{Pic}^d_{C/K}] = [\mathrm{Pic}^0_{C/K}] = 0 \in \mathrm{WC}(\mathrm{Pic}^0_{C/K})$. Taking a broader view, the subgroup $[\mathrm{Pic}_{C/K}]$ of $\mathrm{WC}(\mathrm{Pic}^0_{C/K})$ generated by the isomorphism classes of all the torsors $\mathrm{Pic}^d_{C/K}$ provides a group that is an obstruction to the existence of 0-cycles on C defined over K.

Our Theorem A allows one to construct similar obstructions for cycles of any dimension on any smooth projective variety over K. For instance, given a regular homomorphism $\phi\colon \mathrm{A}^i(X_{\overline{K}}) \to A(\overline{K})$ over K and any family of cycle classes $Z \in \mathrm{CH}^1(T \times X)$ parameterized by a smooth variety T over K, the class $[A^{(Z)}]$ in $\mathrm{WC}(A)$, if nontrivial, is an obstruction to the existence of a K-point of T. Moreover, if the K-morphism $T \to A^{(Z)}$ from Theorem A is an isomorphism, then T has a K-point if and only if $[A^{(Z)}] = 0$ in $\mathrm{WC}(A/K)$. This is the obstruction used in the applications below.

Bearing this in mind, consider now the algebraic representative $\phi : \mathrm{A}^i(X_{\overline{K}}) \to \mathrm{Ab}^i_{X/K}(\overline{K})$, if it exists (e.g., $i = 1, 2, \dim X$). The subgroup of $\mathrm{WC}(\mathrm{Ab}^i_{X/K})$ generated by the isomorphism classes of all the torsors under $\mathrm{Ab}^i_{X/K}$ (as in the first paragraph) provides a group that can be viewed as an obstruction to the existence of certain cycles on X defined over K.

3.2. **Applications to work of Hassett–Tschinkel and Benoist–Wittenberg.** Torsors under intermediate Jacobians (and algebraic representatives) were used by Hassett and Tschinkel, and by Benoist and Wittenberg, to study rationality of certain geometrically rational threefolds, namely smooth complete intersections of two quadrics in $\mathbb{P}^5_K$.

In the former case, Hassett and Tschinkel work over subfields of $\mathbb{C}$, and phrase their arguments in terms of torsors under the distinguished model of an intermediate Jacobian. In the latter case, Benoist and Wittenberg work over an arbitrary field. As we recalled in the introduction, in the special case of a smooth geometrically rational threefold X, Benoist and Wittenberg are able to construct a codimension-2 Chow scheme $\mathbf{CH}^2_{X/K}$; its geometrically irreducible components are then naturally torsors under the connected component of identity which is an abelian variety, isomorphic to $\mathrm{Ab}^2_{X/K}$ if K is perfect, and this is the torsor structure they use. In §3.1, above, we describe an alternative approach that works without any hypotheses on the threefold X.

The analysis of torsors given here is actually sufficient to recover, in the case where K is perfect, a beautiful rationality application uncovered in [BW23, HT21].

Theorem 3.1 (Benoist–Wittenberg, Hassett–Tschinkel)**.** *Assume K is perfect, and let $X \subseteq \mathbb{P}^5_K$ be a smooth complete intersection of two quadrics. Then X is rational if and only if it contains a K-rational line.*

Proof. If X contains a line defined over K, then projection from the line shows that X is rational over K. The converse is proven in [HT21, Thm. 6.5] if $K \subseteq \mathbb{C}$, and in [BW23, Thm. 4.7] over an arbitrary (i.e., not necessarily perfect) field K.

We recall the argument here, in the context of our results in this paper. Let $X \subseteq \mathbb{P}^5_K$ be any smooth complete intersection of two quadrics over a field K, and let F be the Fano variety of lines on X; it is a smooth surface (see e.g., [BW23, Lem. 4.1]). Let $\mathrm{Ab}^2_{X/K}$ be the algebraic representative for codimension-2 cycles over K. From Theorem A, the universal line over the Fano variety F determines a K-morphism $F \to P_{F/K}$ to a torsor $P_{F/K}$ under $\mathrm{Ab}^2_{X/K}$. The theorem will therefore be proved if we show that, provided X is K-rational, then (1) $[P_{F/K}] = 0$ in $\mathrm{WC}(\mathrm{Ab}^2_{X/K})$, and (2) the natural map $F \to P_{F/K}$ is an isomorphism.

