

# PUTTING THE P BACK IN PRYM

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ABSTRACT. After Jacobians of curves, Prym varieties are perhaps the next most studied abelian varieties. They turn out to be quite useful in a number of contexts. For technical reasons, there does not appear to be any systematic treatment of Prym varieties in characteristic 2, and due to our recent interest in this topic, the purpose of this paper is to fill in that gap. Our main result is a classification of branched covers of curves in characteristic 2 that give rise to Prym varieties. We are also interested in the case of Prym varieties in the relative setting, and so we develop that theory here as well, including an extension of Welters' Criterion.

## INTRODUCTION

Consider an étale double cover of smooth complex projective curves  $f: C \rightarrow C'$ . It turns out that the principal polarization of  $\text{Pic}^0(C)$  defines a canonical principal polarization on the Prym variety  $P(C/C')$ , a sub-abelian variety of  $\text{Pic}^0(C)$  complementary to  $f^* \text{Pic}^0(C')$ . This phenomenon, discovered in the late 19th century by complex geometers, was recast in purely algebro-geometric language by Mumford in [Mum74]; see [Far12] for a social and mathematical history of these ideas.

After Jacobians of curves themselves, Pryms are one of the most accessible classes of principally polarized abelian varieties. It is difficult to improve on the assessment by Welters [Wel87]: *Jacobi varieties are certainly the best understood principally polarized abelian varieties (PPAV). In this sense they are followed immediately by Prym varieties. Beyond these two classes, however, darkness predominates generally in the geometry of PPAV.* In particular, for small values of  $g$ , the Prym construction lets one understand the geometry  $\mathcal{A}_g$  in terms of curves. The Torelli map identifies  $\mathcal{M}_g$  with an open subspace of  $\mathcal{A}_g$  only when  $g \leq 3$ ; but for  $g \in \{4, 5\}$ , one at least knows that a dense subset of principally polarized abelian varieties are Prym varieties. For details of this and similar geometric insights afforded by Pryms, we refer the reader to [Bea89] and [Far12].

On one hand, the literature is rich with arithmetic applications of the theory of Prym varieties. For example, in [Bea77a], Beauville constructs a moduli stack of Prym covers over  $\mathbb{Z}[1/2]$ , together with a so-called Prym map to the moduli stack of principally polarized abelian schemes, and uses Prym varieties in [Bea77b] to study the Chow groups and rationality of fibrations in quadrics over algebraically closed fields whose characteristic is not two. Faber and van der Geer [FvdG04] build on Beauville's work to analyze complete families of curves, again away from characteristic two. Over finite fields of odd characteristic, point counts sometimes are able to distinguish Prym varieties from arbitrary abelian varieties (e.g., [Per06]).

On the other hand, while it is natural, of course, to hope to study these phenomena in characteristic two as well, the foundational literature is notably quiet on the question of constructing Pryms in general contexts. For example, Mumford's original algebraic treatment [Mum74] works over an algebraically closed field  $k$  of characteristic other than two. Beauville works in the same

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setting in [Bea77b] and [Bea77a], although in the former, he does allude to results over an algebraically closed field of characteristic 2. Tyurin [Tju72] retreats to the case where  $k = \mathbb{C}$ , while Welters [Wel87] allows an arbitrary algebraically closed field of characteristic zero. The most self-contained exposition the authors are aware of, namely the treatment by Birkenhake and Lange [BL04], is resolutely over the complex numbers.

Speaking from personal experience, it seems to the authors that the further the base field is from the complex numbers (for example, if the field is not algebraically closed, or not even perfect, or has characteristic two), the less certain one might be about the status of certain constructions and assertions regarding Prym varieties. Our intention in this paper is to develop a comprehensive theory of Prym, and Prym–Tyurin, varieties that works over an arbitrary base. Even in the special case where the base scheme is  $S = \text{Spec } \overline{\mathbb{F}}_2$  our results, while perhaps not surprising, do not appear in the literature.

For better or worse, our main result shows that there is nothing new under the sun, even if the landscape is a scheme  $S$  on which 2 is not invertible.

**Theorem (A).** *Let  $S$  be a connected scheme. Let  $f : C \rightarrow C'$  be a finite  $S$ -morphism of degree  $d$  of smooth proper curves over  $S$  with geometrically connected fibers of respective genera  $g > g' \geq 1$ , and suppose that  $f$  is fiberwise separable.*

- (1) *There exists a canonical complementary abelian scheme  $P(C/C')$  for  $f^* \text{Pic}_{C'/S}^0$  in  $\text{Pic}_{C/S}^0$  of relative dimension  $h = g - g'$  over  $S$ .*
- (2) *The principal polarization  $\Theta_{C/S}$  of  $\text{Pic}_{C/S}^0$  restricts to a multiple  $e \cdot \Xi$  of a principal polarization  $\Xi$  of  $P(C/C')$  if and only if, for some geometric point  $s$  of  $S$ , the morphism  $f_s$  has one of the following types:*
  - (a)  *$d = 2$  and  $f_s$  is étale, in which case  $e = 2$  and  $h = g' - 1$ ;*
  - (b)  *$d = 2$  and the ramification divisor of  $f_s$  has degree 2, in which case  $e = 2$  and  $h = g'$ ;*
  - (c)  *$d = 3$ ,  $f_s$  is étale and noncyclic, and  $g' = 2$ , in which case  $e = 3$  and  $h = 2$ ;*
  - (d)  *$g = 2$  and  $g' = 1$ , in which case  $e = \deg f_s / \deg f_s^{\text{abn}}$  and  $h = g' = 1$ , where  $f_s^{\text{abn}}$  is the maximal abelian subcover of  $f_s$  (see (5.4) and Lemma 5.10).*

This is proven in Theorem 5.12 and Theorem 6.4. The abelian scheme  $P(C/C')$  of part (1) is constructed as follows. The morphism  $f : C \rightarrow C'$  induces a morphism of abelian schemes  $f^* : \text{Pic}_{C'/S}^0 \rightarrow \text{Pic}_{C/S}^0$ , whose image  $Y$  is an abelian scheme. The principal polarization of  $\text{Pic}_{C'/S}^0$  restricts to a polarization on  $Y$ , whose exponent we denote by  $e$  (§1.1.1). Using the polarization on  $\text{Pic}_{C'/S}^0$  we define a norm endomorphism  $N_Y : \text{Pic}_{C/S}^0 \rightarrow \text{Pic}_{C'/S}^0$  with image  $Y$  (§2.1), and define  $P(C/C')$  to be the image  $\text{Im}([e]_X - N_Y)$ . In this context, we say that  $P(C/C')$  is a Prym scheme of exponent  $e$  (embedded in  $\text{Pic}_{C/S}^0$ ). In the situation of part (2), when the induced polarization on  $P(C/C')$  is the  $e^{\text{th}}$  multiple of a principal polarization  $\xi$ , we call the principally polarized abelian scheme  $(P(C/C'), \xi)$  a Prym–Tyurin Prym scheme of exponent  $e$ . In much of the literature, as in the beginning of this introduction, the phrase “Prym variety” often refers to case 2(a).

We also recover an arithmetic version of a criterion of Welters; here we work over a field  $K$ .

**Theorem (B).** *Let  $C$  be a smooth pointed proper curve over a field  $K$ , which we take to be embedded via the Abel–Jacobi map in its principally polarized Jacobian  $(\text{Pic}_{C/K}^0, \Theta_{C/K})$ , let  $Z \hookrightarrow \text{Pic}_{C/K}^0$  be a sub-abelian variety of dimension  $g_Z$  with principal polarization  $\Xi$ , and let  $\beta$  be the canonical composition*

$$\beta : C \longrightarrow \text{Pic}_{C/K}^0 \xrightarrow{\sim} \widehat{\text{Pic}_{C/K}^0} \longrightarrow \widehat{Z} \xrightarrow{\sim} Z.$$

*Then  $\Theta_{C/K}|_Z = e \cdot \Xi$  if and only if there is a numerical (or equivalently, homological) equivalence*

$$\beta_* [C_{\overline{K}}] \equiv e \frac{[\Xi]^{g_Z - 1}}{(g_Z - 1)!}$$

of 1-cycles on  $Z_{\bar{K}}$ .

In fact, our result does not require  $C$  to be pointed, and is actually phrased in terms of a morphism  $\beta : C \rightarrow T$  to a torsor under  $Z$ ; see Theorem 5.3 for the complete statement.

The reader simply curious about what sorts of Prym–Tyurin and Prym constructions are valid over arbitrary fields is invited to directly turn to the results of §5. Here is how we work up to these statements. In §1, we start by collecting some basic facts on abelian schemes, their polarizations and their endomorphisms.

In §2, we define the norm endomorphism mentioned above, and use this to get a general theory of complements for polarized abelian schemes. Our treatment here follows [BL04, Ch. 5], except that at each stage we make sure our constructions are valid over in the relative setting.

In §3, we collect some results on the Abel map  $\alpha_{C/S} : C \rightarrow \text{Pic}_{C/S}^{(1)}$  associated to a smooth proper curve, and the compatibility of  $\alpha_{C/S}$  and polarizations under finite morphisms  $C \rightarrow C'$ . Since we work in a setting where  $C$  need not admit a section,  $\text{Pic}_{C/S}^{(1)}$  might be a nontrivial torsor under  $\text{Pic}_{C/S}^0$ , and so our results are necessarily expressed in terms of maps between *torsors* under Jacobian varieties, rather than the Jacobians themselves.

In §5, we follow the development in [BL04, Ch. 12] of the theory of Prym–Tyurin and Prym varieties, as well as Welters’ Criterion. One of the tools used in *op. cit.* is a construction due to Matsusaka [Mat59] which, given cycles  $\alpha$  and  $\beta$  of complementary dimension on an abelian variety  $X$ , produces an endomorphism  $\delta(\alpha, \beta) \in \text{End}(X)$ . However, Matsusaka’s work is built upon Weil’s foundations for algebraic geometry, which now seem ill-suited to analyzing objects over a field that is not algebraically closed. Consequently, we take a detour in §4 to recast this work using a modern reformulation of Samuel’s notion of a regular homomorphism ([ACMV23b]). In §6, we consider the existence of Prym–Tyurin and Prym schemes relative to an arbitrary base  $S$ . Finally, some sample applications are sketched out in §7.

After we distributed a first draft of this paper, Tudor Ciuca kindly shared with us his recent work on Prym varieties and cubic threefolds over  $\mathbb{Z}$  [Ciu24]. As part of this work Ciuca gives a proof – completely independent from ours – that if  $C \rightarrow C'$  is an étale cover of curves over an arbitrary field  $K$ , then the induced morphism on Jacobians  $\text{Pic}_{C/K}^0 \rightarrow \text{Pic}_{C'/K}^0$  is smooth; he uses this to deduce the existence of a Prym variety  $P(C/C')$  for an étale double cover of curves in characteristic two. We encourage the interested reader to consult Ciuca’s work directly.

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## 1. PRELIMINARIES ON ABELIAN SCHEMES

In this section we review some standard results about abelian schemes. The main purpose is to fix notation, as well as to collect some results for which we could not find a reference.

We follow the conventions in [MFK94, Ch. 6]. In particular, we work over a connected locally Noetherian scheme  $S$ . An *abelian scheme* over  $S$ , or an *abelian  $S$ -scheme*, is a smooth proper group scheme  $X \rightarrow S$  with geometrically connected fibers. Recall that a smooth proper group scheme  $X \rightarrow S$  with connected fibers has geometrically connected fibers [FC90, Rem. 1.2(c)], and so is an abelian scheme. We refer to abelian schemes over fields as abelian varieties, and recall that every abelian variety is projective; in fact, more generally, if  $S$  is geometrically unibranch (e.g., normal), then every abelian scheme over  $S$  is projective over  $S$  (see [FC90, p.3, (c)]). On the other hand, we recall that, for instance, an abelian scheme over an Artinian ring need not be projective [Ray70, XII.4, p.189].

A *morphism of abelian schemes* over  $S$  is a morphism  $X \rightarrow Y$  of  $S$ -schemes which carries the identity section of  $X$  to that of  $Y$ . It follows from rigidity [MFK94, Prop. 6.1] that such a morphism

is automatically a homomorphism of group schemes [MFK94, Cor. 6.4]. Recall that given a morphism  $f: X \rightarrow Y$  of abelian schemes over  $S$ , the *kernel*  $\ker(f)$  is defined to be the pull-back of  $f$  along the 0-section of  $Y$ . This is naturally a sub- $S$ -group scheme of  $X$ , and by construction, its formation is compatible with base change.

For an abelian scheme  $X$  over  $S$  and an integer  $e$ , we denote by  $[e]_X: X \rightarrow X$  the multiplication by  $e$  map, and by  $X[e]$  its kernel. We denote by  $\widehat{X} := \text{Pic}_{X/S}^0$  the dual abelian  $S$ -scheme ([MFK94, Cor. 6.8] when  $X/S$  is projective, and [FC90, p.3] in general), and for an  $S$ -homomorphism  $f: X \rightarrow Y$  of abelian  $S$  schemes, we denote by  $\widehat{f}: \widehat{Y} \rightarrow \widehat{X}$  the  $S$ -homomorphism induced by pull-back of line bundles. We denote by  $\pi: X \rightarrow S$  the structure map to  $S$ ,  $\mu: X \times_S X \rightarrow X$  the group law,  $p_i: X \times_S X \rightarrow X$ ,  $i = 1, 2$  the projections, and by  $\epsilon: S \rightarrow X$  the zero section.

**1.1. Polarizations of abelian schemes.** Given an abelian scheme  $X$  over  $S$  and a line bundle  $L$  on  $X$ , one obtains a morphism

$$\phi_L: X \longrightarrow \widehat{X}, \quad (1.1)$$

whose definition we recall on points. Given a morphism  $a: T \rightarrow X$  of  $S$ -schemes, i.e., an element  $a \in X(T)$ , denote by  $\pi_T: X_T = T \times_S X \rightarrow T$  the pull-back of  $X/S$  along  $T \rightarrow S$ , by  $L_T$  the pull-back of  $L$  along  $X_T \rightarrow X$ , and we denote the translation by  $a$  map by

$$t_a: X_T = T \times_T X_T \xrightarrow{a \times \text{Id}} X_T \times_T X_T \xrightarrow{\mu_T} X_T.$$

Then we have

$$\phi_L(T): X(T) \longrightarrow \widehat{X}(T) \quad (1.2)$$

$$a \longmapsto t_a^* L_T \otimes L_T^{-1} \otimes \pi_T^* a^* L_T^{-1} \otimes \pi_T^* \epsilon_T^* L_T.$$

By construction, the formation of  $\phi_L$  is compatible with base change. In the usual way, the morphism of schemes  $\phi_L$  in (1.1) is identified with  $\phi_L(X)(\text{Id}_X)$  in (1.2) and, viewing  $\widehat{X}$  as a moduli space of line bundles on  $X$ , we have that  $\phi_L$  is the morphism associated to the line bundle

$$\mu^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1},$$

viewing this line bundle on  $X \times_S X$  as a family of line bundles on  $X$  parameterized by the first copy of  $X$ . Moreover, if  $S$  is the spectrum of an algebraically closed field  $k$ , then for  $a \in X(k)$ , one has  $\phi_L(a) = t_a^* L \otimes L^{-1}$ . Note that, *a priori*, the definition of  $\phi_L(T)$  above only gives a map to  $\text{Pic}_{X/S}(T)$ ; to check that the image is in  $\text{Pic}_{X/S}^0(T) = \widehat{X}(T)$  it suffices to check on geometric fibers, and this is standard (e.g., [Mum08, p.79]).

We use the notation

$$K(L) := \ker \phi_L.$$

We have  $K(L) = X$  if and only if  $L \in \widehat{X}(S)$ , i.e., the map  $S \rightarrow \text{Pic}_{X/S}$  induced by  $L$  has image in the connected component of the identity  $\widehat{X} = \text{Pic}_{X/S}^0$ . Again, it suffices to check this on geometric fibers, which is standard (e.g., [Mum08, Def. p.74 and Diag. p.80]). At the opposite extreme, if  $\pi_* L \neq 0$  then  $K(L)$  is finite over  $S$  if and only if  $L$  is ample; since  $\phi_L$  is stable under base change by construction, and  $X/S$  is proper, it suffices to show that on the geometric fibers  $X_s$  we have that  $H^0(X_s, L_s) \neq 0$  and  $K(L_s)$  is quasi-finite if and only if  $L_s$  is ample, which is a standard result (e.g., [Mum08, p.60, Appl. 1]).

Given an abelian scheme  $X$  over  $S$ , a *polarization* on  $X$  is a homomorphism

$$\lambda: X \longrightarrow \widehat{X}$$

such that for every geometric point  $s$  of  $S$ , there is an ample line bundle  $L_s$  on  $X_s$  such that the morphism  $\lambda_s = \phi_{L_s}: X_s \rightarrow \widehat{X}_s$ . This is equivalent to the existence of an étale cover  $S' \rightarrow S$  such

that there exists a relatively ample line bundle  $L'$  on  $X_{S'} = S' \times_S X$  such that  $\lambda_{S'} = \phi_{L'} : X_{S'} \rightarrow \widehat{X}_{S'}$  [FC90, p.4]. Note in particular that: (1) every polarized abelian scheme is étale locally projective, (2) if  $X$  is projective and  $L$  is relatively ample, then  $\phi_L$  is a polarization, and (3) every polarization on an abelian variety  $X$  over an algebraically closed field is of the form  $\phi_L$  for an ample line bundle  $L$  on  $X$ . Finally, given a polarization  $\lambda$ , there is a moduli space parameterizing bundles  $L$  on  $X$  for which  $\phi_L = \lambda$ ; this admits a coarse moduli space which, if nonempty, defines a closed subscheme of  $\text{Pic}_{X/S}$  that is a torsor under  $\widehat{X}$  [FC90, p.4].

A polarization is necessarily finite and surjective [MFK94, p.120], as well as faithfully flat, being a surjective morphism of abelian schemes [MFK94, Lem. 6.12]. Note that this implies that  $\ker \lambda$  is a finite flat commutative group scheme over  $S$ . A polarization is called principal if it is an isomorphism. We say that a line bundle  $L$  (resp. Cartier divisor  $D$ ) on  $X$  that is ample over  $S$  induces the polarization  $\lambda$  if  $\lambda = \phi_L$  (resp.  $\lambda = \phi_{\mathcal{O}_X(D)}$ ).

1.1.1. *Degree, exponent, and type of a polarization.* A polarized abelian scheme  $(X, \lambda)$  over  $S$  has an associated *degree*  $d$ . Recall that this is defined so that the degree of the finite flat morphism  $\lambda$  is  $d^2$ , or equivalently, that the order of the finite flat commutative group scheme  $\ker \lambda$  over  $S$  is  $d^2$ , and that after an étale cover  $S' \rightarrow S$  such that  $\lambda = \phi_L$  for some ample line bundle  $L$  on  $X_{S'}$ , one has that  $\pi_{S'}^* L$  is a vector bundle of rank  $d$  on  $S$ , and the finite flat group scheme  $K(L)$  has order  $d^2$  over  $S'$  [FC90, p.4], [Mum08, p.150]. The degree of a polarization is stable under base change. We denote by  $\mathcal{A}_{g,d}$  the separated algebraic stack of finite type over  $\text{Spec } \mathbb{Z}$  whose objects over  $S$  are polarized abelian schemes  $(A, \lambda)$  of dimension  $g$  and degree  $d$  over  $S$  (see e.g., [dJ93, p.486]).

For a finite flat commutative group scheme  $G$  of order  $n$  over  $S$ , one has that  $G$  is contained in  $G[n]$ , the kernel of the multiplication by  $n$  map on  $G$  [TO70, Thm. (Deligne), p.4]. One defines the *exponent* of  $G$  to be the smallest positive integer  $e$  such that  $G \subseteq G[e]$ . The exponent of a polarized abelian scheme  $(X, \lambda)$  is defined to be the exponent of the finite flat commutative group scheme  $\ker(\lambda)$ . Since the multiplication map is proper, and therefore closed, by considering generic fibers it is easy to see that the exponent is lower semi-continuous on fibers, and stable under faithfully flat base change.

In order to describe the behavior of the exponent under base change in more detail, it is helpful to discuss the type of a polarization. We refer to [dJ93] for references. First recall that for a polarized abelian scheme  $(A, \lambda)$  of dimension  $g$  and degree  $d$  over  $S$  there is a nondegenerate alternating pairing (e.g., [FC90, p.5], [dJ93, p.487]):

$$e^\lambda : \ker(\lambda) \times_S \ker(\lambda) \longrightarrow \mathbf{G}_{m,S}.$$

For every sequence of positive integers  $\delta = (\delta_1, \dots, \delta_g)$  with  $\delta_1 \mid \delta_2 \cdots \mid \delta_g$  and  $\prod_{i=1}^g \delta_i = d$ , define a scheme over  $\text{Spec } \mathbb{Z}$ :

$$K(\delta) := \prod_{i=1}^g \mu_{\delta_i, \mathbb{Z}} \times_{\mathbb{Z}} (\mathbb{Z} / \delta_i \mathbb{Z})$$

and the nondegenerate alternating pairing

$$e_\delta : K(\delta) \times_{\mathbb{Z}} K(\delta) \longrightarrow \mathbf{G}_{m, \mathbb{Z}}$$

$$((\zeta_i, n_i)_{i=1, \dots, g}, (\eta_i, m_i)_{i=1, \dots, g}) \mapsto \prod_{i=1}^g \zeta_i^{m_i} \cdot \eta_i^{-n_i}.$$

We denote by  $\mathcal{A}_{g,\delta}$  the separated algebraic stack of finite type over  $\text{Spec } \mathbb{Z}$  whose objects over  $S$  are polarized abelian schemes  $(A, \lambda)$  of dimension  $g$  and degree  $d$  over  $S$  such that there exists a faithfully flat base change  $S' \rightarrow S$  such that  $(\ker(\lambda)_{S'}, e_{S'}^\lambda) \cong (K(\delta)_{S'}, e_{\delta, S'})$ ; it is equivalent to require simply that  $\ker(\lambda)_{S'} \cong K(\delta)_{S'}$  ([dJ93, Rem. 1.4(2)]). We say that such a polarization  $\lambda$  is of *type*  $\delta$ . For a polarization of type  $\delta = (\delta_1, \dots, \delta_g)$ , the exponent is  $\delta_g$ .

The stacks  $\mathcal{A}_{g,\delta}$  are locally closed substacks of  $\mathcal{A}_{g,d}$  [dJ93, Prop. 1.5], and the irreducible components of  $\mathcal{A}_{g,d}$  are the closures  $(\mathcal{A}_{g,\delta})^c$  of the  $\mathcal{A}_{g,\delta}$  in  $\mathcal{A}_{g,d}$  [dJ93, Thm. 1.12(1)]. If  $k$  is an algebraically closed field of characteristic  $p = \text{char}(k)$  and  $p \nmid d$ , then the  $\mathcal{A}_{g,\delta,k}$  are closed in  $\mathcal{A}_{g,d,k}$ , and  $\mathcal{A}_{g,d,k}$  is the disjoint union of the  $\mathcal{A}_{g,\delta,k}$ ; if  $\text{char}(k) = 0$ , this is classical, and for  $\text{char}(k) = p > 0$ , this is [dJ93, Thm. 1.12(2), Rem. 1.13(2)]. If  $p \mid d$ , then one sees from the definition of  $K(\delta)$  that each  $\mathcal{A}_{g,\delta,k}$  avoids the  $p$ -rank zero locus of  $\mathcal{A}_{g,d,k}$ , and thus the union of the various  $\mathcal{A}_{g,d,k}$  cannot be all of  $\mathcal{A}_{g,d,k}$ . (See, e.g., Remark 5.11 for a reminder on the notion of  $p$ -rank.)

Moreover, if  $p \mid d$ , but  $p^2 \nmid d$ , then the  $(\mathcal{A}_{g,\delta,k})^c$  are still disjoint, but if  $p^2 \mid d$ , then the  $(\mathcal{A}_{g,\delta,k})^c$  are not disjoint [dJ93, Rem. 1.13(2)]. In particular, there are examples due to Norman ([Nor78]; see also [dJ93, Exa. 3.5.1], and Remark 1.2 below) showing that  $(\mathcal{A}_{2,(p,p),\mathbb{F}_p})^c$  and  $(\mathcal{A}_{2,(1,p^2),\mathbb{F}_p})^c$  have a nontrivial intersection.

In summary, in terms of exponents, this implies that in characteristic 0, or in positive characteristic  $p$  if  $p \nmid d$ , the exponent is stable under base change. (One can also see this directly, since then the kernel of the polarization is an étale group scheme over  $S$ .) However, in positive characteristic, if  $p^2 \mid d$ , then there are examples where the exponent is *not* stable under base change. The following lemma shows, however, that in the situation we frequently will be in, namely the case where our polarization is a multiple of a principal polarization, the exponent is in fact stable under base change:

**Lemma 1.1** (Exponent under base change). *Let  $(X, \lambda)$  be a polarized abelian scheme of dimension  $g$  and degree  $d$  over  $S$ . The exponent of  $\lambda$  is stable under base change if one of the following conditions hold:*

- (a) *The associated morphism  $S \rightarrow \mathcal{A}_{g,d}$  factors through the natural inclusion  $\mathcal{A}_{g,\delta} \subseteq \mathcal{A}_{g,d}$  for some polarization type  $\delta$ ; e.g.,  $S$  is a scheme over a field  $K$  of characteristic  $p$  and  $p \nmid d$ .*
- (b) *We have  $\lambda = e\xi$  for a positive integer  $e$  and a principal polarization  $\xi$  on  $X$ .*

*Proof.* Case (a) is clear as the exponent of  $K(\delta)$  is  $\delta_g$ , and thus the exponent of  $(X, \lambda)$  is  $\delta_g$  after any base change. For case (b) we have  $\ker \lambda = X[e]$ . Clearly the exponent is at most  $e$ , and since  $\deg[n]_X = n^{2g}$  (e.g., [Mum08, p.65]), the exponent of  $X[e]$  is not less than  $e$ .  $\square$

*Remark 1.2.* Condition (b) does not imply condition (a). Indeed, Norman's example [Nor78] showing that  $(\mathcal{A}_{2,(p,p),\mathbb{F}_p})^c$  and  $(\mathcal{A}_{2,(1,p^2),\mathbb{F}_p})^c$  are not disjoint is obtained in the following way. Let  $(E, \lambda)/k$  be a (principally polarized) supersingular elliptic curve over an algebraically closed field of characteristic  $p > 0$ . Then  $(E \times_k E, p \cdot (\lambda \times \lambda))$  deforms into both  $\mathcal{A}_{2,(p,p),k}$  and  $\mathcal{A}_{2,(1,p^2),k}$ .

**1.2. Short exact sequences of abelian schemes.** The following seems to be well-known, but we are not aware of a reference:

**Lemma 1.3.** *If  $0 \rightarrow X \xrightarrow{\iota} Y \rightarrow Z \rightarrow 0$  is a short exact sequence of abelian schemes over  $S$ , then so is*

$$0 \longrightarrow \widehat{Z} \longrightarrow \widehat{Y} \xrightarrow{\widehat{\iota}} \widehat{X} \longrightarrow 0.$$

*Proof.* It suffices to verify that the sequence is exact on fibers over geometric points, so we assume that  $S$  is the spectrum of an algebraically closed field. While this case appears to be well-known, for lack of a better reference we include a proof, following Conrad on [MathOverflow \(37536\)](#): Let  $N$  be any positive integer. Since the multiplication by  $N$  map  $[N]$  is surjective, by the snake lemma we have an exact sequence of group schemes  $0 \rightarrow X[N] \xrightarrow{\iota[N]} Y[N] \rightarrow Z[N] \rightarrow 0$ . Taking Cartier duals, and recalling that  $\text{Ext}^1(G, \mathbb{G}_m) = 0$  for a finite commutative group scheme  $G$  over an algebraically closed field, we obtain an exact sequence of finite group schemes  $0 \rightarrow Z[N]^\vee \rightarrow Y[N]^\vee \rightarrow X[N]^\vee \rightarrow 0$ . Using the canonical identifications  $\widehat{X}[N] \cong X[N]^\vee$ , etc. (e.g., [Oda69, Cor. 1.3(ii)]), we find that we have an exact sequence  $0 \rightarrow \widehat{Z}[N] \rightarrow \widehat{Y}[N] \xrightarrow{\widehat{\iota}[N]} \widehat{X}[N] \rightarrow 0$ . In

particular, for each  $N$  we have  $\ker(\hat{i})[N] = \hat{Z}[N]$ , and thus the canonical morphism  $\hat{Z} \rightarrow \ker \hat{i}$  obtained via pull-back of line bundles is an isomorphism, i.e.,  $\hat{Z} = \ker \hat{i}$ , as the union of the  $N$ -torsion in a sub-group scheme of an abelian variety is dense in the sub-group scheme. A similar argument using the surjection  $Y[N] \xrightarrow{\hat{i}[N]} \hat{X}[N]$  implies that the image of  $\hat{i}$  is equal to  $X$ .  $\square$

One application of this is to provide a condition for a morphism of abelian schemes to be a closed embedding:

**Corollary 1.4.** *Let  $f : X \rightarrow Y$  be a morphism of abelian schemes over  $S$ . Then the following are equivalent:*

- (a)  $f$  is a closed embedding;
- (b)  $\hat{f}$  is surjective, and  $\ker(\hat{f})$  is an abelian scheme.

*Proof.* If  $f$  is a closed embedding, then we obtain a short exact sequence of abelian schemes over  $S$ ,  $0 \rightarrow X \xrightarrow{f} Y \rightarrow X/Y \rightarrow 0$ . Dualizing this short exact sequence, Lemma 1.3 implies that  $\hat{f}$  is surjective and  $\ker(\hat{f}) \cong \widehat{X/Y}$  is an abelian scheme over  $S$ . Conversely, if (b) holds, then we have an exact sequence of abelian varieties  $0 \rightarrow \ker(\hat{f}) \rightarrow \hat{Y} \xrightarrow{\hat{f}} \hat{X} \rightarrow 0$ ; now invoke Lemma 1.3 again.  $\square$

In §1.5, we review some criteria via Nori fundamental groups for checking the condition in Corollary 1.4(b), that the kernel of a morphism of abelian schemes be an abelian scheme.

**1.3. Isogenies and endomorphisms of abelian schemes.** Given abelian schemes  $X/S$  and  $Y/S$ , an *isogeny*  $f : X \rightarrow Y$  is a finite surjective homomorphism of abelian  $S$ -schemes. The kernel  $\ker(f)$  is a finite flat commutative  $S$ -group scheme, being the kernel of a surjective morphism of abelian schemes [MFK94, Lem. 6.12], and therefore has an *exponent*,  $e = e(f)$ , i.e., the smallest integer  $e$  such that  $\ker(f) \subseteq \ker([e]_X)$  (see §1.1.1).

*Remark 1.5* (Exponent of isogenies and base change). Since the multiplication map is proper, and therefore closed, by considering generic fibers it is easy to see that the exponent of an isogeny is lower semi-continuous on fibers, and stable under faithfully flat base change. However, the exponent of an isogeny need not be stable under base change, even if the isogeny is a polarization (see §1.1.1). Nevertheless, our primary interest will be exponents of polarizations, and there are cases of interest where the exponent is stable under base change (see Lemma 1.1).

A basic tool we will use is:

**Proposition 1.6.** *Let  $X, Y$ , and  $Y'$  be abelian  $S$ -group schemes, and suppose that we are given  $S$ -homomorphisms  $\phi : X \rightarrow Y$  and  $\phi' : X \rightarrow Y'$ . If  $\phi$  is surjective, then there is an  $S$ -homomorphism  $\psi : Y \rightarrow Y'$  making the following diagram commute*

$$\begin{array}{ccc} X & \xrightarrow{\phi'} & Y' \\ \phi \downarrow & \nearrow \psi & \\ Y & & \end{array}$$

*if and only if  $\ker \phi \subseteq \ker \phi'$ . Moreover, in this situation,  $\psi$  is unique.*

*Proof.* Since  $\phi$  is surjective it is flat [MFK94, Lem. 6.12], and so  $\ker(\phi)$  is flat over  $S$ . The proposition then follows from [GP11, Exp. V, Cor. 10.1.3], observing that  $Y = X/\ker(\phi)$  and represents the fppf quotient sheaf.  $\square$

**Corollary 1.7** ([BL04, Prop. 1.2.6]). *Let  $f : X \rightarrow Y$  be an isogeny of abelian schemes over  $S$ . If  $\ker(f) \subseteq X[N]$  for some positive integer  $N$ , then there is an isogeny  $g_N : Y \rightarrow X$ , unique up to isomorphisms, such that  $g_N f = [N]_X$  and  $f g_N = [N]_Y$ . The formation of  $g_N$  is compatible with base change.*

*Proof.* For existence, one applies Proposition 1.6 to  $f$  and  $[N]_X$  to see that  $[N]_X$  factors through  $f$  to give  $g_N$  such that  $g_N f = [N]_X$ . Proposition 1.6 also gives the uniqueness statement for  $g_N$ . That the formation of  $g_N$  is compatible with base change follows from the uniqueness of  $g_N$ .

To show that  $f g_N = [N]_Y$ , one argues as follows. One can check from the construction that  $\ker g_N \subseteq Y[N]$ , and so by the same argument,  $[N]_Y$  factors through  $g_N$  to give  $f_N$  such that  $f_N g_N = [N]_Y$ . Then we have  $f_N [N]_X = f_N g_N f = [N]_Y f = f [N]_X$ . Since  $[N]_X$  is surjective, we can conclude that  $f_N = f$ , completing the proof.  $\square$

*Remark 1.8.* If the exponent of the isogeny  $f: X \rightarrow Y$  is  $e(f)$ , then the homomorphism  $g_{e(f)}$  is primitive. (A morphism  $a: X \rightarrow Y$  is called primitive if there is no morphism  $b: X \rightarrow Y$  and integer  $n \geq 2$  such that  $a = nb$ .) Indeed, if there were an integer  $n \geq 2$  such that  $h = \frac{g_{e(f)}}{n} \in \text{Hom}(Y, X)$ , then we would have  $h f = [e(f)/n]_X$ , and so  $\ker f \subseteq \ker [e(f)/n]_X$ , contradicting the definition of the exponent of  $f$ . Recall that while the exponent  $e(f)$  of  $f$  is stable under *faithfully flat* base change, the exponent  $e(f)$  is not necessarily stable under *arbitrary* base change (§1.1.1). In particular, if  $S' \rightarrow S$  is a morphism of schemes and  $f_{S'}: X_{S'} \rightarrow Y_{S'}$  is the morphism obtained by base change, it is possible that  $(g_{e(f)})_{S'}$  may not equal  $g_{e(f_{S'})}$ .

*Remark 1.9.* The exponent of  $g_e$  must divide the exponent of  $f$  by construction. But considering the case  $f = [n]_X: X \rightarrow X$  for any natural number  $n > 1$ , which obviously has exponent  $e = n$  and  $g_e = \text{Id}_X$ , one sees that the exponent of  $g_e$  (equal to 1 in this case) may be strictly smaller than the exponent of  $f$ .

Recall that for abelian  $S$ -schemes  $X$  and  $Y$ , we have that  $\text{Hom}(X, Y) (:= \text{Hom}_S(X, Y))$  is a free  $\mathbb{Z}$ -module of finite rank; indeed, by rigidity [MFK94, Ch. 6, §1, Prop. 6.1, p.115], we have an injection  $\text{Hom}(X, Y) \hookrightarrow \text{Hom}(X_s, Y_s)$  for any geometric point  $s$  in  $S$ , and then one concludes using the case of abelian varieties (e.g., [Mum08, §19, Thm. 3, p.176]). We set  $\text{Hom}_{\mathbb{Q}}(X, Y) = \text{Hom}(X, Y) \otimes_{\mathbb{Z}} \mathbb{Q}$ .

**Corollary 1.10** ([BL04, Cor. 1.2.7]). *In the notation of Corollary 1.7:*

- (a) *Isogenies define an equivalence relation for abelian  $S$ -schemes.*
- (b) *An element of  $\text{End}(X/S)$  is an isogeny if and only if it is invertible in  $\text{End}_{\mathbb{Q}}(X/S)$ .*

*Proof.* This is clear from what is above.  $\square$

In light of Corollary 1.10(b), for an isogeny  $f: X \rightarrow Y$  of abelian schemes over  $S$  with  $\ker(f) \subseteq X[N]$ , if  $g_N: Y \rightarrow X$  is the isogeny from Corollary 1.7, we use the notation

$$f^{-1} := \frac{g_N}{N} \in \text{Hom}_{\mathbb{Q}}(Y, X); \quad (1.3)$$

i.e.,  $f^{-1}$  is the inverse in  $\text{Hom}_{\mathbb{Q}}(Y, X)$  of  $f$ . Note that this definition of  $f^{-1}$  is independent of the choice of  $N$  such that  $\ker(f) \subseteq X[N]$ . Indeed, if  $\ker(f) \subseteq X[M]$ , then  $(M g_N) f = [NM]_X = (N g_M) f$  and  $f(M g_N) = [NM]_Y = f(N g_M)$ , implying by uniqueness of  $g_{NM}$  that  $M g_N = g_{NM} = N g_M$ , so that dividing by  $NM$  we have  $\frac{g_N}{N} = \frac{g_{NM}}{NM} = \frac{g_M}{M}$ . Since the formation of  $g_N$  is stable under base change, so is the formation of  $f^{-1}$ .

Every  $h \in \text{End}_{\mathbb{Q}}(X/S)$  can be written as  $h = r f$  for some  $r \in \mathbb{Q}$  and some  $f \in \text{End}(X/S)$ , and since  $\hat{h} := r \hat{f} \in \text{End}_{\mathbb{Q}}(X/S)$  does not depend on the choice of  $r$  and  $f$  such that  $h = r f$ , the assignment  $h \mapsto \hat{h}$  defines an involution on  $\text{End}_{\mathbb{Q}}(X/S)$ . The involution is stable under base change.

From this one obtains the Rosati involution for  $h \in \text{End}_{\mathbb{Q}}(X/S)$  associated to a polarization  $\lambda$ :

$$h^{(+)} := \lambda^{-1} \hat{h} \lambda \in \text{End}_{\mathbb{Q}}(X/S).$$

If  $\lambda$  is principal, then  $\text{End}(X/S)$  is stable under the Rosati involution; and in all cases, formation of the Rosati involution is compatible with base change.

**1.4. Multiples of principal polarizations.** One of our main focuses is studying abelian schemes together with polarizations that are multiples of a principle polarization. We will frequently use the following elementary lemma, whose proof we include for lack of a reference:

**Lemma 1.11.** *Let  $(X, \lambda)$  be a polarized abelian scheme over  $S$  and let  $e$  be a positive integer. The following are equivalent:*

- (a)  $\lambda$  is equal to  $e$  times a principal polarization  $\xi$  on  $X$ .
- (b) For every point  $s$  of  $S$  we have  $\lambda_s$  is  $e$  times a principal polarization on  $X_s$ .
- (c) For every geometric point  $\bar{s}$  of  $S$  we have  $\lambda_{\bar{s}}$  is  $e$  times a principal polarization on  $X_{\bar{s}}$ .
- (d)  $\ker \lambda = X[e]$ .