In fact, the arguments in the proof of [BW23, Thm. 4.5(ii)] show that for any smooth complete intersection of two quadrics X (regardless of whether X is rational), the natural map $F \to P_{F/K}$ is an isomorphism, establishing (2) above. Indeed, it suffices to show that this is an isomorphism after base change to the separable closure, which is a classical result outside of characteristic 2 (we direct the reader to [BW23, Thm. 4.5(ii)] for references and details). Therefore, all that remains to show is (1), above, that $[P_{F/K}] = 0$.

We now recall the geometry used to show this, assuming K is perfect. For now, again, assume only that X is a smooth complete intersection of two quadrics (i.e., with no assumption on the rationality of X). From [BW23, Thm. 4.5(iv)], which goes back to Wang [Wan18] in characteristic not 2, together with [BW23, Thm. 3.1(vi)] there is a smooth projective geometrically integral curve D over K of genus 2 and an isomorphism of principally polarized abelian varieties

$$\mathrm{Ab}^2_{X/K} \cong \mathrm{Pic}^0_{D/K}\,. \tag{3.1}$$

We recall that under certain hypotheses on a threefold X defined over a *perfect* field, there is a canonically defined polarization on $\mathrm{Ab}^2_{X/K}$, and this is the polarization mentioned above. In the case at hand, this is [BW23, Thm. 3.1]. Alternatively, by invoking less about the geometry of X (i.e., the existence of a decomposition of the diagonal) but at the cost of acknowledging liftability, one can use [ACMV25, §§4-5]; since X is a complete intersection, and therefore liftable to characteristic 0, the symmetric isogeny constructed in [ACMV25] is a polarization (see [ACMV25, §13.2(3)]).

The curve D can be described geometrically as follows. As X is geometrically rational, Bloch–Srinivas [BS83, Thm. 1] implies that the algebraic representative gives an isomorphism $\mathrm{A}^2(X_{\overline{K}}) \cong \mathrm{Ab}^2_{X/K}(\overline{K})$; the classes in $\mathrm{A}^2(X_{\overline{K}})$ that are represented by conics on $X_{\overline{K}}$ translated by a fixed conic, determine a curve in $(\mathrm{Ab}^2_{X/K})_{\overline{K}} = \mathrm{Ab}^2_{X_{\overline{K}}/\overline{K}}$. In fact, this canonically determines a curve in the base change to $\overline{K}$ of another torsor P' under $\mathrm{Ab}^2_{X/K}$, corresponding to components of Chow parameterizing such cycles (without the choice of a fixed conic; see §3.1 for how this works in general). The main result is that this curve descends to a curve D contained in P'.

In addition, it is shown in [BW23, Thm. 4.5(iv)] that

$$2[P_{F/K}] = [\mathrm{Pic}^1_{D/K}] \tag{3.2}$$

in $\mathrm{WC}(\mathrm{Ab}^2_{X/K}) = \mathrm{WC}(\mathrm{Pic}^0_{D/K})$. Specifically, one has $[P'] = [\mathrm{Pic}^1_{D/K}]$; note that with the canonical factorization of the inclusion $D \subseteq P'$ through the Albanese torsor, it suffices to show the natural morphism $\mathrm{Pic}^1_{D/K} \to P'$ is an isomorphism after base change to the separable closure, which is a classical result outside of characteristic 2; indeed, it is equivalent to the isomorphism (3.1). The identification $2[P_{F/K}] = [P']$ comes from Theorem A(b), as $P_{F/K}$ is the torsor associated with lines, and P' is the torsor associated with conics.

Now assume that X is K-rational. Then, by [BW23, Thm. 3.11(iii)], there is some degree d such that

$$[P_{F/K}] = [\mathrm{Pic}^d_{D/K}]. \tag{3.3}$$

Here we mention that the identification above requires the isomorphism of *polarized* abelian varieties (3.1) in an important way; without the isomorphism of polarized abelian varieties one can only conclude that there is some smooth projective geometrically integral curve C/K of genus 2 and some degree d' so that $[P_{F/K}] = [\mathrm{Pic}^{d'}_{C/K}]$. To conclude that one may take $C = D$, one uses the canonical polarization on $\mathrm{Ab}^2_{X/K}$ in the isomorphism (3.1) and then invokes the Torelli theorem for curves.

Subtracting (3.3) from (3.2) in the Weil–Chatelet group gives the additional description:

$$[P_{F/K}] = [\mathrm{Pic}^1_{D/K}] - [\mathrm{Pic}^d_{D/K}] = [\mathrm{Pic}^{1-d}_{D/K}].$$

Since D has genus 2, it has a degree-2 zero-cycle defined over K, namely its canonical divisor, and so we have that $[\mathrm{Pic}^{2n}_{D/K}] = 0$ for any integer n. As one of d and $1-d$ is even, we must have that $[P_{F/K}] = 0$, and we are done. □

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