*Proof.* The implications (a)  $\implies$  (b)  $\implies$  (c) are immediate (as is (a)  $\implies$  (d)).

Next let us show (c)  $\implies$  (d). First, we claim that  $e \ker \lambda = 0$ . We just need to check that the zero section  $\epsilon: S \rightarrow e \ker \lambda$  and the projection  $\pi|_{e \ker \lambda}: e \ker \lambda \rightarrow S$  are inverses to one another. In fact, it suffices to show that  $\pi|_{e \ker \lambda}$  is a closed immersion. This can be checked on every fiber ([Gro67, 17.2.6] gives the equivalence of monomorphisms with maps with fibers that are empty or isomorphisms, and [Gro67, 18.12.6] gives that proper monomorphisms are closed immersions). Since being a closed immersion satisfies faithfully flat descent (e.g., [GW20, Prop. 14.53]), it in fact suffices to check after base change to the geometric fiber. Therefore we have that  $e \ker \phi = 0$ , and we can conclude that  $\ker \lambda \subseteq X[e]$ . The degree of  $\ker \lambda$  over  $S$  can be checked on fibers, and therefore, since both  $\ker \lambda$  and  $X[e]$  are finite flat groups schemes of the same degree, the inclusion is an isomorphism.

We now show (d)  $\implies$  (a). Assuming (d), then Proposition 1.6 implies that  $\lambda$  factors as  $\lambda = \psi \circ [e]$  for a unique morphism of abelian  $S$ -schemes  $\psi: X \rightarrow \hat{X}$ . We just need to show that  $\psi$  is a principal polarization, and for degree reasons, it suffices to show that  $\psi$  is a polarization. From the definition, we must show that for each  $s$  in  $S$  there is an ample line bundle  $M_{\bar{s}}$  on the geometric fiber  $X_{\bar{s}}$  such that  $\psi_{\bar{s}} = \phi_{M_{\bar{s}}}$ . By definition, there exists an ample line bundle  $L_{\bar{s}}$  on  $X_{\bar{s}}$  such that  $\lambda_{\bar{s}} = \phi_{L_{\bar{s}}}$ . From [Mum08, §23, Thm. 3, p.231], the containment  $K(L_{\bar{s}}) \supseteq X_{\bar{s}}[e]$  implies that there exists a line bundle  $M_{\bar{s}}$  such that  $M_{\bar{s}}^{\otimes e} \cong L_{\bar{s}}$ . We then have that  $\lambda_{\bar{s}} = \phi_{L_{\bar{s}}} = \phi_{M_{\bar{s}}^{\otimes e}} = \phi_{M_{\bar{s}}} \circ [e]$ . Since  $\psi$  is unique, and stable under base change, we have that  $\psi_{\bar{s}} = \phi_{M_{\bar{s}}}$ , and we are done.  $\square$

*Remark 1.12.* Given a polarized abelian scheme  $(X, \lambda)$ , for degree reasons one may conclude that if for one geometric point  $\bar{s}$  of  $S$  the fiber  $\lambda_{\bar{s}}$  is a principal polarization, then  $\lambda$  is a principal polarization. In characteristic 0, if for one geometric fiber  $\lambda_{\bar{s}}$  is  $e$  times a principal polarization, then  $\lambda$  is  $e$  times a principal polarization; this follows, for instance, from the consideration of the type of the polarization (see §1.1.1). However, if we do not restrict to characteristic 0, Norman's example (see Remark 1.2) shows that it is possible for one fiber  $\lambda_s$  to be  $e$  times a principal polarization ( $e \geq 2$ ), while  $\lambda$  is *not*  $e$  times a principal polarization.

**1.5. Kernels, abelian schemes, and the geometric Nori fundamental group.** In order to apply Corollary 1.4(b), it will be convenient to have a criterion for when the kernel of a surjective morphism of abelian schemes  $f: X \rightarrow Y$  is actually an abelian scheme. As motivation, consider the case of complex abelian varieties. In this case having the kernel be an abelian scheme is equivalent to the kernel being connected; said in another way, the question is whether  $f$  factors through any isogeny over  $Y$ . For complex abelian varieties this question can be understood purely in terms of its effect on the period lattices of the abelian varieties, or, equivalently, on their fundamental groups via the theory of covering spaces; i.e., the kernel of  $f$  is an abelian scheme if and only if  $f_*(\pi_1(X, 0_X)) = \pi_1(Y, 0_Y)$ .

In positive characteristic, one can study the effect of a morphism of abelian varieties on their étale fundamental groups, but this misses phenomena related to inseparability; e.g., over an algebraically closed field of characteristic  $p > 0$ , one may have  $\ker f$  being connected but non-reduced,

so that the kernel would fail to be an abelian scheme, but the fundamental group would not detect this. Working instead with Nori's fundamental group scheme allows one to recover the connection between fundamental groups and kernels. Recall that for an abelian variety  $X/K$  over a field  $K$ , its geometric étale fundamental group may be computed as  $\pi_1^{\text{ét}}(X_{\bar{K}}, 0) = \lim_{\leftarrow N} X[N](\bar{K})$ , while its geometric Nori fundamental group scheme is  $\pi_1^{\text{Nori}}(X_{\bar{K}}) = \lim_{\leftarrow N} X[N]$  [Nor83]. In characteristic zero, these notions coincide; in positive characteristic, the geometric Nori fundamental group scheme is a strictly richer object. Note that if  $f: Y \rightarrow X$  is an isogeny of abelian varieties, then  $Y$  is a torsor over  $X$  under the finite group scheme  $\ker(f)$ ; and that  $f$  is separable if and only if  $\ker(f)$  is étale.

More generally, recall that if  $(X, P)/K$  is a proper connected reduced pointed scheme, then the Nori fundamental group scheme  $\pi_1^{\text{Nori}}(X, P)$  is a profinite group scheme over  $K$  which classifies torsors over  $X$  (pointed over  $P$ ) under finite group schemes; such a torsor  $(\tilde{X}, \tilde{P})$  under a group scheme  $G$  corresponds to homomorphisms  $\pi_1^{\text{Nori}}(\tilde{X}, \tilde{P}) \hookrightarrow \pi_1^{\text{Nori}}(X, P) \twoheadrightarrow G$  [Gar09, Thm. 4] [Nor83]. Similarly, there is a profinite group scheme  $\pi_1^{\text{Nori}}(X, P)^{\text{abn}}$  which classifies torsors under finite commutative group schemes [Ant11, §3]. There is a surjection  $\pi_1^{\text{Nori}}(X, P) \twoheadrightarrow \pi_1^{\text{Nori}}(X, P)^{\text{abn}}$ , and a cover  $(\tilde{X}, \tilde{P}) \rightarrow (X, P)$  is abelian, i.e., is a torsor under a finite commutative group scheme  $G$ , if and only if the corresponding classifying map  $\pi_1^{\text{Nori}}(X, P) \twoheadrightarrow G$  factors through  $\pi_1^{\text{Nori}}(X, P)^{\text{abn}}$ . In a given setting (such as for abelian varieties, or pointed curves), if the choice of basepoint is clear from context, we will often write  $\pi_1^{\text{Nori}}(X)$  for  $\pi_1^{\text{Nori}}(X, P)$ .

Recall that if  $f: X \rightarrow Y$  is a morphism of abelian varieties over a field, then  $\ker(f)$  admits a maximal subabelian variety  $\ker(f)^{\text{ab}}$  (Lemma 1.17).

**Lemma 1.13.** *Let  $f: X \rightarrow Y$  be a surjective morphism of abelian schemes over  $S$ . Then  $\ker(f)$  is an abelian scheme over  $S$  if and only if for each geometric point  $s$  of  $S$ , the induced map*

$$\pi_1(f_s): \pi_1^{\text{Nori}}(X_s) \rightarrow \pi_1^{\text{Nori}}(Y_s)$$

*is surjective.*

*Proof.* It suffices to prove this for the geometric fibers, so we may and do assume that  $S = \text{Spec } k$  for an algebraically closed field  $k$ .

If  $\ker(f)$  is an abelian variety, then we have a short exact sequence of abelian varieties

$$0 \longrightarrow \ker(f) \longrightarrow X \xrightarrow{f} Y \longrightarrow 0,$$

and in the proof of Lemma 1.3, we saw that this induces a short exact sequence of  $k$ -group schemes

$$0 \longrightarrow \ker(f)[N] \longrightarrow X[N] \xrightarrow{f[N]} Y[N] \longrightarrow 0$$

for each  $N \geq 2$ . Since  $N$ -torsion in an abelian variety forms a surjective system, we see that in the limit, the exact sequence stays exact, and we obtain a surjection  $\pi_1(f): \pi_1^{\text{Nori}}(X) \rightarrow \pi_1^{\text{Nori}}(Y)$ .

Conversely, suppose that  $\ker(f)$  is not an abelian variety. We want to show that  $\pi_1(f)$  is not surjective. Initially, suppose  $f$  is an isogeny. Then we may assume that  $\ker(f)$  is nontrivial, and suppose it has exponent  $N > 1$ . From the exact sequence of group schemes

$$0 \longrightarrow \ker(f) \longrightarrow X \longrightarrow Y \longrightarrow 0$$

and the snake lemma applied to the multiplication-by- $N$  maps, we deduce that there is an exact sequence of group schemes

$$0 \longrightarrow \ker(f) \longrightarrow X[N] \longrightarrow Y[N] \longrightarrow \ker(f)/[N]\ker(f) \cong \ker(f) \longrightarrow 0, \quad (1.4)$$

where we are using that the multiplication by  $N$  maps are surjective for abelian varieties, and that we have chosen  $N$  so that  $\ker(f)[N] = \ker(f)$ . (It is important here that we are using the group schemes, and not just the  $k$ -points, since otherwise the argument would fail if  $f$  were a purely inseparable isogeny.) Therefore,  $\pi_1(f): \pi_1^{\text{Nori}}(X) \rightarrow \pi_1^{\text{Nori}}(Y)$  is not surjective.

For the general case, let  $Z = \ker(f)^{\text{ab}} \subsetneq \ker(f)$  be the maximal abelian subvariety (lemma 1.17). By factoring  $f$  as  $X \rightarrow X/Z \rightarrow Y$ , we find that  $X/Z \rightarrow Y$  is a necessarily non-trivial isogeny, and therefore, from the previous paragraph, we have that  $\pi_1^{\text{Nori}}(X/Z) \rightarrow \pi_1^{\text{Nori}}(Y)$  is not surjective, so that  $f_*: \pi_1^{\text{Nori}}(X) \rightarrow \pi_1^{\text{Nori}}(X/Z) \rightarrow \pi_1^{\text{Nori}}(Y)$  fails to be surjective, as well.  $\square$

More generally we have the following result, which is well-documented in the case where  $f$  is an isogeny.

**Lemma 1.14.** *Let  $f: X \rightarrow Y$  be a surjective morphism of abelian schemes over  $S$ . Then for each geometric point  $s$  of  $S$ , we have canonical isomorphisms*

$$\text{coker}(\pi_1^{\text{Nori}}(f_s)) \cong \ker(f_s) / \ker(f_s)^{\text{ab}} \quad \text{and} \quad \ker(\widehat{f}_s) \cong (\ker(f_s) / \ker(f_s)^{\text{ab}})^{\vee} \cong \text{coker}(\pi_1^{\text{Nori}}(f_s))^{\vee}.$$

*Proof.* It clearly suffices to prove this when  $S = \text{Spec}(k)$  is the spectrum of an algebraically closed field. Let  $\overline{X} = X / \ker(f)^{\text{ab}}$ , and factor  $f$  as a composition of surjections  $f: X \xrightarrow{\phi} \overline{X} \xrightarrow{\overline{f}} Y$ .

First observe that from Lemma 1.13, since by construction the kernel of  $\phi$  is an abelian scheme,  $\pi_1^{\text{Nori}}(\phi)$  is surjective, so that  $\text{coker}(\pi_1^{\text{Nori}}(f)) = \text{coker}(\pi_1^{\text{Nori}}(\overline{f}))$ . Let  $N$  be any integer which annihilates the finite flat group scheme  $\ker(\overline{f}) = \ker(f)^{\text{ff}}$ ; i.e.,  $\ker(\overline{f}) \subseteq \overline{X}[N]$ . As in (1.4), we obtain an exact sequence of finite flat group schemes

$$0 \longrightarrow \ker(\overline{f}) \longrightarrow \overline{X}[N] \xrightarrow{\overline{f}[N]} Y[N] \longrightarrow \ker(\overline{f}) \longrightarrow 0. \quad (1.5)$$

so that for sufficiently divisible  $N$ , we have that  $\text{coker}(\pi_1^{\text{Nori}}(\overline{f})) = \text{coker}(\overline{X}[N] \rightarrow Y[N]) \cong \ker(\overline{f})$ . To prove the first claim of the lemma, it now suffices to recall that  $\ker(\overline{f}) = \ker(f) / \ker(f)^{\text{ab}}$ .

Now, to prove the second claim of the lemma, take Cartier duals in (1.5) and recall [Oda69, Cor. 1.3(ii)] that this yields an exact sequence

$$0 \longrightarrow \ker(\overline{f})^{\vee} \longrightarrow \widehat{Y}[N] \xrightarrow{\widehat{\overline{f}}[N]} \widehat{\overline{X}}[N] \longrightarrow \ker(\overline{f})^{\vee} \longrightarrow 0.$$

Since this is true for all  $N$  divisible by the exponent of  $\ker(\overline{f})$ , we find that  $\ker(\widehat{\overline{f}})$  is canonically isomorphic to  $\ker(\overline{f})^{\vee}$ . Because  $\widehat{f}$  factors as  $\widehat{f}: \widehat{Y} \xrightarrow{\widehat{\overline{f}}} \widehat{\overline{X}} \xrightarrow{\widehat{\phi}} \widehat{X}$  (with the inclusion coming by virtue of Lemma 1.3), we conclude that  $\ker(\widehat{f}) = \ker(\widehat{\overline{f}}) \cong \ker(\overline{f})^{\vee}$ .  $\square$

**1.6. Images and preimages of abelian schemes.** Consider a morphism  $f: X \rightarrow Y$  of abelian schemes over  $S$ . We will tentatively say that  $f$  is *epi-abelian* if the schematic image of  $f$  is again an abelian scheme. Certain arguments we develop here are more transparent under the following hypothesis on a scheme  $S$ .

**Condition 1.15.** *Any morphism of  $S$ -abelian schemes is epi-abelian.*

On one hand, we have:

**Lemma 1.16** ([ACMW24, Thm. A]). *Suppose that  $S$  is either:*

- (a) *the spectrum of a field, or*
- (b) *the spectrum of an unramified DVR of mixed characteristic, or*
- (c) *of characteristic zero.*

*Then  $S$  satisfies Condition 1.15.*

On the other hand we note that, in contrast, both  $\text{Spec } \mathbb{F}_p[[T]]$  and  $\text{Spec } \mathbb{Z}_{(p)}[\zeta_p]$  fail to satisfy Condition 1.15 [ACMW24, §4].

Fortunately, it turns out that for the construction of Prym schemes, we can often circumvent such charming pathologies.

Let  $X/S$  be a group scheme over  $S$ . A maximal abelian subscheme of  $X$  is an abelian subscheme  $X^{\text{ab}}$ , maximal among all abelian subschemes of  $X$ , whose construction is compatible with base change  $T \rightarrow S$ .

We will use the following criterion to assert the existence of a maximal abelian subscheme in the context of constructing complementary abelian subschemes of a separably polarized abelian scheme.

**Lemma 1.17.** *Let  $\lambda : X \rightarrow Y$  be a surjection of abelian schemes over  $S$ , and let  $Z \hookrightarrow Y$  be an abelian subscheme. Let  $\tilde{Z} = \lambda^{-1}(Z) = Z \times_Y X$ . If  $\lambda$  is smooth or if  $S$  is the spectrum of a field, then  $\tilde{Z}$  admits a maximal abelian subscheme,  $\tilde{Z}^{\text{ab}}$ .*

*Proof.* Call a morphism  $f : W \rightarrow S$  cohomologically flat if it is cohomologically flat in dimension zero, i.e., if formation of  $f_*\mathcal{O}_X$  is compatible with base change. If  $f$  is proper, flat and smooth, then it is cohomologically flat as well [Gro63, Prop. 7.8.6]. If  $W$  is a proper, flat finitely presented  $S$ -group scheme, then  $W$  admits a maximal abelian subscheme if and only if it is cohomologically flat [Bro21, Prop. 2.16].

With these preliminaries dispatched, the proof goes quickly. The case where  $S$  is the spectrum of a field is immediate – indeed, any scheme over a field is cohomologically flat – so suppose  $\lambda$  is smooth. It is also proper and flat, thus cohomologically flat. Pulling back  $Z$  by  $\lambda$  shows that  $\tilde{Z} \rightarrow Z$  is also proper, flat and smooth (and cohomologically flat). Since  $Z \rightarrow S$  is also proper, flat and smooth, the structural morphism  $\tilde{Z} \rightarrow Z \rightarrow S$  is proper, flat and smooth, thus cohomologically flat. In particular,  $\tilde{Z}$  admits a maximal abelian subscheme.  $\square$

*Remark 1.18.* If  $X$  is an abelian variety, and  $G \subseteq X$  is a closed algebraic subgroup, then  $G^{\text{ab}} = (G^\circ)_{\text{red}}$ , the reduction of the connected component of the identity. Indeed, over a perfect field, the reduction of an algebraic group is an algebraic group, while over an arbitrary field, the same holds for proper algebraic groups [Gro62, Exp. VI, Prop. 3.1].

**1.7. Picard and Albanese schemes.** We now turn our attention to Picard and Albanese schemes.

**1.7.1. Picard schemes for families of smooth projective varieties with vanishing  $H^2$ .** Assume that  $f : C \rightarrow S$  is a smooth and projective morphism with geometrically integral fibers, and  $R^2f_*\mathcal{O}_C = 0$ ; e.g.,  $C/S$  is a smooth projective curve over  $S$ . We use [Kle05] as a reference for the following standard results on Picard schemes. First, using that  $C/S$  is projective and flat with geometrically integral fibers, we can invoke [Kle05, Thm. 9.4.8] to conclude the étale sheafification of the Picard functor is representable by a separated scheme  $\text{Pic}_{C/S}$ . Then using the additional assumption that  $R^2f_*\mathcal{O}_C = 0$ , we can invoke [Kle05, Prop. 9.5.19] to conclude that  $\text{Pic}_{C/S}$  is smooth. Denoting by  $\text{Pic}_{C/S}^0$  the connected component of the identity, we can use [Kle05, Thm. 9.5.4] to conclude that the fiber of  $\text{Pic}_{C/S}^0$  over every point is projective. It then follows from [Kle05, Prop. 9.5.20] that  $\text{Pic}_{C/S}^0$  is proper over  $S$ . In other words, it is an abelian scheme over  $S$ . By construction it is stable under base change.

In fact, even if  $C \rightarrow S$  is merely proper (but not projective; this can happen if the fibers of  $C$  have genus one), it is still true that  $\text{Pic}_{C/S}^0$  is an abelian scheme. Indeed, after an étale base change  $T \rightarrow S$ ,  $C_T \rightarrow T$  is projective, and thus  $\text{Pic}_{C_T/T}^0$  is an abelian scheme. A descent argument (see Cesnavicius on [MathOverflow \(204182\)](#)) then shows that  $\text{Pic}_{C_T/T}^0$  descends to  $S$  as an abelian scheme.

1.7.2. *Picard schemes for torsors under abelian schemes.* Let  $T/S$  be a torsor under an abelian scheme  $X$  over  $S$ . The same argument as above implies the existence of the Picard scheme  $\text{Pic}_{T/S}$  representing the étale sheafification of the Picard functor. The fact that in this situation the connected component  $\text{Pic}_{T/S}^0$  is an abelian scheme is for instance [Ols08, Prop. 2.1.5]. In fact, given any  $S$ -scheme  $S' \rightarrow S$ , the set of isomorphisms of  $X_{S'}$ -torsors  $\iota: X_{S'} \rightarrow T_{S'}$  is canonically in bijection with the set  $T(S')$ . Choosing an étale cover  $S'/S$  over which  $T_{S'}$  is isomorphic to  $X_{S'}$  we obtain a non-empty set  $T(S')$ , and by descent, any such isomorphism  $\iota$  then induces an isomorphism  $\iota_*: \widehat{X} \rightarrow \text{Pic}_{T/S}^0$ . This construction gives a canonical morphism  $\widehat{X} \times_S T \rightarrow \text{Pic}_{T/S}^0$ ,  $(L, \iota) \mapsto \iota_*L$ , which factors uniquely through the first projection  $\widehat{X} \times_S T \xrightarrow{pr_1} \widehat{X} \xrightarrow{\zeta} \text{Pic}_{T/S}^0$  [Ols08, Prop. 2.1.5], giving a canonical isomorphism:

$$\zeta: \widehat{X} \longrightarrow \text{Pic}_{T/S}^0. \quad (1.6)$$

Note that given a Poincaré line bundle over  $\text{Pic}_{T/S}^0 \times_S T$ , we can view it alternatively as a family of line bundles over  $\text{Pic}_{T/S}^0 = \widehat{X}$  parameterized by  $T$ , and we obtain a morphism  $T \rightarrow \text{Pic}_{\widehat{X}/S}$ , which identifies  $T$  with a component of  $\text{Pic}_{\widehat{X}/S}$ . Indeed, it suffices to check this on geometric fibers, in which case the torsor is trivial, and the assertion comes down to the fact that  $\widehat{X} = X$ . Note that if the universal object for  $\text{Pic}_{T/S}^0$  is not a line bundle, so that there is not a Poincaré line bundle, one can make an étale base change, and then use descent, to obtain the same result. In other words, a torsor under  $X$  over  $S$  can be viewed as a component of  $\text{Pic}_{\widehat{X}/S}$ .

1.7.3. *Albanese schemes for families with  $\text{Pic}^0$  an abelian scheme.* We now assume that  $f: C \rightarrow S$  is a smooth projective morphism with geometrically integral fibers, or that it is a smooth proper morphism whose fibers are geometrically integral curves; either implies that the étale sheafification of the Picard functor is representable by a scheme  $\text{Pic}_{C/S}$  (see the discussion in §1.7.1), and we assume further that the connected component  $\text{Pic}_{C/S}^0$  is an abelian scheme (e.g., this holds in the situation of §1.7.1 or 1.7.2).

Given an abelian scheme  $X/S$ , then via the definition of the Picard scheme, there is a bijection between:

- $S$ -morphisms  $C \rightarrow T$  where  $T/S$  is some torsor under  $X$ .
- $S$ -morphisms  $\widehat{X} \rightarrow \text{Pic}_{C/S}$ .

Indeed, given a Poincaré line bundle on  $\text{Pic}_{C/S} \times_S C$ , a morphism  $\widehat{X} \rightarrow \text{Pic}_{C/S}$  corresponds to a line bundle  $L$  on the product  $\widehat{X} \times_S C$ . Now viewing  $L$  instead as a family of line bundles on  $\widehat{X}$  parameterized by  $C$ , we have a morphism  $C \rightarrow \text{Pic}_{\widehat{X}/S}$ . Looking at the component containing the image of  $C$  under this map, we get a map  $C \rightarrow T$  for some torsor  $T$  under  $\text{Pic}_{\widehat{X}/S}^0 = X$ . Again, if the universal object for  $\text{Pic}_{C/S}$  is not a line bundle, so that there is not a Poincaré line bundle, one can obtain the same result by taking an étale base change, and then using descent. As every torsor under  $X$  over  $S$  can be viewed as a component of  $\text{Pic}_{\widehat{X}/S}$  (§1.7.2), one can reverse the above construction, giving the asserted equivalence.

From this equivalence and the identity map  $\widehat{\widehat{\text{Pic}}}_{C/S}^0 = \text{Pic}_{C/S}^0 \rightarrow \text{Pic}_{C/S}^0 \subseteq \text{Pic}_{C/S}$ , one obtains a canonical morphism

$$\text{alb}^1: C \rightarrow \text{Alb}_{C/S}^1$$

where  $\text{Alb}_{C/S}^1$  is a torsor under  $\text{Alb}_{C/S}^0 := \widehat{\widehat{\text{Pic}}}_{C/S}^0$ , and moreover, one obtains [Gro62, Exp. VI, Thm. 3.3(iii)] that  $\text{alb}^1: C \rightarrow \text{Alb}_{C/S}^1$  is an Albanese morphism, meaning that it is initial for  $S$ -morphisms of  $C$  into torsors under abelian  $S$ -schemes. A section of  $C/S$  gives an identification

$\text{Alb}_{C/S}^1 \cong \text{Alb}_{C/S}^0$  so that the composition of  $\text{alb}^1$  with this isomorphism gives an  $S$ -morphism

$$\text{alb}^0: C \rightarrow \text{Alb}_{C/S}^0$$

that is initial for  $S$ -morphisms of  $C$  into abelian  $S$ -schemes taking the section of  $C/S$  to the zero section. These constructions are stable under base change.

1.7.4. *Albanese schemes for curves.* By a *curve* we will mean a geometrically integral separated scheme of dimension one over a field, and by a curve over  $S$  we will mean a flat morphism  $C \rightarrow S$  such that every fiber is a curve.

Suppose  $f: C \rightarrow S$  is a smooth proper curve over  $S$ , and let  $\text{Div}_{C/S}$  be the functor of relative effective divisors on  $C$  [Kle05, Def. 9.3.6]; there is a natural transformation of functors  $C \rightarrow \text{Div}_{C/S}$ , where  $\sigma \in C_T(T)$  gets sent to the corresponding relative effective divisor on  $C_T$ . Composing this with the Abel map, a natural transformation  $\text{Div}_{C/S} \rightarrow \text{Pic}_{C/S}$  [Kle05, Def. 9.4.6], yields a morphism of functors, and thus of schemes,  $\alpha: C \rightarrow \text{Pic}_{C/S}$ . Let  $\text{Pic}_{C/S}^{(1)}$  be the component of  $\text{Pic}_{C/S}$  which contains the image of  $\alpha$ ; it is a torsor under  $\text{Pic}_{C/S}^0$ . We denote the induced  $S$ -morphism

$$\alpha^{(1)}: C \longrightarrow \text{Pic}_{C/S}^{(1)},$$

and this agrees with the Albanese torsor discussed in §1.7.3. Moreover, a section  $P: S \rightarrow C$  of  $C/S$  gives an identification  $\text{Pic}_{C/S}^{(1)} \cong \text{Pic}_{C/S}^0$  so that the composition of  $\alpha^{(1)}$  with this isomorphism gives an  $S$ -morphism

$$\alpha_P: C \longrightarrow \text{Pic}_{C/S}^0$$

which agrees with the pointed Albanese defined above (see also [Mil08, Prop. III.6.1] in the case where  $S$  is the spectrum of a field). When  $S = \text{Spec } k$  for an algebraically closed field  $k$ , then  $\alpha^{(1)}$  is given by  $Q \in C(k) \mapsto \mathcal{O}_C(Q)$ , the identification  $\text{Pic}_{C/k}^{(1)} \cong \text{Pic}_{C/k}^0$  is given by  $\mathcal{L} \mapsto \mathcal{L}(-P)$ , and  $\alpha_P$  is given by  $Q \in C(k) \mapsto \mathcal{O}_C(Q - P)$ .

## 2. NORM ENDOMORPHISMS, COMPLEMENTS, AND PRYM SCHEMES

The goal of this section is to review the notion of projectors for sub-abelian schemes of an abelian scheme. More precisely, given an inclusion  $Y \hookrightarrow X$  of abelian schemes over  $S$ , one would like to have an endomorphism  $\epsilon_Y$  of  $X$  such that  $\text{Im}(\epsilon_Y) = Y$  and  $\epsilon_Y^2 = \epsilon_Y$ . The norm endomorphism  $N_Y$  of  $Y$ , constructed via an auxiliary polarization  $\lambda$  of  $X$ , achieves this, up to the exponent  $e$  of the polarization restricted to  $Y$ , in the sense that  $\text{Im } N_Y = Y$  and  $N_Y^2 = eN_Y$ . Consequently, one may take  $\epsilon_Y := \frac{1}{e}N_Y \in \text{End}_{\mathbb{Q}}(X/S)$  and obtain a projector in the rational endomorphism ring. If the exponent  $e$  of the polarization restricted to  $Y$  is stable under base change, e.g., the restricted polarization is a positive integer multiple of a principal polarization (see Lemma 1.1), then so is the norm map  $N_Y$  and the projector  $\epsilon_Y$ . We review this here, as we are not aware of a reference for this in the relative setting.

To carry out the full construction, we will need to make at least one of the following assumptions:

**Condition 2.1.** *The abelian scheme  $X/S$  is equipped with a polarization  $\lambda: X \rightarrow \hat{X}$  and at least one of the following holds:*

- (a)  $S$  satisfies Condition 1.15;
- (b)  $\text{deg}(\lambda)$  is invertible on  $S$ .

Note that if  $\lambda$  is principal then  $(X, \lambda)/S$  automatically satisfies Condition 2.1 without additional hypothesis.

**2.1. Norm endomorphisms and complements.** Let  $\iota = \iota_Y: Y \hookrightarrow X$  be an inclusion of abelian schemes over  $S$ , and assume that  $X$  admits a polarization  $\lambda: X \rightarrow \widehat{X}$ . Starting in Lemma 2.5, and until the end of this section, we will assume that Condition 2.1 holds.

Let

$$e = e_Y = e_{Y \subseteq X, \lambda} \quad (2.1)$$

be the exponent (§1.1.1) of the polarization  $\iota_Y^* \lambda := \widehat{\iota} \circ \lambda \circ \iota$ , the pullback of  $\lambda$  by  $\iota_Y$ :

$$\begin{array}{ccc} Y & \xrightarrow{\iota_Y^* \lambda} & \widehat{Y} \\ \downarrow \iota & & \uparrow \widehat{\iota} \\ X & \xrightarrow{\lambda} & \widehat{X} \end{array}$$

Define the norm map  $N_Y = N_{Y \subseteq X, \lambda}: X \rightarrow X$  as the composition

$$\begin{array}{ccccccc} & & & N_Y & & & \\ & & & \curvearrowright & & & \\ Y & \xrightarrow{\iota} & X & \xrightarrow{\lambda} & \widehat{X} & \xrightarrow{\widehat{\iota}} & \widehat{Y} & \xrightarrow{e_Y \cdot (\iota_Y^* \lambda)^{-1}} & Y & \xrightarrow{\iota} & X \\ & & & \curvearrowleft & & & & & & & \\ & & & M_Y & & & & & & & \end{array} \quad (2.2)$$

We similarly define the map  $M_Y: X \rightarrow Y$  as indicated in the diagram.

*Remark 2.2* (Norm maps and base change). The norm map  $N_Y$  is stable under faithfully flat base change (since the exponent  $e_Y$  is), and if the exponent  $e_Y$  is stable under arbitrary base extension, e.g.,  $\iota_Y^* \lambda$  is a positive integer multiple of a principal polarization (see Lemma 1.1), then so is the norm map  $N_Y$ . All of the constructions and results that follow in §2.1 have the same base change properties.

**Lemma 2.3.** *We have*

- (a)  $M_Y \iota = N_Y|_Y = [e_Y]_Y$ .
- (b)  $N_Y^2 = e_Y N_Y$ .

*Proof.* Assertion (b) is a straight forward computation from the definitions (see [BL04, Lem. 5.3.1], where the same argument works verbatim); (a) then follows (see e.g., [BL04, p. 125, eq. (1)]).  $\square$

**Lemma 2.4.** *We have*

$$[e_Y](\ker N_Y) \subseteq \text{im}([e_Y] - N_Y) \subseteq \ker N_Y.$$

*Proof.* On one hand, since  $N_Y|_{\ker(N_Y)} = [0]$ , we have

$$[e_Y](\ker N_Y) = ([e_Y] - N_Y)(\ker N_Y) \subseteq ([e_Y] - N_Y)(X) = \text{im}([e_Y] - N_Y).$$

On the other hand, using Lemma 2.3(b), we see that

$$N_Y([e_Y] - N_Y) = N_Y[e_Y] - N_Y^2 = e_Y N_Y - e_Y N_Y = 0,$$

and so  $N_Y(\text{im}([e_Y] - N_Y)) = (0)$ .  $\square$

To proceed, we place a modest condition on the polarized abelian scheme  $(X, \lambda)$ .

**Lemma 2.5.** *If  $(X, \lambda)/S$  satisfies Condition 2.1, then the group scheme  $\ker(N_Y)$  admits a maximal abelian subscheme  $\ker(N_Y)^{\text{ab}}$ , and  $\dim \ker(N_Y)^{\text{ab}} = \dim \ker(N_Y)$ .*

*Proof.* The exact sequence of abelian schemes

$$0 \longrightarrow Y \xrightarrow{\iota} X \longrightarrow X/Y \longrightarrow 0$$

yields a dual exact sequence

$$0 \longrightarrow \widehat{X/Y} \longrightarrow \widehat{X} \longrightarrow \widehat{Y} \longrightarrow 0.$$

Let  $Z = \lambda^{-1}(\widehat{X/Y})^{\text{ab}}$  (Lemma 1.17). Then  $Z \subseteq \ker(N_Y)$ , and  $\dim Z = \dim \lambda^{-1}(\widehat{X/Y}) = \dim \ker(N_Y)$ . (The last equality holds because  $e_Y(\iota^*\lambda)^{-1}$  is an isogeny.)  $\square$

For reasons we explain below in §2.1.1, we call the abelian scheme

$$Z := \ker(N_Y)^{\text{ab}} = \lambda^{-1}(\widehat{X/Y})^{\text{ab}} \quad (2.3)$$

constructed in Lemma 2.5 *the complement of  $Y$  in  $X$  (with respect to  $\lambda$ )*.

Alternatively, we could define  $Z$  using the following equivalent formulation.

**Lemma 2.6.** *We have*

$$\ker(N_Y)^{\text{ab}} = \text{im}([e_Y]_X - N_Y).$$

*In particular,  $N_Y$  and  $[e_Y]_X - N_Y$  are epi-abelian.*

*Proof.* Using Lemma 2.4, we see that

$$[e]Z = Z \subseteq \text{im}([e] - N_Y) \subseteq \ker(N_Y).$$

Since  $\dim_S Z = \dim_S \ker(N_Y)$ , the result follows from [ACMW24, Thm. C].  $\square$

*Remark 2.7.* When  $S = \text{Spec } K$  for a field  $K$ , we have  $Z = ((\ker N_Y)^\circ)_{\text{red}}$  (Remark 1.18), and in particular, over a field of characteristic 0 we have  $Z = (\ker N_Y)^\circ$ .

**Lemma 2.8.** *We have  $(\ker N_Y) \times_X Y = Y[e]$ .*

*Proof.* We have  $(\ker N_Y) \times_X Y = \ker(N_Y|_Y) = Y[e]$  (Lemma 2.3(a)).  $\square$

2.1.1. *Complements are complementary.* We now establish a few results justifying the terminology that  $Z$  is complementary to  $Y$ , and showing that  $Y$  is the complement of  $Z$ .

**Lemma 2.9.** *The sub-abelian schemes  $Y$  and  $Z$  are complementary in the sense that the addition map*

$$\mu := \iota_Y + \iota_Z : Y \times_S Z \longrightarrow X \quad (2.4)$$

*is an isogeny of abelian schemes over  $S$ .*

*The kernel of  $\mu$  is*

$$\ker(\mu) = Y \times_X Z \subseteq Y \times_X (\ker N_Y) = Y[e], \quad (2.5)$$

*so that  $\deg \mu \leq e^{2 \dim Y}$ , and moreover, equality holds in (2.5), as well as for the degree inequality, if  $\ker N_Y$  is an abelian  $S$ -scheme; i.e., if the natural inclusion  $Z \subseteq \ker N_Y$  is an equality.*

*Proof.* Since  $\ker(\mu) = Y \times_X Z \subseteq Y \times_X (\ker N_Y) = Y[e]$  (Lemma 2.8) has relative dimension zero over  $S$ , it follows that  $\mu$  is an isogeny onto its image. Considering the definition of the norm map (2.2), and Lemma 2.3(b), it follows that  $\text{Im } N_Y = Y$ , and therefore  $\dim_S Y + \dim_S (\ker N_Y) = \dim_S X$ . Thus, as  $\dim_S(Y) + \dim_S(Z) = \dim_S(Y) + \dim_S(\ker N_Y) = \dim_S(X)$ , the image of  $\mu$  is all of  $X$ . The assertion on degrees now follows from the containment  $\ker(\mu) \subseteq Y[e]$ . Finally, if  $\ker N_Y$  is an abelian scheme, then we have  $Z = \ker N_Y$  (Lemma 2.6), so that  $Y \times_X Z = Y \times_X (\ker N_Y) = Y \times_X (\ker N_Y)$ , which equals  $Y[e]$  by Lemma 2.8.  $\square$

Now consider the norm map  $N_Z = N_{Z \subseteq X, \lambda}$  for the complement  $Z$  of  $Y$  in  $X$ . For later reference, we collect a few results about the interactions of  $N_Y$  and  $N_Z$ :

**Lemma 2.10.** *We have*

- (a)  $N_Y|_Y = [e_Y]$  and  $N_Z|_Z = [e_Z]$ .
- (b)  $N_Y^{(\dagger)} = N_Y$ , and  $N_Z^{(\dagger)} = N_Z$ , where  $(\dagger)$  is the Rosati involution induced by  $\lambda$ .
- (c)  $N_Z N_Y = 0$  and  $N_Y N_Z = 0$ , so that  $N_Y|_Z = 0$  and  $N_Z|_Y = 0$ .
- (d)  $e_Y N_Z + e_Z N_Y = [e_Y e_Z]_X$ .

*Proof.* (a) is just a restatement of Lemma 2.3(a). (b) is a straight forward computation from the definitions (see [BL04, Lem. 5.3.1], where the same argument works verbatim). For (c), since the image of  $N_Z$  is  $Z$ , and  $Z$  is contained in the kernel of  $N_Y$ , we have  $N_Y N_Z = 0$ . For the other claim, use the fact that  $(N_Z N_Y)^{(\dagger)} = N_Y^{(\dagger)} N_Z^{(\dagger)} = N_Y N_Z$ . For (d), this follows immediately on restriction to  $Y$  and  $Z$  from (a) and (c). Then, using the fact that the addition map  $\mu : Y \times_S Z \rightarrow X$  is surjective, (d) follows for all of  $X$ . (See also [BL04, p. 125, eqs. (1)–(4)] for  $S = \text{Spec } \mathbb{C}$ .)  $\square$

**Lemma 2.11.** *The image of  $[e_Z] - N_Z$  is  $Y$ .*

*Proof.* The image of  $[e_Z] - N_Z$  is the largest abelian scheme contained in the kernel of  $N_Z$  (apply Lemma 2.4 and Lemma 2.6 to  $Z \subseteq X$ ); now use a dimension count.  $\square$

Thus,  $(Y_0, Y_1) := (Y, Z)$  form a complementary pair in a symmetric sense:

$$N_{Y_i}(Y_i) = Y_i, \quad (2.6)$$

$$([e_{Y_i}] - N_{Y_i})(Y_i) = Y_{1-i}. \quad (2.7)$$

*Remark 2.12.* The map

$$(N_Y, N_Z) : X \longrightarrow Y \times_S Z$$

is an isogeny. In fact, this follows from Lemma 2.10(d), which says that  $(e_Y N_Y, e_Z N_Z)$  is an isogeny inverse of  $\mu$ , in the sense that  $\mu \circ (e_Y N_Y, e_Z N_Z) = [e_Y e_Z]_X$ , or alternatively, that

$$\mu^{-1} = \frac{1}{e_Y e_Z} (e_Y N_Y, e_Z N_Z) \in \text{Hom}_{\mathbb{Q}}(X, Y \times_S Z).$$

(If  $S = \text{Spec } \mathbb{C}$ , this is [BL04, Thm. 5.3.5].)

**2.2. Poincaré reducibility.** We can use these ideas to prove Poincaré reducibility for polarizable abelian schemes (abelian schemes that admit a polarization). The starting point is:

**Theorem 2.13.** *For a polarized abelian scheme  $(X, \lambda)$  over  $S$ , and an abelian subscheme  $Y \subseteq X$  with complement  $Z \subseteq X$ , the addition map*

$$\mu : (Y, \iota_Y^* \lambda) \times (Z, \iota_Z^* \lambda) \longrightarrow (X, \lambda)$$

*is an isogeny of polarized abelian schemes.*

*Proof.* The proof of [BL04, Cor. 5.3.6] holds in this situation, using Lemma 2.10.  $\square$

From this one obtains:

**Corollary 2.14.** *Suppose that  $S$  satisfies Condition 1.15. Given a polarizable abelian scheme  $X$  over  $S$ , there is an isogeny*

$$X_1^{n_1} \times_S \cdots \times_S X_r^{n_r} \longrightarrow X \quad (2.8)$$

*with simple polarizable abelian schemes  $X_v$  over  $S$  not isogenous to each other. Moreover, the polarizable abelian schemes  $X_v$  and the integers  $n_v$  are uniquely determined up to isogenies and permutations. Finally, if  $X$  is projective over  $S$ , so are the  $X_v/S$ .*

*Proof.* (The cases  $S = \text{Spec } \mathbb{C}$ ,  $\text{Spec } \bar{K}$  and  $\text{Spec } K$  may be found, respectively, in [BL04, Thm. 5.7], [Mum08, §19 Thm. 1], [Con06, Cor. 3.20].) We may and do fix a polarization on  $X$ . Utilizing Theorem 2.13, one obtains such a decomposition by induction on dimension (since a non-simple abelian scheme admits a nontrivial sub-abelian scheme, which in turn admits a complement), and replacing isogenous abelian schemes with copies of one another.

The uniqueness can be shown as follows. Given another such isogeny  $Y_1^{m_1} \times_S \cdots \times_S Y_t^{m_t} \rightarrow X$ , one obtains an isogeny  $X \rightarrow Y_1^{m_1} \times_S \cdots \times_S Y_t^{m_t}$  from Corollary 1.7; then by considering the composition  $X_1^{n_1} \times_S \cdots \times_S X_r^{n_r} \rightarrow X \rightarrow Y_1^{m_1} \times_S \cdots \times_S Y_t^{m_t}$  and the projections onto each component, one concludes uniqueness, as a nonzero map between simple abelian schemes is an isogeny. Note that this also shows that the decomposition is independent of the choice of polarization of  $X$ .

Finally, if  $L$  on  $X$  is relatively ample, then the pull-backs of  $L$  via (2.8) are also relatively ample.  $\square$

*Remark 2.15.* Note that the decomposition (2.8) in Corollary 2.14 is *not* stable under base change. For instance one can construct families of principally polarized abelian varieties over  $\mathbb{C}$  (e.g., Jacobians of curves) whose very general fibers are simple, but which degenerate to products of abelian varieties. Such a family must be simple, since the very general fiber is simple, but the special fiber is not simple. In fact, in contrast to other base change properties we have been discussing, the decomposition in Corollary 2.14 is not even stable under faithfully flat base change. Indeed, there are examples of abelian varieties that are simple, but not geometrically simple. For instance, let  $L/K$  be a nontrivial finite separable extension of fields, and let  $X$  be a geometrically simple abelian variety over a field  $L$  which admits no model over  $L$  (e.g., let  $K = \mathbb{Q}$ , let  $L = \mathbb{Q}(j)$  for some irrational algebraic number  $j$ , and let  $X/L$  be an elliptic curve with  $j$ -invariant  $j$ ). Let  $Y = R_{L/K} X$  be the Weil restriction of  $X$  to  $K$ . Then  $Y$  is simple, but not geometrically simple. For example, if  $L/K$  is Galois and  $K$  is finitely generated, then  $H^1(Y_{\bar{K}}, \mathbb{Q}_\ell)$  is the induced representation  $\text{Ind}_{\text{Gal}(L)}^{\text{Gal}(K)}(H^1(X_{\bar{L}}, \mathbb{Q}_\ell))$ , and thus irreducible as a representation of  $\text{Gal}(K)$ , while  $Y_L \cong \prod_{\sigma \in \text{Gal}(L/K)} X^\sigma$ ; see, e.g., [Mil72, §1].

We also note the following consequence of Theorem 2.13; recall that a polarized abelian scheme is said to be *indecomposable* if it is not isomorphic as a polarized abelian scheme to a product of polarized abelian schemes.

**Corollary 2.16.** *If  $e_Y = 1$ , then  $(X, \lambda) \cong (Y, \iota_Y^* \lambda) \times_S (Z, \iota_Z^* \lambda)$ . In particular, if  $(X, \lambda)$  is indecomposable, then  $e_Y = 1$  if and only if  $Y = 0$  or  $Y = X$ ; if  $Y = X$ , then  $\lambda$  is principal.*

*Proof.* This is a direct consequence of Theorem 2.13 and Lemma 2.9.  $\square$

**2.3. Complements inside principally polarized abelian varieties.** In the special, but extremely useful, case where  $\lambda$  is a principal polarization, we can say more. Since  $\lambda$  is an isomorphism, the complement  $Z$  of  $Y$  is simply

$$Z = \lambda^{-1}(\ker \widehat{\iota_Y}) \cong \widehat{X/Y}. \quad (2.9)$$

In fact, there is a short exact sequence

$$0 \longrightarrow Y \xrightarrow{\iota_Y} X \xrightarrow{\widehat{\iota_Z} \lambda} \widehat{Z} \longrightarrow 0 \quad (2.10)$$

and under the isomorphism  $\widehat{Z} \cong X/Y$ , the composition  $\iota_Z^* \lambda : Z \xrightarrow{\iota_Z} X \xrightarrow{\widehat{\iota_Z} \lambda} \widehat{Z}$  is identified with the composition  $Z \xrightarrow{\iota_Z} X \rightarrow X/Y$ .

Moreover, we can characterize the kernel of the induced polarization on  $Y$ .

**Corollary 2.17.** *If  $\lambda$  is principal, then  $K(\iota_Y^* \lambda) = Y \times_X Z \subseteq Y$ .*

*Proof.* (See [BL04, Cor. 12.1.4] for the case  $S = \text{Spec } \mathbb{C}$ .) We have  $K(\iota_Y^* \lambda) := \ker \iota_Y^* \lambda = \ker(\widehat{\iota}_Y \lambda_{\iota_Y}) = \iota_Y^{-1} \lambda^{-1}(\ker \widehat{\iota}_Y) = \iota_Y^{-1} Z := Y \times_X Z$ .  $\square$

*Remark 2.18.* Note by symmetry (e.g., (2.7)) that if  $\lambda$  is principal we also have

$$K(\iota_Z^* \lambda) = Z \times_X Y = Z \times_X Y \subseteq Z,$$

so that

$$K(\iota_Z^* \lambda) \cong K(\iota_Y^* \lambda).$$

In particular, the exponents of the kernels of  $\iota_Y^* \lambda$  and  $\iota_Z^* \lambda$  are the same; i.e.,

$$e_Y = e_Z.$$

In addition, it follows that if either of  $\iota_Z^* \lambda$  or  $\iota_Y^* \lambda$  has a type (see §1.1.1), then so does the other. In fact, without loss of generality, if  $\dim Z \leq \dim Y$ , and  $Z$  has type  $\delta = (\delta_1, \dots, \delta_{\dim Z})$ , then  $Y$  has type  $(1, \dots, 1, \delta_1, \dots, \delta_{\dim Z})$ . (See [BL04, Cor. 12.1.5] for the case  $S = \text{Spec } \mathbb{C}$ .)

**Corollary 2.19.** *If  $\lambda$  is principal and  $e_Y = 1$  (or equivalently,  $e_Z = 1$ ), then  $\iota_Y^* \lambda$  and  $\iota_Z^* \lambda$  are principal polarizations and  $(X, \lambda) \cong (Y, \iota_Y^* \lambda) \times_S (Z, \iota_Z^* \lambda)$ . In particular, if  $(X, \lambda)$  is indecomposable, then  $e_Y = e_Z = 1$  if and only if  $Y = X$  or  $Z = X$ .*

*Proof.* This follows immediately from Corollary 2.16 and Remark 2.18.  $\square$

**2.4. Symmetric idempotents, projectors, and abelian subschemes.** We now show that norm endomorphisms and abelian subschemes are equivalent.

We start with the observation that if we set  $\epsilon_Y = \frac{1}{e_Y} N_Y \in \text{End}_{\mathbb{Q}}(X)$ , then as a consequence of Lemma 2.3(b) and Lemma 2.10(b), we have

$$\epsilon_Y^{(+)} = \epsilon_Y \quad \epsilon_Y^2 = \epsilon_Y.$$

We call any endomorphism  $\epsilon \in \text{Hom}_{\mathbb{Q}}(X)$  such that  $\epsilon^{(+)} = \epsilon$  and  $\epsilon^2 = \epsilon$  a *symmetric idempotent*, or a symmetric projector. For a symmetric idempotent  $\epsilon$  there is a positive integer  $n$  such that  $n\epsilon \in \text{End}(X)$ , and we define

$$X^\epsilon := \text{Im}(n\epsilon)$$

to be the image of  $n\epsilon$ ; this is independent of the choice of  $n$ . This construction is only useful to us if the image is an abelian scheme, in which case we will say that  $\epsilon$  is a *symmetric epi-abelian idempotent*. (The reader should not be too alarmed at this apparent restriction. On one hand, in many natural settings, an idempotent, like any morphism of abelian schemes, is epi-abelian (Lemma 1.16); on the other hand, the authors have never encountered a symmetric idempotent which is not epi-abelian.)

We therefore have maps of sets

$$\begin{array}{ccc} & \xrightarrow{\epsilon_{(-)}} & \\ \{\text{Abelian subschemes of } X\} & & \{\text{Symmetric epi-abelian idempotents in } \text{End}_{\mathbb{Q}}(X)\} \\ & \xleftarrow{X^{(-)}} & \end{array}$$

**Theorem 2.20.** *The maps  $X^{(-)} \circ \epsilon_{(-)}$  and  $\epsilon_{(-)} \circ X^{(-)}$  are the identity maps.*

*Proof.* The fact that  $X^{(-)} \circ \epsilon_{(-)}$  is the identity map comes down to the fact that by definition, given an abelian subscheme  $Y$  of  $X$ , the image of  $N_Y$  is  $Y$  (see (2.2) and Lemma 2.3(a)). Note that this shows also that  $X^{(-)}$  is surjective.

Therefore, for the converse, it suffices to show that  $X^{(-)}$  is injective; i.e., given two symmetric epi-abelian idempotents  $\epsilon_1$  and  $\epsilon_2$  such that  $X^{\epsilon_1} = X^{\epsilon_2}$ , one must show  $\epsilon_1 = \epsilon_2$ . By rigidity, it suffices to show this when restricted to geometric fibers. The rest of the argument is identical to

[BL04, Thm. 5.3.2] as the Rosati involution is stable under base change, so that the proof reduces to the positive definiteness of the trace pairing on geometric fibers [Mum08, §21, Thm. 1].  $\square$

**Corollary 2.21.** *If  $f \in \text{End}(X)$  and  $Y = \text{Im}(f)$  is an abelian subscheme, then  $f = N_Y$  if and only if  $f^2 = e_Y f$  and  $f^{(+)} = f$ .*

*Proof.* The forward implication is Lemma 2.3(b) and Lemma 2.10. For the converse, we have  $\epsilon := \frac{f}{e}$  is a symmetric epi-abelian idempotent, and then one employs Theorem 2.20. (When  $S = \text{Spec } \mathbb{C}$ , this is [BL04, Cor. 5.3.3, Crit. 5.3.4].)  $\square$

**Corollary 2.22.** *There is an involution on the set of symmetric epi-abelian idempotents given by  $\epsilon \mapsto (1 - \epsilon)$ . If  $(Y_0, Y_1)$  are complementary abelian subschemes, then  $\epsilon_{Y_i} = 1 - \epsilon_{Y_{1-i}}$ .*

*Proof.* The first assertion is clear. The second follows from Theorem 2.20 and (2.7).  $\square$

**Corollary 2.23.** *If  $\lambda$  is principal, then  $N_Y$  is primitive.*

*Proof.* Due to Proposition 1.6, we only have to show that  $\ker N_Y \not\cong X[n]$  for any integer  $n \geq 2$ . This follows from the definition (2.2), since  $\lambda$  is assumed to be an isomorphism,  $\hat{\lambda}$  has connected kernel (Lemma 1.3), and  $e_Y(\iota_Y^* \lambda)^{-1}$  is primitive (Remark 1.8). (If  $S = \text{Spec } \mathbb{C}$ , this is [BL04, Crit. 5.3.4].)  $\square$

**2.5. Complements inside principally polarized abelian varieties: automorphisms.** We return now to the case that  $\lambda$  is a principal polarization, and we assume that the principally polarized abelian  $S$ -scheme  $(X, \lambda)$  admits an automorphism  $\sigma$  over  $S$  of finite order  $n$ . In particular,

$$\hat{\sigma}\lambda = \lambda\sigma \quad \text{and} \quad \sigma^n = 1.$$

To make progress, suppose that the endomorphism

$$\tau := 1 + \sigma + \sigma^2 + \cdots + \sigma^{n-1} \in \text{End}(X)$$

is epi-abelian. (This holds in the key case of interest where  $X = \text{Pic}_{C/S}^0$  is the Picard scheme of a smooth projective curve and  $\sigma$  is induced by an automorphism of  $C$ , as in Remark 2.30, as well as over any base scheme  $S$  which satisfies Condition 1.15.) Now consider the abelian subscheme

$$Y := \text{im}(\tau) \subseteq X.$$

For context, define the  $\sigma$ -fixed subgroup scheme

$$X^\sigma := \ker(1 - \sigma),$$

and observe that  $(1 - \sigma)(1 + \sigma + \sigma^2 + \cdots + \sigma^{n-1}) = 0$ , so that  $Y \subseteq X^\sigma$ . At the same time, since on  $X^\sigma$  we have that  $\sigma = 1$ , it follows that on  $X^\sigma$  we have  $\tau|_{X^\sigma} = [n]_{X^\sigma}$ . Therefore,  $[n]X^\sigma = \tau X^\sigma \subseteq \tau X = Y$ . In other words,

$$[n]X^\sigma \subseteq Y \subseteq X^\sigma.$$

As in the proof of Lemma 2.6, we conclude that  $Y = (X^\sigma)^{\text{ab}}$ .

Next observe that since  $(1 + \sigma + \sigma^2 + \cdots + \sigma^{n-1})^2 = n(1 + \sigma + \sigma^2 + \cdots + \sigma^{n-1})$ , and  $\sigma^{(+)} = \lambda^{-1}\hat{\sigma}\lambda = \lambda^{-1}\lambda\sigma = \sigma$  we have that

$$\frac{1}{n} \left( 1 + \sigma + \sigma^2 + \cdots + \sigma^{n-1} \right) = \frac{1}{n} \tau = \epsilon_Y$$

is the symmetric idempotent associated to  $Y$  (Theorem 2.20).

Now setting  $Z$  to be the complement of  $Y$  in  $X$ , we have from Corollary 2.22 that  $\epsilon_Z = 1 - \epsilon_Y$ . Consequently,  $Z$  is the image of

$$n\epsilon_Z = (n - 1) - \sigma - \sigma^2 - \cdots - \sigma^{n-1}. \tag{2.11}$$

Since  $n e_Y$  is an endomorphism of  $X$  and  $N_Y$  is primitive (Corollary 2.23), we have  $e_Y \mid n$ , and if  $e_Y = 1$  then both  $Y$  and  $Z$  are principally polarized, and  $(X, \lambda) \cong (Y, \iota_Y^* \lambda) \times_S (Z, \iota_Z^* \lambda)$  as principally polarized abelian schemes (Corollary 2.19). As a special case, if  $n$  is prime,  $(X, \lambda)$  is indecomposable as a principally polarized abelian scheme, and  $Y \neq 0$ , then  $e_Y = n = e_Z$ ; moreover, in this case, if  $\iota_Y^* \lambda$  has a type  $\delta$ , then  $\delta = (1, \dots, 1, n, \dots, n)$ , and the type of  $\iota_Z^* \lambda$  is  $(1, \dots, 1, n, \dots, n)$ , where the number of times  $n$  appears in each type is the same (Remark 2.18).

**2.6. Definitions of Prym schemes.** For clarity, we review several definitions of Prym schemes related to various usages in the literature, where there is some ambiguity, depending on the context. This section serves to fix the terminology we will be using. Roughly speaking, for varieties over a field  $K$ , an abelian subvariety  $Z \subseteq \text{Pic}_{C/K}^0$  of a Jacobian of a curve  $C/K$  will be called a Prym–Tyurin variety if the restriction to  $Z$  of the canonical polarization on the Jacobian is a multiple of a principal polarization on  $Z$ , and will be called a Prym variety if there is a finite cover of curves  $f : C \rightarrow C'$  so that  $Z$  is the complement of  $f^* \text{Pic}_{C'/K}^0$ , whether or not the restriction of the canonical polarization to  $Z$  is a multiple of a principal polarization.

**2.6.1. Prym–Tyurin scheme.** An embedded Prym–Tyurin scheme of exponent  $e$  over  $S$  is a tuple

$$(Z, \xi, C, \iota_Z)$$

such that  $(Z, \xi)$  is a principally polarized abelian scheme over  $S$ ,  $C$  is a smooth proper curve over  $S$ , and

$$\iota_Z : Z \longrightarrow \text{Pic}_{C/S}^0$$

is a closed subabelian  $S$ -scheme such that

$$\iota_Z^* \lambda_C = e \xi,$$

where  $\lambda_C$  is the canonical principal polarization on  $\text{Pic}_{C/S}^0$ .

*Remark 2.24.* We note that  $e_Z = e$ ; i.e., the exponent  $e_Z$  of  $Z$  as a subvariety of  $\text{Pic}_{C/S}^0$  with respect to  $\lambda_C$ , in the sense of §2.1, is  $e$ , explaining the terminology. Indeed, the exponent  $e_Z$  was defined as the exponent of the isogeny  $\iota_Z^* \lambda_C$  (§1.3).

*Remark 2.25.* Given a smooth proper curve  $C/S$  and an abelian subscheme  $\iota_Z : Z \hookrightarrow \text{Pic}_{C/S}^0$  over  $S$ , we have from Lemma 1.11 that  $(Z, \frac{1}{e} \iota_Z^* \lambda_C, C, \iota_Z)$  is an embedded Prym–Tyurin scheme of exponent  $e$  over  $S$  if and only if

$$\mathbb{K}(\iota_Z^* \lambda_C) = Z[e] := \ker[e]_Z,$$

if and only if for every point  $s$  of  $S$  we have  $(Z_s, \frac{1}{e} \iota_{Z_s}^* \lambda_{C_s}, C_s, \iota_{Z_s})$  is an embedded Prym–Tyurin variety of exponent  $e$ .

A Prym–Tyurin scheme of exponent  $e$  over  $S$  is a principally polarized abelian scheme  $(Z, \xi)$  over  $S$  such that there exists a tuple  $(Z, \xi, C, \iota_Z)$  that is an embedded Prym–Tyurin scheme of exponent  $e$ . In this situation, we also say that  $(Z, \xi)$  is a Prym–Tyurin scheme for the curve  $C/S$  and that  $e$  is the exponent of the Prym–Tyurin scheme  $(Z, \xi)$ .

**2.6.2. Generalized Prym–Tyurin scheme.** An embedded generalized Prym–Tyurin scheme of generalized exponent  $m$  is a tuple

$$(Z, \xi, C, \iota_Z, \sigma)$$

where  $(Z, \xi, C, \iota_Z)$  is an embedded Prym–Tyurin scheme of some exponent  $e$ , and  $\sigma \in \text{End}(\text{Pic}_{C/S}^0)$  is an epi-abelian endomorphism, symmetric with respect to the Rosati involution given by the canonical polarization  $\lambda_C$  on  $\text{Pic}_{C/S}^0$ , such that

$$\sigma^2 + (m-2)\sigma - (m-1) = (1-\sigma)((1-\sigma) - m) = 0, \tag{2.12}$$

and  $\iota_Z(Z) = \text{Im}(1 - \sigma)$ . (In particular,  $1 - \sigma$  is epi-abelian.) For simplicity of notation, we will identify  $Z$  with  $\iota_Z(Z)$  going forward.

**Warning 2.26.** Note that the terminology in the definition for generalized Prym–Tyurin schemes may be confusing, in that one need not have the generalized exponent  $m$  equal to the exponent  $e$  (see Lemma 2.28).

*Remark 2.27.* When  $S = \text{Spec } K$  for a field  $K$ , this definition is due to Bloch–Murre [BM79] and Kanev [Kan87]. Note that over a field there is a surjection (discussed in §4.2)  $\text{CH}^1(C \times_K C) \rightarrow \text{End}(\text{Pic}_{C/K}^0)$  given by taking a correspondence to the associated endomorphism. Bloch–Murre and Kanev give their definition in terms of correspondences, but we prefer to work directly with the associated endomorphism.

It is worth noting that (2.12) is equivalent to  $(1 - \sigma)^2 = m(1 - \sigma)$ , so that by virtue of Theorem 2.20, the condition (2.12) is equivalent to the statement that  $\frac{1}{m}(1 - \sigma) \in \text{End}_{\mathbb{Q}}(\text{Pic}_{C/S}^0)$  is the symmetric epi-abelian idempotent  $\epsilon_Z = \frac{1}{e}N_Z$  associated to  $Z$ .

**Lemma 2.28** (Prym–Tyurin is generalized Prym–Tyurin). *Given an embedded Prym–Tyurin scheme  $(Z, \xi, C, \iota_Z)$  of exponent  $e$ , then for every positive integer  $a$ , setting  $\sigma = 1 - aN_Z \in \text{End}(\text{Pic}_{C/K}^0)$ , we have that  $(Z, \xi, C, \iota_Z, \sigma)$  is an embedded generalized Prym–Tyurin scheme of generalized exponent  $ae$ ; i.e., (2.12) holds with  $m = ae$ .*

*Conversely, if  $(Z, \xi, C, \iota_Z, \sigma)$  is an embedded generalized Prym–Tyurin scheme of exponent  $m$  such that  $(Z, \xi, C, \iota_Z)$  is an embedded Prym–Tyurin scheme of exponent  $e$ , then  $e \mid m$  and  $\sigma = 1 - \frac{m}{e}N_Z$ .*

*Proof.* On one hand, suppose  $(Z, \xi, C, \iota_Z)$  is an embedded Prym–Tyurin scheme of exponent  $e$ , and let  $a$  be a positive integer. Consider the endomorphism  $\sigma = 1 - aN_Z$ . Since  $N_Z|_Y = 0$  and  $Y = \text{Im}([e]_X - N_Z)$ , we have that  $N_Z([e]_X - N_Z) = 0$ . It follows that  $aN_Z([e]_X - aN_Z) = 0$ , and thus  $\sigma$  satisfies (2.12) with  $m = ae$ .

On the other hand, suppose  $(Z, \xi, C, \iota_Z, \sigma)$  is an embedded generalized Prym–Tyurin scheme of exponent  $m$  such that  $(Z, \xi, C, \iota_Z)$  is an embedded Prym–Tyurin scheme of exponent  $e$ . Then, as observed above, due to Theorem 2.20, we have  $\frac{1}{m}(1 - \sigma) = \frac{1}{e}N_Z$ . Thus  $1 - \sigma = \frac{m}{e}N_Z$ . Since the left hand side is an endomorphism and  $N_Z$  is primitive (Corollary 2.23), we have that  $e \mid m$ , and  $\sigma = 1 - \frac{m}{e}N_Z$ .  $\square$

A generalized Prym–Tyurin scheme of exponent  $m$  over  $S$  is a principally polarized abelian scheme  $(Z, \xi)$  over  $S$  such that there exists a tuple  $(Z, \xi, C, \iota_Z, \sigma)$  that is an embedded generalized Prym–Tyurin scheme of exponent  $m$ . From the lemma above, this is the same as a Prym–Tyurin scheme of exponent  $e$  for some  $e \mid m$ .

*Remark 2.29* (Welters). In [Wel87, Prop. 1.1], Welters gives an equivalent formulation of the condition (2.12); namely, the data of a symmetric endomorphism  $\sigma$  satisfying (2.12) is equivalent to the data of an abelian subscheme  $\iota: Z \hookrightarrow \text{Pic}_{C/S}^0$  such that  $K(\iota^*\lambda_C) \subseteq Z[m]$ . Again, in this situation  $Z = \text{Im}(1 - \sigma)$ . Welters’ proof is given over an algebraically closed field of characteristic 0, but the proof as written holds over any  $S$  which satisfies Condition 1.15, if one replaces connected components of kernels with largest abelian subschemes and images of abelian varieties with images of abelian schemes. Note that the condition that the data give a generalized Prym–Tyurin scheme is that  $K(\iota^*\lambda_C) = Z[e] \subseteq Z[m]$  for some  $e$ .

2.6.3. *Prym scheme.* An embedded Prym scheme of exponent  $e$  over  $S$  is a tuple

$$(Z, f: C \rightarrow C', \iota_Z)$$

such that  $Z$  is an abelian scheme over  $S$ ,  $C$  is a smooth proper curve over  $S$ ,  $\iota_Z: Z \hookrightarrow \text{Pic}_{C/S}^0$  is a closed subabelian  $S$ -scheme,  $f: C \rightarrow C'$  is a finite, thus flat (see §3.2),  $S$ -morphism of smooth

proper  $S$ -curves, and setting  $Y := f^* \text{Pic}_{C'/S}^0 \subseteq \text{Pic}_{C/S}^0$  (Lemma 3.7), we have  $e_Y = e$  and, identifying  $Z$  with  $\iota_Z(Z)$ , we have that  $Z$  is the complement of  $Y$ :

$$Z := \text{Im}([e]_X - N_Y). \quad (2.13)$$

A *Prym scheme of exponent  $e$*  over  $S$  is an abelian scheme  $Z/S$  such that there exists a tuple  $(Z, f: C \rightarrow C', \iota_Z)$  that is an embedded Prym scheme of exponent  $e$ .

Let  $f: C \rightarrow C'$  be a finite  $S$ -morphism of proper  $S$ -curves. We will see in Lemma 3.7 that  $Y := f^* \text{Pic}_{C'/S}^0$  is an abelian scheme. Setting  $e = e_Y$ , we have the associated Prym scheme of exponent  $e$ , which is defined to be the complement of  $Y$ :

$$P(C/C') := \text{Im}([e]_X - N_Y) = (\ker N_Y)^{\text{ab}} = (\text{Pic}_{C/S}^0 / f^* \text{Pic}_{C'/S}^0)^\wedge. \quad (2.14)$$

(The second equality follows from Lemma 2.6, while the third is our definition of the Prym.) In Corollary 3.10 we give yet another equivalent formulation. If we have  $\lambda_C|_{P(C/C')} = e\xi$  for some positive integer  $e$  and some principal polarization  $\xi$  on  $P(C/C')$ , then we have also a Prym–Tyurin scheme of exponent  $e$

$$(P(C/C'), \xi),$$

which we call the *Prym–Tyurin Prym scheme of exponent  $e$*  associated to the cover  $C/C'$ . In particular,  $(P(C/C'), \xi, C, \iota)$ , where  $\iota: P(C/C') \hookrightarrow \text{Pic}_{C/S}^0$  is the natural inclusion, is an embedded Prym–Tyurin scheme of exponent  $e$ .

*Remark 2.30 (Curves with automorphisms).* Let  $C/S$  be a smooth projective curve over  $S$ , which admits a finite étale group of  $S$ -automorphisms  $G \subseteq \text{Aut}_S(C)$ . The quotient  $C' := C/G$  exists as an algebraic space over  $S$ , being the quotient of a scheme by an étale equivalence relation, and the geometric fibers, being quotients of smooth projective curves by finite groups, are smooth projective curves. Provided the fibers have genus  $g' \geq 2$ , then  $C'$  is a projective scheme over  $S$ , via the relatively ample relative canonical bundle. In this case one has a morphism  $f: C \rightarrow C'$  as in the discussion above, and one can consider the associated Prym scheme  $P(C/C')$ . In the situation where  $G$  is generated by a single automorphism  $\sigma$  of order  $n$  (see §2.5), we note that  $Y = \text{Im}(1 + \sigma^* + (\sigma^*)^2 + \dots + (\sigma^*)^{n-1})$ . Indeed, by descent,  $Y = f^* \text{Pic}_{C'/S}^0$  parameterizes exactly those line bundles invariant under the action of  $G$ . (Note that in this case, the endomorphism  $\tau$  of §2.5 is automatically epi-abelian.) Consequently, in addition to the descriptions of  $P(C/C')$  in (2.14), one has  $P(C/C') = \text{Im}((n-1) - \sigma^* - (\sigma^*)^2 - \dots - (\sigma^*)^{n-1})$  (2.11). In this case, since  $\lambda_C$  is an indecomposable polarization, and  $g' \geq 2$  implies that  $Y \neq 0$  (see e.g., Lemma 3.5 for a reminder), one has that  $2 \leq e \mid n$ , and therefore, if  $n$  is prime one has  $e = n$  (see §2.5). See also Corollary 3.10 for another description of  $P(C/C')$ , and §2.5 for a discussion of the polarization type.

### 3. ABEL MAPS FOR CURVES

Let  $C/S$  be a smooth proper curve over  $S$ , with canonically principally polarized Jacobian  $(\text{Pic}_{C/S}^0, \lambda_C)$ . Recall from §1 that there is an Albanese torsor of  $C/S$ :

$$\alpha^{(1)}: C \longrightarrow \text{Pic}_{C/S}^{(1)}$$

with  $\text{Pic}_{C/S}^{(1)}$  a torsor under  $\text{Pic}_{C/S}^0$ , which is initial for  $S$ -morphisms from  $C$  into torsors under abelian schemes over  $S$ . If  $C/S$  admits a section  $P: S \rightarrow C$ , then there is a pointed Albanese scheme

$$\alpha_P: C \longrightarrow \text{Pic}_{C/S}^0$$

which is initial for  $S$ -morphisms from  $C$  into abelian schemes over  $S$  taking  $P$  to the zero section.

Recall that there is a canonical isomorphism (1.6):

$$\widehat{\text{Pic}}_{C/S}^0 \xrightarrow[\sim]{\zeta} \text{Pic}_{\text{Pic}_{C/S}^{(1)}/S}^0. \quad (3.1)$$

### 3.1. Basic properties of the Abel map.

**Lemma 3.1.** *There is an equality of morphisms*

$$\text{Pic}_{C/S}^0 \xrightarrow[\alpha^{(1)*\zeta^{-1}}]{-\lambda_C} \widehat{\text{Pic}}_{C/S}^0.$$

In particular,  $\widehat{\text{Pic}}_{C/S}^0 \xrightarrow{\alpha^{(1)*\zeta}} \text{Pic}_{C/S}^0$  is an isomorphism.

*Proof.* Using rigidity (e.g., [MFK94, Prop. 6.1]), it suffices to check equality of the morphisms along a single fiber over  $S$ . Having restricted to a fiber, it suffices to verify the claims after passage to the algebraic closure, at which point the result is standard; see [Mil08, Lem. III.6.9]. (See also [BL04, Lem. 11.3.1, Prop. 11.3.5] for the case  $S = \text{Spec } \mathbb{C}$ .)  $\square$

The universal property of the Albanese torsor means that if  $\beta: C \rightarrow T$  is an  $S$ -morphism to a torsor under an abelian scheme  $Z/S$ , then there is a uniquely determined  $S$ -morphism  $\tilde{\beta}: \text{Pic}_{C/S}^{(1)} \rightarrow T$  which makes the following diagram commute:

$$\begin{array}{ccc} C & \xrightarrow{\beta} & T \\ \alpha^{(1)} \downarrow & \nearrow \tilde{\beta} & \\ \text{Pic}_{C/S}^{(1)} & & \end{array} \quad (3.2)$$

Pullback of line bundles by  $\beta$  induces an  $S$ -morphism  $\text{Pic}_{T/S} \rightarrow \text{Pic}_{C/S}$  which restricts to give the morphism of abelian schemes  $\beta^*: \text{Pic}_{T/S}^0 \rightarrow \text{Pic}_{C/S}^0$ . Moreover, pullback of line bundles by  $\tilde{\beta}$  induces a morphism  $\tilde{\beta}^*: \text{Pic}_{T/S}^0 \rightarrow \text{Pic}_{\text{Pic}_{C/S}^{(1)}/S}^0$ . Letting  $\zeta_Z: \widehat{Z} \rightarrow \text{Pic}_{T/S}^0$  be the canonical isomorphism (1.6), we denote by  $\widehat{\tilde{\beta}}$  the composition

$$\widehat{\tilde{\beta}} := \zeta^{-1} \tilde{\beta}^* \zeta_Z: \widehat{Z} \longrightarrow \widehat{\text{Pic}}_{C/S}^0. \quad (3.3)$$

**Lemma 3.2.** *There is an equality of morphisms*

$$\widehat{\tilde{\beta}} = -\lambda_C \beta^* \zeta_Z;$$

i.e., the following diagram commutes:

$$\begin{array}{ccccc} \widehat{Z} & \xrightarrow{\zeta_Z} & \text{Pic}_{T/S}^0 & \xrightarrow{\beta^*} & \text{Pic}_{C/S}^0 \\ & & \downarrow \tilde{\beta}^* & & \downarrow -\lambda_C \\ & & \text{Pic}_{\text{Pic}_{C/S}^{(1)}/S}^0 & \xrightarrow[\sim]{\zeta^{-1}} & \widehat{\text{Pic}}_{C/S}^0 \end{array} \quad (3.4)$$

*Proof.* The commutativity of (3.4) follows immediately from (3.2) and Lemma 3.1. Indeed, using from Lemma 3.1 that  $-\lambda_C = (\alpha^{(1)*\zeta})^{-1}$ , it is equivalent to show that  $\alpha^{(1)*\tilde{\beta}^*\zeta_Z = \beta^*\zeta_Z$ . As  $\zeta_Z$  is an isomorphism, it suffices to show  $\alpha^{(1)*\tilde{\beta}^* = \beta^*$ ; but this follows from the commutativity of (3.3). (The case  $S = \text{Spec } \mathbb{C}$  is [BL04, Cor. 11.4.2].)  $\square$

**3.2. Covers of curves.** Let  $C$  and  $C'$  be smooth proper curves over  $S$ , and let  $f: C \rightarrow C'$  be a finite  $S$ -morphism of degree  $d$ . Then  $f$  is a flat  $S$ -morphism. Indeed, since  $S$  is by hypothesis locally Noetherian, it suffices to verify that for each  $s \in S$ ,  $f_s: C_s \rightarrow C'_s$  is flat (e.g., [GW20, Cor. 14.27]); but this is immediate, since  $f_s$  is a finite morphism of smooth curves (e.g., [GW20, Prop. 14.14]).

On one hand, pullback gives an  $S$ -morphism

$$\mathrm{Pic}_{C'/S}^0 \xrightarrow{f^*} \mathrm{Pic}_{C/S}^0.$$

On the other hand, the universal property of the Albanese torsor means that there is a unique morphism  $f_*^{(1)}$  that makes the following diagram commute:

$$\begin{array}{ccc} C & \xrightarrow{\alpha^{(1)}} & \mathrm{Pic}_{C/S}^{(1)} \\ \downarrow f & & \downarrow f_*^{(1)} \\ C' & \xrightarrow{\alpha^{(1)}} & \mathrm{Pic}_{C'/S}^{(1)} \end{array} \quad (3.5)$$

and moreover,  $f_*^{(1)}$  is a morphism of torsors over a homomorphism

$$f_*: \mathrm{Pic}_{C/S}^0 \longrightarrow \mathrm{Pic}_{C'/S}^0.$$

*Remark 3.3.* The homomorphism  $f_*$  is denoted by  $N_f$  in [BL04, §11.4], and agrees with the norm map associated to  $f$  (e.g., [GW20, §12.6]). When  $S = \mathrm{Spec} k$  for an algebraically closed field  $k$ , then on  $k$ -points  $f_*$  is the restriction of the map  $\mathrm{Pic}_{C/S}(k) \rightarrow \mathrm{Pic}_{C'/S}(k)$  given by  $\sum a_P P \mapsto \sum a_P f(P)$ ; i.e., the morphism induced by proper push forward on codimension-1 cycle classes.

The homomorphisms  $f_*$  and  $f^*$  are essentially dual:

**Lemma 3.4.** *We have*

$$\widehat{f}_* \lambda_{C'} = \lambda_C f^*;$$

i.e., the following diagram commutes

$$\begin{array}{ccc} \mathrm{Pic}_{C'/S}^0 & \xrightarrow[\sim]{\lambda_{C'}} & \widehat{\mathrm{Pic}}_{C'/S}^0 \\ \downarrow f^* & & \downarrow \widehat{f}_* \\ \mathrm{Pic}_{C/S}^0 & \xrightarrow[\sim]{\lambda_C} & \widehat{\mathrm{Pic}}_{C/S}^0. \end{array}$$

In particular,  $\ker(\widehat{f}_*)$  and  $\ker(f^*)$  are isomorphic group schemes.

*Proof.* (If  $S = \mathrm{Spec} \mathbb{C}$ , this is [BL04, p. 331, eq(2)].) Consider the diagram (3.5) and apply  $\mathrm{Pic}_{(-)}^0$  and the canonical isomorphism  $\zeta$  (3.1). Then using Lemma 3.1, this reduces to showing that pull back of line bundles along  $f_*^{(1)}: \mathrm{Pic}_{C/S}^{(1)} \rightarrow \mathrm{Pic}_{C'/S}^{(1)}$  corresponds to pull back of line bundles along  $f^*: \mathrm{Pic}_{C/S}^0 \rightarrow \mathrm{Pic}_{C'/S}^0$ . By rigidity it suffices to check equality of the morphisms along a geometric fiber; over an algebraically closed field this is standard (e.g., [Mum74, p.328]).  $\square$

We further detail the relation between  $f_*$  and  $f^*$ .

**Lemma 3.5.** *We have*

$$f_* f^* = [d]_{\mathrm{Pic}_{C'/S}^0}.$$

*Proof.* Again, by rigidity we may check along a geometric fiber. This is then standard, say viewing  $f_*$  and  $f^*$  as being induced by proper push forward and flat pull back, respectively, in the Chow group; i.e., one looks at the effect on divisors, remembering that  $f$  is a finite flat morphism of degree  $d$ .  $\square$

We can track the effect on polarizations.

**Lemma 3.6.** *We have  $(f^*)^*(\lambda_C) = d\lambda_{C'}$ .*

*Proof.* As in [BL04, Lem. 12.3.1], observe that

$$d\lambda_{C'} = \lambda_{C'} \circ [d]_{\text{Pic}_{C'/K}^0} = \lambda_{C'} \circ f_* \circ f^* = \widehat{f^*} \circ \lambda_C \circ f^* =: (f^*)^*(\lambda_C)$$

where the second equality is Lemma 3.5, and the third equality comes from applying  $\widehat{(-)}$  to Lemma 3.4.  $\square$

In Lemma 5.10 below, we calculate  $\ker(f^*)$  when  $f$  is a finite morphism of smooth projective curves over a field, and show that there is an abelian cover  $C^{\text{abn}} \rightarrow C'$  which is initial among all abelian covers of  $C'$  through which  $f$  factors. Under a separability hypothesis, we can use this to show that even over an arbitrary base scheme  $S$ , the image of  $f^*$  is an abelian scheme. (In all applications known to the authors, even if the inseparable degree of  $f_s$  is nonconstant – as in, for example, the case of the multiplication-by- $p$  morphism for a family of elliptic curves with both supersingular and ordinary fibers – it is clear that  $\ker(f^*)$  is still a finite flat group scheme, and thus the image of  $\text{Pic}_{C'/S}^0$  is an abelian scheme.)

**Lemma 3.7.** *Let  $f: C \rightarrow C'$  be a finite morphism of smooth projective curves over  $S$  such that either  $f$  is fiberwise separable or Condition 1.15 holds. Then  $\ker(f^*: \text{Pic}_{C'/S}^0 \rightarrow \text{Pic}_{C/S}^0)$  is a finite flat group scheme, and so the image  $f^* \text{Pic}_{C'/S}^0$  is an abelian subscheme of  $\text{Pic}_{C/S}^0$ .*

*Proof.* Since the claim is tautologically true if Condition 1.15 holds, we assume  $f$  fiberwise separable and proceed. By Lemma 3.8,  $f$  factors as  $C \xrightarrow{j} C^{\text{abn}} \xrightarrow{h} C'$  where  $C^{\text{abn}}$  is a smooth proper curve over  $S$ ;  $j$  and  $h$  are finite;  $h$  is an abelian étale morphism; and for each  $s \in S$ ,  $(C^{\text{abn}})_s$  is initial among all abelian étale covers through which  $C_s \rightarrow C'_s$  factors.

By Lemma 5.10(d),  $\ker(j^*)$  is fiberwise trivial, and thus trivial, while Lemma 5.10(a) shows that  $\ker(h^*)$  is fiberwise the covering group of the morphism  $C^{\text{abn}} \rightarrow C'$ .  $\square$

In the previous argument, we used the fact that the formation of  $C^{\text{abn}}$  makes sense in families:

**Lemma 3.8.** *Let  $f: C \rightarrow C'$  be a finite fiberwise separable morphism of smooth proper curves over  $S$ . Then  $f$  factors through an abelian étale cover  $C^{\text{abn}} \rightarrow C'$  such that, for each  $s \in S$ ,  $(C^{\text{abn}})_s \cong (C_s)^{\text{abn}}$ .<sup>1</sup>*

*Proof.* By fppf descent, we may and do assume that  $S$  is the spectrum of an Artinian local ring, with closed point  $0$ . Recall that, since the fundamental group is insensitive to nilpotents, if  $D/S$  is a smooth proper curve, then an étale cover of  $D_0$  extends uniquely to an étale cover of  $D$ .

It now suffices to show that if there is an abelian étale cover  $g_0: C_0^a \rightarrow C_0'$  through which  $f_0$  factors, and if  $g: C^a \rightarrow C'$  is the unique extension of  $C_0^a$  to an étale cover of  $C'$ , then  $f$  factors through  $g$ .

<sup>1</sup>If the fibers of  $C^{\text{abn}}$  have genus 1, then it is possible that  $C^{\text{abn}}$  is not a scheme, but merely an algebraic space. All our techniques and results still apply, and we will elide this distinction.

To see this, let  $D_0 = C_0^a \times_{C'_0} C_0$  be the fiber product

$$\begin{array}{ccc} D_0 & \longrightarrow & C_0 \\ \downarrow & & \downarrow f_0 \\ C_0^a & \xrightarrow{g_0} & C'_0. \end{array}$$

It is an étale cover of  $C_0$ .

On one hand, let  $D$  be the extension of  $D_0$  as an étale cover of  $C$ . On the other hand,  $C^a \times_{C'} C$  is also an étale cover of  $C$ . Since their special fibers coincide, these covers of  $C$  are isomorphic;

$$D \cong C^a \times_{C'} C.$$

However, we can analyze  $D$  in greater detail. Let  $G$  be the covering group of  $g_0$ , and let  $r = \deg(g_0) = |G|$ . Like any torsor, the  $G$ -torsor  $C_0^a \rightarrow C'_0$  trivializes itself, so that  $C_0^a \times_{C'_0} C_0^a \cong \sqcup_{\gamma \in G} C_0^a$ , and so

$$D_0 = C_0^a \times_{C'_0} C_0 \cong C_0^a \times_{C'_0} C_0^a \times_{C_0^a} C_0 \cong \sqcup_{\gamma \in G} C_0^a \times_{C_0^a} C_0 \cong \sqcup_{\gamma \in G} C_0.$$

Moreover  $D$ , the unique extension of  $D_0$  to a cover of  $C$ , is visibly the disjoint union of  $r$  copies of  $C$ . We thus find that  $C^a \times_{C'} C$  is isomorphic to a disjoint union of  $r$  copies of  $C$ , and thus  $f$  factors through  $g$ .  $\square$

In Lemma 3.5, we computed  $f_* f^*$ . In a complementary fashion, we have

**Lemma 3.9.** *Let  $Y := f^* \text{Pic}_{C'/S}^0 \subseteq \text{Pic}_{C/S}^0$ . Then  $e_Y | d$  and*

$$f^* f_* = \frac{d}{e_Y} N_Y.$$

*Proof.* The proof of the stated equality in [BL04, Prop. 12.3.2] holds in this setting, as well. Indeed, if we factor  $f^*$  as an isogeny  $j$  and an inclusion  $\iota_Y$ , then using Lemma 3.6 we obtain the diagram:

$$\begin{array}{ccccc} & & f^* & & \\ & & \curvearrowright & & \\ \text{Pic}_{C'/S}^0 & \xrightarrow{j} & Y & \xrightarrow{\iota_Y} & \text{Pic}_{C/S}^0 \\ \downarrow d\lambda_{C'} & & \downarrow \iota_Y^* \lambda_C & & \downarrow \lambda_C \\ \widehat{\text{Pic}}_{C'/S}^0 & \xleftarrow{\widehat{j}} & \widehat{Y} & \xleftarrow{\widehat{\iota}_Y} & \widehat{\text{Pic}}_{C/S}^0 \\ & & \widehat{f^*} = \lambda_{C'} f_* \lambda_C^{-1} & & \end{array}$$

where the identification of  $\widehat{f^*}$  comes from Lemma 3.4. Therefore, we have

$$f^* \circ f_* = f^* \circ \lambda_{C'}^{-1} \circ (\lambda_{C'} \circ f_* \circ \lambda_C^{-1}) \circ \lambda_C = (\iota_Y \circ j) \circ \lambda_{C'}^{-1} \circ (\widehat{j} \circ \widehat{\iota}_Y) \circ \lambda_C.$$

Now from the left hand square we have in  $\text{End}_{\mathbb{Q}}(Y)$  that  $\iota_Y^* \lambda_C = \widehat{j}^{-1} \circ d\lambda_{C'} \circ j^{-1}$  so that  $\lambda_{C'}^{-1} = (\frac{1}{d} \widehat{j} \circ (\iota_Y^* \lambda_C) \circ j)^{-1} = dj^{-1} \circ (\iota_Y^* \lambda_C)^{-1} \circ \widehat{j}^{-1}$ , and we have in  $\text{End}_{\mathbb{Q}}(\text{Pic}_{C/S}^0)$  that

$$f^* \circ f_* = d\iota_Y \circ (\iota_Y^* \lambda_C)^{-1} \circ \widehat{\iota}_Y \circ \lambda_C = d \frac{1}{e_Y} N_Y.$$

We conclude that  $e_Y | d$  as  $N_Y$  is primitive (Corollary 2.23).  $\square$

**Corollary 3.10.** *We have*

$$P(C/C') = (\ker f_*)^{\text{ab}}.$$

*Proof.* From the definition (2.14) and Lemma 3.9 we have

$$P(C/C') = (\ker N_Y)^{\text{ab}} = (\ker \frac{d}{e_Y} N_Y)^{\text{ab}} = (\ker(f^* f_*))^{\text{ab}}$$

Clearly we have  $(\ker f_*)^{\text{ab}} \subseteq (\ker(f^* f_*))^{\text{ab}}$ . From Lemma 3.5 we have that  $\ker f^*$  is finite over  $S$ , being a subscheme of the finite group scheme  $\text{Pic}_{C'/S}^0[d]$ . Therefore, the inclusion  $(\ker f_*)^{\text{ab}} \subseteq (\ker(f^* f_*))^{\text{ab}}$  is an equality, and we are done.  $\square$

*Remark 3.11.* If  $S = \text{Spec } K$  is the spectrum of a field  $K$ , we have  $P(C/C') = ((\ker f_*)^\circ)_{\text{red}}$ , and if moreover,  $\text{char}(K) = 0$ , then  $P(C/C') = (\ker f_*)^\circ$  (Remark 1.18).

#### 4. CYCLES AND ENDOMORPHISMS OVER FIELDS

The main goal of this section is to review the construction and basic properties of the endomorphism  $\delta(\alpha, \beta)$  of an abelian variety  $X$  over a field  $K$  induced by cycle classes  $\alpha$  and  $\beta$  of complementary dimension on  $X$ . Since the construction uses intersection theory, we prefer to work over a field  $K$ ; one can do intersection theory over slightly more general bases, but for our purposes we get more general results working over a field and then spreading, and so it is more convenient to simply do the constructions over a field.

The endomorphism  $\delta(\alpha, \beta)$  was constructed [Mat59], and described and analyzed in the language of Weil's foundations for algebraic geometry. Thus, it was originally discovered as an object over an arbitrary, "sufficiently large" field. It is studied in [BL04] in the special case of complex abelian varieties. In the present treatment, we rely on a modern reformulation of Samuel's notion of a regular homomorphism for the initial construction of  $\delta(\alpha, \beta)$ . Once the existence of this endomorphism has been secured, it turns out that the analysis of its properties in [BL04, §11.6] is equally valid in the algebraic category.

**4.1. Equivalences of algebraic cycles on abelian varieties.** We recall here the agreement of certain notions of equivalence between cycles on an abelian variety  $X$  over an algebraically closed field  $k$ . We refer the reader to [MNP13, §1.2 and App. A] for references.

To fix terminology and notation, we fix a cohomology theory  $\mathcal{H}^\bullet$  with coefficient ring  $R_{\mathcal{H}}$ . This can be any Weil cohomology theory, but we will also work with several cohomology theories with "integral" coefficients, which give rise to standard Weil cohomology theories. For instance, for  $\ell \neq \text{char}(k)$ , we will consider  $\ell$ -adic cohomology with  $\mathbb{Z}_\ell$ -coefficients. In positive characteristic, we will also consider crystalline cohomology with coefficients in the Witt ring  $W(k)$ , and for  $k = \mathbb{C}$ , we also consider Betti cohomology with  $\mathbb{Z}$ -coefficients. Note that tensoring with the field of fractions of the coefficient ring of any of these cohomology theories gives rise to a Weil cohomology theory. Unless a particular cohomology theory is specified, if we say a cycle or cycle class is homologically trivial, we mean its class in  $\mathcal{H}^\bullet$  is trivial for all of the applicable cohomology theories.

Now, for a divisor  $D$  on  $X$ , we have that

- $D$  is algebraically trivial  $\iff D$  is homologically trivial  $\iff D$  is numerically trivial.

Indeed, on any smooth projective variety over an algebraically closed field, for cycles of any dimension one has the forward implications. Moreover, for divisors, Matsusaka's theorem (e.g., [MNP13, App. A, Thm., p.13]) states that numerical triviality of a divisor implies that some multiple is homologically trivial, and homological triviality of a divisor implies that some multiple is algebraically trivial. On an abelian variety, however, the cohomology is free of finite rank over the ring of coefficients, and the Néron–Severi group is a free  $\mathbb{Z}$ -module of finite rank, and so numerical (resp. homological) triviality of  $D$  implies homological (resp. algebraic) triviality of  $D$ . We also recall that  $D$  is algebraically trivial if and only if  $\mathcal{O}_X(D) \in \text{Pic}_{X/k}^0(k)$ .

Similarly, for a 1-cycle  $\Gamma$  on  $X$ , we have

- $\Gamma$  is algebraically trivial  $\implies \Gamma$  is homologically trivial  $\iff \Gamma$  is numerically trivial.

As mentioned above, the forward implication holds for cycles of any dimension on any smooth projective variety over an algebraically closed field. The reverse implication follows from [Seb13, Cor. 10], which states that any numerically trivial 1-cycle on a variety dominated by a product of curves has a positive integer multiple that is homologically trivial. Since every abelian variety is dominated by the Jacobian of a smooth projective curve (see the proof of Corollary 5.7 for a reminder), and the Jacobian of a smooth projective curve of genus  $g$  is dominated by the product of  $g$  copies of the curve, Sebastian's result applies to abelian varieties. We thus conclude that some positive multiple of  $\Gamma$  is homologically trivial. However, again, since the cohomology of an abelian variety is free of finite rank over the ring of coefficients, this implies that  $\Gamma$  itself is homologically trivial.

**4.2. Regular homomorphisms and traces of endomorphisms.** Let  $Z/K$  be a smooth geometrically irreducible projective variety, and let  $A/K$  be an abelian variety. Denote by  $A^i(Z) \subseteq \text{CH}^i(Z)$  denote the group of algebraically trivial cycle classes. A (Galois-equivariant) regular homomorphism (in codimension  $i$ ) is an  $\text{Aut}(\bar{K}/K)$ -equivariant group homomorphism

$$A^i(Z_{\bar{K}}) \xrightarrow{\phi} A(\bar{K})$$

such that, for each pair  $((T, t_0), Z)$  with  $(T, t_0)$  a pointed smooth variety over  $K$ , and each  $\Xi \in \text{CH}^i(T \times Z)$ , the map of points

$$\begin{aligned} T(\bar{K}) &\longrightarrow A(\bar{K}) \\ t &\longmapsto \Phi(\Xi_t - \Xi_{t_0}) \end{aligned}$$

is induced by a  $K$ -morphism of varieties

$$T \xrightarrow{\psi_{\Xi}} A.$$

In particular, if  $(Z, z_0)$  is a pointed  $K$ -variety of dimension  $d$ , then the Albanese map is a universal regular homomorphism in codimension  $d$ . We refer the reader to [ACMV23b] for more details on regular homomorphisms.

We now turn our attention to recalling Weil's trace formula (4.1). For this we further specialize the discussion above to the case where  $Z = C$  is a pointed smooth projective curve over  $K$ , and recall the Albanese property of the Picard variety (§1.7.4). In this situation, we recover the relation between correspondences on curves and endomorphisms of the Jacobian which Weil constructed ([Wei48b]; see [Kah20, §3.2.2] and [Sch94, §3] for modern treatments): there is a canonical surjection

$$\begin{aligned} \text{CH}^1(C \times_K C) &\twoheadrightarrow \text{End}(\text{Pic}_{C/K}^0) \\ D &\longmapsto \gamma_D, \end{aligned}$$

with kernel the subgroup of divisors generated by horizontal and vertical fibers of the product  $C \times_K C$ . Below, we will need to use Weil's formulation of the Lefschetz formula for self-correspondences on  $C$ ; it is expressed in terms of the trace of the corresponding endomorphism on the Jacobian.

Suppose  $X$  is a  $g$ -dimensional abelian variety and  $f \in \text{End}(X)$ . Recall that there is a unique monic polynomial  $P_f \in \mathbb{Z}[T]$  of degree  $2g$ , the characteristic polynomial of  $f$ , such that for each integer  $r$  we have  $P_f(r) = \deg(f - [r]_X)$ . If  $X$  is a complex abelian variety, this is the characteristic polynomial of the action of  $f$  on  $H_1(X, \mathbb{Q})$ ; more generally,  $P_f(T)$  is the characteristic polynomial

of the linear operator induced by  $f$  on  $T_\ell X$  for any  $\ell \neq \text{char}(K)$ . The trace of  $f$ ,  $\text{tr}(f)$ , can be read off from  $P_f(T)$ ; indeed,  $P_f(T) = T^{2g} - \text{tr}(f)T^{2g-1} + \dots$ .

For  $D \in \text{CH}^1(C \times_K C)$ , Weil was able to calculate the trace of the associated endomorphism using intersection theory on  $C \times_K C$ . More precisely for  $D \in \text{CH}^1(C \times_K C)$ , if one defines the indices  $d_1(D)$  and  $d_2(D)$  by  $\text{pr}_i(D) = d_i(D)[C]$  (e.g., [Ful98, Ex. 16.1.4], [Kah20, Def. 2.19]), then (e.g., [Kah20, Ex. 6.45] [Lan59, Thm. VI.3.6] [Wei48a, §2.II])

$$\text{tr}(\gamma_D) = d_1(D) + d_2(D) - (\Delta_C \cdot D), \quad (4.1)$$

where  $\Delta_C \in \text{CH}^1(C \times_K C)$  is the class of the diagonal.

**4.3. Definition of  $\delta$  via intersection theory and regular homomorphisms.** Let  $X/K$  be an abelian variety over a field  $K$ . Let  $\alpha \in \text{CH}^a(X)$  and  $\beta \in \text{CH}^b(X)$  be cycle classes of complementary (co)dimension, so that  $a + b = \dim X$ . On the product  $X \times_K X$ , with projections  $\pi_1, \pi_2$ , and addition map  $\mu : X \times_K X \rightarrow X$ , we can consider  $\pi_2^* \alpha \in \text{CH}^a(X \times_K X)$ , and  $(\text{Id}_X \times \mu)_* \pi_1^* \beta \in \text{CH}^b(X \times_K X)$ , and the intersection product

$$\Gamma(\alpha, \beta) := (\pi_2^* \alpha) \cdot ((\text{Id}_X \times \mu)_* \pi_1^* \beta) \in \text{CH}^{\dim X}(X \times_K X). \quad (4.2)$$

Since for an abelian variety the identity map is an Albanese map, the fact that the Albanese is a regular homomorphism implies that the cycle class  $\Gamma(\alpha, \beta)$  induces a morphism of varieties over  $K$ :

$$\hat{\delta}(\alpha, \beta) : X \longrightarrow X.$$

We adjust  $\hat{\delta}$  to construct a  $K$ -homomorphism, i.e., a morphism of abelian varieties:

$$\delta(\alpha, \beta) = \hat{\delta}(\alpha, \beta) - \hat{\delta}(\alpha, \beta)(0) : X \longrightarrow X.$$

If  $V, W \in Z^\bullet(X)$  are cycles with cycle classes  $[V], [W]$ , then we define  $\delta(V, W) = \delta([V], [W])$ .

*Remark 4.1.* As regular homomorphisms and intersection products are stable under base change of field [ACMV23b, Thm. 1], so is  $\delta$ . In particular,  $\delta(\alpha_{\bar{K}}, \beta_{\bar{K}}) = (\delta(\alpha, \beta))_{\bar{K}}$ .

From the definition it is clear that  $\delta(-, -)$  is additive in each entry. We also have:

**Proposition 4.2.** *In the notation above, suppose that  $\alpha_{\bar{K}}$  or  $\beta_{\bar{K}}$  is one of the following:*

- (1) *A torsion cycle class,*
- (2) *A homologically trivial cycle class,*
- (3) *A numerically trivial 1-cycle class or divisor class.*

*Then  $\delta(\alpha, \beta) = 0$ .*

*Proof.* (See [BL04, Prop. 5.4.3] for the case  $S = \text{Spec } \mathbb{C}$ .) A morphism of abelian varieties is zero if and only if the morphism is zero after base change to the algebraic closure; thus from Remark 4.1, we may assume we are working over an algebraically closed field.

(1) If for some positive integer  $n$  we have  $\delta(n\alpha, \beta) = 0$  or  $\delta(\alpha, n\beta) = 0$ , then  $\delta(\alpha, \beta) = 0$ , due to the additivity of  $\delta(-, -)$  in each entry, together with the fact that  $\text{End}(X)$  is a free  $\mathbb{Z}$ -module of finite rank (e.g., §1.3).

(2) Note first that if  $\alpha$  is algebraically trivial, the assertion follows from the definitions and rigidity of endomorphisms of abelian varieties (e.g., [MFK94, Prop. 6.1]). For the case where  $\alpha$  is only assumed to be homologically trivial, we use a recent result [ACMV23a, Prop. 8.1], which shows that for fields of any characteristic, regular homomorphisms depend only on the cohomology class of the correspondence inducing the regular homomorphism (this is well-known over  $\mathbb{C}$ ). Then, if  $\alpha$  is homologically trivial, one has that  $\pi_2^* \alpha$  is homologically trivial, and, since intersection product in Chow is taken to cup product in cohomology, one has from the definition of  $\Gamma(\alpha, \beta)$  in (4.2) that

$\Gamma(\alpha, \beta)$  is homologically trivial. Therefore, [ACMV23a, Prop. 8.1] implies that  $\hat{\delta}(\alpha, \beta)$  is the zero map, and consequently,  $\delta(\alpha, \beta) = 0$ , as well. The same argument works for  $\beta$ , as well.

(3) This follows from (2) since numerically trivial divisors and 1-cycles on abelian varieties are homologically trivial (§4.1).  $\square$

**4.4. Geometric construction of  $\delta$  over an algebraically closed field.** Over an algebraically closed field we can give a geometric description of  $\delta$ . Let  $X$  be an abelian variety over an algebraically closed field  $k$ . Suppose  $V, W \subseteq X$  are integral cycles of complementary dimension, so that  $\dim V + \dim W = \dim X$ .

If the intersection  $V \cap W$  is nonempty of pure dimension 0, and the intersection is  $V \cdot W = \sum_{i=1}^n r_i x_i$ , then we define

$$S(V, W) = r_1 x_1 + \cdots + r_n x_n \quad (4.3)$$

where the sum above is addition in the abelian variety  $X$ . Note that  $S(-, -)$  is clearly additive in each entry (under the hypotheses on the intersections of the cycles), and symmetric.

For  $P \in X(k)$ , let  $t_P: X \rightarrow X$  be the corresponding translation map. Recall from the Moving Lemma ([Kle74, Thm. 2, Lem. 1]) that there is a nonempty open subset  $U \subseteq X$  such that, if  $P \in U$ , then  $V \cap t_P^* W$  is empty or of pure dimension zero. If for a general  $P \in U$  the intersection is of pure dimension 0, then we define for  $P \in U$

$$\hat{\delta}(V, W)(P) = S(V, t_P^* W)$$

giving a morphism  $\hat{\delta}(V, W): U \rightarrow X$ , which extends to give a morphism  $\hat{\delta}(V, W): X \rightarrow X$ . We define  $\delta(V, W) = \hat{\delta}(V, W) - \hat{\delta}(V, W)(0)$ . Otherwise, if for general  $P \in U$  the intersection  $V \cap t_P^* W$  is empty, we set  $\delta(V, W) = 0$ . Clearly  $\hat{\delta}$  and  $\delta$  agree with the definition in the previous section. For clarity we note that when  $V \cap t_P^* W$  and  $V \cap W$  are of pure dimension 0 we have

$$\delta(V, W)(P) = S(V, t_P^* W) - S(V, W). \quad (4.4)$$

**4.5. Formal properties of  $\delta$ .** Here we review some of the formal properties of  $\delta$  established in [Mat59, §1], following the treatment in [BL04, §5.4].

**Proposition 4.3.** *In the notation above:*

(a) For algebraic cycles  $V, W$  on  $X$  with  $\dim V + \dim W = \dim X$ , we have

$$\delta(V, W) + \delta(W, V) = -(V \cdot W)[1]_X.$$

(b) For algebraic cycles  $V_0, \dots, V_r$  on  $X$  with  $\sum \dim V_i = r \dim X$ , we have

$$\delta(V_0, V_1 \cdot \dots \cdot V_r) = \sum_{i=1}^r \delta(V_0 \cdot V_1 \cdot \dots \cdot \check{V}_i \cdot \dots \cdot V_r, V_i).$$

(c)

$$\delta(D^r, D^{g-r}) = -\frac{g-r}{g}(D^g)[1]_X.$$

*Proof.* (a) It suffices to prove the equality after base change to the algebraic closure. Then this is a straight forward argument using (4.4) and (4.3) (see [BL04, Lem. 5.4.4]).

(b) Again, it suffices to prove the equality after base change to the algebraic closure. Then this is a straight forward argument using (4.4) and (4.3) (see [BL04, Lem. 5.4.6]).

(c) This follows formally from (a) and (b); see [BL04, Prop. 5.4.7].  $\square$

**4.6. Interaction of  $\delta$  with the Abel map.** Here we review some of the results from [BL04, §11.6]. We start with a  $K$ -morphism  $\beta : C \rightarrow Z$  from a smooth projective genus  $g$  curve  $C$  over  $K$  to an abelian variety  $Z$  over  $K$ , and we will use the notation from (3.2) and (3.3).

**Proposition 4.4.** *For a divisor  $D$  on  $Z$ ,*

$$\delta(\beta_*[C], D) = \widehat{\widehat{\beta}} \circ \lambda_C^{-1} \circ \widehat{\beta} \circ \phi_D, \quad (4.5)$$

$$Z \xrightarrow{\phi_D} \widehat{Z} \xrightarrow{\widehat{\beta}} \widehat{\text{Pic}}_{C/K}^0 \xrightarrow{\lambda_C^{-1}} \text{Pic}_{C/K}^0 \xrightarrow{\widehat{\beta}} Z$$

*Proof.* It suffices to check equality of the morphisms over the algebraic closure. In this case, using the definition of  $\delta$  given in §4.4, the proof follows formally as in [BL04, Prop. 11.6.1].  $\square$

**Proposition 4.5.** *With notation as above,*

- (a)  $\text{tr}(\delta(\beta_*[C], D)) = -2(\beta_*[C] \cdot D)$
- (b)  $\text{tr}(\delta(D, \beta_*[C])) = -(2g - 2)(\beta_*[C] \cdot D)$

*Proof.* As usual, it suffices to prove this over an algebraically closed field. The proof in [BL04, Prop. 11.6.2] proceeds without incident, provided one uses the intrinsic trace reviewed in §4.2, and then invokes Weil's trace formula (4.1) in the analysis of a certain self-correspondence on  $C$ .  $\square$

**Theorem 4.6** ([Mat59, §2]; see also [BL04, Thm. 11.6.4]). *Let  $Z$  be an abelian variety over a field  $K$ , let  $D$  be a divisor on  $Z$ , and let  $\Gamma$  be an algebraic 1-cycle on  $Z$ .*

- (a) *If  $D$  is non-degenerate (i.e.,  $\phi_D$  is an isogeny), then  $\delta(\Gamma, D) = 0$  if and only if  $\Gamma_{\overline{K}}$  is numerically equivalent to zero.*
- (b) *If  $\Gamma$  is an integral curve and non-degenerate (i.e., generates  $Z$ ), then  $\delta(\Gamma, D) = 0$  if and only if  $D_{\overline{K}}$  is algebraically equivalent to zero.*

*Proof.* (a) We have already shown one direction (Proposition 4.2), so assume that  $\delta(\Gamma, D) = 0$ . It suffices to show the result after base change to the algebraic closure (see Remark 4.1). The argument of [BL04, Thm. 11.6.4(a)], which relies only on Proposition 4.4 and Proposition 4.5, is valid in the algebraic setting, too. (In that argument, the curve  $C$  ranges among the desingularizations of the various components of  $\Gamma$ .)

(b) We have already shown one direction (Proposition 4.2), so assume that  $\delta(\Gamma, D) = 0$ . Again, it suffices to show the result after base change to the algebraic closure. The argument of [BL04, Thm. 11.6.4(b)] holds here, as well.  $\square$

## 5. PRYM VARIETIES OVER FIELDS

In this section we work over a field  $K$ . We show that every principally polarized abelian variety is a Prym–Tyurin variety of some exponent, and we classify Prym–Tyurin Prym varieties of all exponents.

**5.1. Prym–Tyurin varieties and Welters' criterion.** Recall from §2.6.1 that a Prym–Tyurin variety of exponent  $e$  over a field  $K$  is a principally polarized abelian variety  $(Z, \xi)/K$  together with a smooth projective curve  $C/K$  and an injective  $K$ -homomorphism  $\iota_Z : Z \hookrightarrow \text{Pic}_{C/K}^0$  of abelian varieties such that  $\iota_Z^* \lambda_C = e\xi$ , where  $\lambda_C$  is the canonical principal polarization on  $\text{Pic}_{C/K}^0$ . The condition  $\iota_Z^* \lambda_C = e\xi$  is equivalent to the condition that  $\text{K}(\iota_Z^* \lambda_C) = Z[e] := \ker[e]_Z$  (Remark 2.25); if there exist divisors  $\Theta$  and  $\Xi$  such that  $\phi_\Theta = \lambda_C$  and  $\phi_\Xi = \xi$ , e.g., if  $K = \overline{K}$ , then this condition is also equivalent to the condition

$$\iota_Z^* \Theta \equiv e\Xi.$$

The equivalence above is taken to be homological or numerical; these notions all agree for divisors on abelian varieties, as they are equivalent to homological or numerical equivalence over the algebraic closure (§4.1).

The main tool in classifying Prym–Tyurin varieties is the following:

**Lemma 5.1.** *Let  $\beta: C \rightarrow T$  be a  $K$ -morphism from a smooth projective curve to a torsor under a principally polarized abelian variety  $(Z, \zeta)/K$  of dimension  $g_Z$ , and let  $\zeta_Z: \widehat{Z} \xrightarrow{\sim} \text{Pic}_{T/K}^0$  be the canonical isomorphism (1.6). Fix an identification of  $T_{\overline{K}}$  with  $Z_{\overline{K}}$ , and fix divisors  $\Theta_{\overline{K}}$  and  $\Xi_{\overline{K}}$  such that  $\phi_{\Theta_{\overline{K}}} = (\lambda_C)_{\overline{K}}$  and  $\phi_{\Xi_{\overline{K}}} = \zeta_{\overline{K}}$ .*

*Then the following are equivalent:*

- (a)  $(\beta^* \circ \zeta_Z \circ \zeta)^* \lambda_C = e\zeta$ .
- (b) *The following diagram commutes:*

$$\begin{array}{ccc}
 Z & \xrightarrow{e\zeta} & \widehat{Z} \\
 \zeta \downarrow & & \uparrow \zeta = \widehat{\zeta} \\
 \widehat{Z} & \xrightarrow{(\beta^* \zeta_Z)^* \lambda_C} & Z \\
 \beta^* \zeta_Z \downarrow & & \uparrow \widehat{\beta^* \zeta_Z} \\
 \text{Pic}_{C/K}^0 & \xrightarrow{\lambda_C} & \widehat{\text{Pic}}_{C/K}^0
 \end{array} \tag{5.1}$$

(c)  $\delta(\beta_{\overline{K}*}[C_{\overline{K}}], \Xi_{\overline{K}}) = -[e]_{Z_{\overline{K}}}$ .

- (d)  $\beta_{\overline{K}*}[C_{\overline{K}}] \equiv e \frac{[\Xi_{\overline{K}}]^{g_Z-1}}{(g_Z-1)!}$ , where equivalence is taken to be homological or numerical; these notions agree for 1-cycles on abelian varieties over algebraically closed fields (§4.1).

*Proof.* (See also [BL04, Lem. 12.2.3].)

(a)  $\iff$  (b). This is clear from the definition of a restriction of a polarization.

(b)  $\iff$  (c). The diagram (5.1) commutes if and only if it commutes over  $\overline{K}$ . Therefore, we will work over  $\overline{K}$ . Recalling that we have identified  $T_{\overline{K}}$  with  $Z_{\overline{K}}$ , so that  $\beta_{\overline{K}}: C_{\overline{K}} \rightarrow Z_{\overline{K}}$ , and  $(\zeta_Z)_{\overline{K}}$  is the identity, we always have:

$$\begin{aligned}
 \delta(\beta_{\overline{K}*}[C_{\overline{K}}], \Xi_{\overline{K}}) &= \widehat{\widehat{\beta_{\overline{K}}}} \lambda_{C_{\overline{K}}}^{-1} \widehat{\widehat{\beta_{\overline{K}}}} \zeta_{\overline{K}} && \text{(Proposition 4.4)} \\
 &= -(\widehat{\widehat{\beta_{\overline{K}}}} \lambda_{C_{\overline{K}}}) \lambda_{C_{\overline{K}}}^{-1} (\lambda_{C_{\overline{K}}} \widehat{\widehat{\beta_{\overline{K}}}}) \zeta_{\overline{K}} && \text{(Lemma 3.2)} \\
 &= -\widehat{\widehat{\beta_{\overline{K}}}} \lambda_{C_{\overline{K}}} \widehat{\widehat{\beta_{\overline{K}}}} \zeta_{\overline{K}}
 \end{aligned}$$

as well as the fact that

$$-\widehat{\widehat{\zeta}}^{-1} e\zeta = -[e]_Z,$$

so that (b) and (c) are equivalent.

- (c)  $\iff$  (d) By Proposition 4.3(c)

$$\delta\left(\frac{e}{(g_Z-1)!} \Xi_{\overline{K}}^{g_Z-1}, \Xi_{\overline{K}}\right) = -[e]_{Z_{\overline{K}}}.$$

Now use Theorem 4.6(a) to see that, since  $\Xi_{\overline{K}}$  is ample,  $\delta(\Gamma_1, \Xi_{\overline{K}}) = \delta(\Gamma_2, \Xi_{\overline{K}})$  if and only if  $\Gamma_1 \equiv \Gamma_2$ .  $\square$

*Remark 5.2* (Welters). For an abelian variety  $A$  over a field  $K$ , denote by  $e_{A,e}: A[e] \times A[e] \rightarrow \mu_{e,K}$  the Weil pairing on  $e$ -torsion. In the setting of Lemma 5.1, set  $W = \zeta \beta^* \zeta_Z(Z)$  to be the image of  $Z$  in  $\text{Pic}_{C/K}^0$ , so that we have a factorization  $\beta^* \circ \zeta_Z \circ \zeta: Z \twoheadrightarrow W \hookrightarrow \text{Pic}_{C/K}^0$ , and let  $\zeta|_{\widehat{W}}: \widehat{W} \rightarrow W$  be the

pull-back of the principal polarization  $\zeta$  along  $\widehat{W} \rightarrow \widehat{Z} \xrightarrow{\zeta^{-1}} Z$ . If  $K = \overline{K}$  is an algebraically closed field of characteristic 0, [Wel87, Prop. 1.17] shows that Lemma 5.1(a), (b), (d) are equivalent to the condition that  $\ker \lambda_C|_W \subseteq W[e]$  (see Remark 2.29) together with the data of a maximal isotropic subgroup  $\ker \lambda_C|_W \subseteq H \subseteq W[e]$  with respect to the Weil pairing  $e_e: \text{Pic}_{C/K}^0[e] \times \text{Pic}_{C/K}^0[e] \rightarrow \mu_e$ . Welters' arguments hold over an arbitrary algebraically closed field provided one works with  $\ker(-)^{\text{ab}}$  instead of  $\ker(-)^\circ$  in [Wel87, (1.4) and (1.13)], and in the calculation of the Weil pairing at the bottom of [Wel87, p.90], one works with the group schemes, rather than just  $\overline{K}$  points.

From Lemma 5.1 we obtain what is known as Welters' Criterion [Wel87]. Welters works over an algebraically closed field of characteristic 0, but his arguments hold more generally (see Remark 5.2) over an arbitrary algebraically closed field. Our presentation follows that of [BL04], which in turn uses strategies from [Mat59, §2].

**Theorem 5.3** (Welters' Criterion [Wel87, (1.18–1.19)], [BL04, Crit. 12.2.2]). *Let  $(Z, \zeta)/K$  be a principally polarized abelian variety of dimension  $g_Z$  and let  $C/K$  be a smooth projective curve. Fix divisors  $\Theta_{\overline{K}}$  and  $\Xi_{\overline{K}}$  such that  $\phi_{\Theta_{\overline{K}}} = (\lambda_C)_{\overline{K}}$  and  $\phi_{\Xi_{\overline{K}}} = \zeta_{\overline{K}}$ .*

(a) *Suppose there is a  $K$ -morphism  $\beta: C \rightarrow T$  to a torsor under  $Z$  over  $K$  such that*

(i) *the composition  $\widehat{Z} \xrightarrow[\sim]{\zeta_Z} \text{Pic}_{T/K}^0 \xrightarrow{\beta^*} \text{Pic}_{C/K}^0$  is an injective  $K$ -homomorphism of abelian varieties, where  $\zeta_Z$  is the canonical isomorphism (1.6), and*

(ii) *after identifying  $T_{\overline{K}}$  and  $Z_{\overline{K}}$ , we have  $\beta_{\overline{K}*}[C_{\overline{K}}] \equiv e^{\frac{[\Xi_{\overline{K}}]^{g_Z-1}}{(g_Z-1)!}}$ .*

*Then the inclusion  $\iota_Z := \beta^* \zeta_Z \zeta: Z \hookrightarrow \text{Pic}_{C/K}^0$  makes  $(Z, \zeta)$  a Prym–Tyurin variety of exponent  $e$ ; i.e.,  $\iota_Z^* \lambda = e\zeta$ , so that  $(Z, \zeta, C, \iota_Z)$  is an embedded Prym–Tyurin variety of exponent  $e$ .*

(b) *Conversely, suppose there is an inclusion  $\iota_Z: Z \hookrightarrow \text{Pic}_{C/K}^0$  making  $(Z, \zeta)$  a Prym–Tyurin variety of exponent  $e$ ; i.e.,  $(Z, \zeta, C, \iota_Z)$  is an embedded Prym–Tyurin variety of exponent  $e$ . Let  $Y$  be the complement of  $Z$ . Then, under the isomorphism  $Z \xrightarrow[\sim]{-\zeta} \widehat{Z} \xrightarrow[\sim]{} \text{Pic}_{C/K}^0/Y$ , (see (2.10))  $T := \text{Pic}_{C/K}^{(1)}/Y$  is a torsor under  $Z$ , and the composition  $\beta: C \xrightarrow{\alpha^{(1)}} \text{Pic}_{C/K}^{(1)} \rightarrow T$  of the Abel map with the quotient map satisfies conditions (i) and (ii) above.*

*Moreover, these constructions are inverse to one another, up to canonical isomorphisms.*

*Proof.* It suffices to check the assertions of (a) and (b) after base change to the algebraic closure, so we will assume that  $K = \overline{K}$ , and we fix a point  $P \in C(K)$ . (a) follows immediately from Lemma 5.1.

For (b), suppose conversely that there is an inclusion  $\iota_Z: Z \hookrightarrow \text{Pic}_{C/K}^0$  making  $(Z, \zeta)$  a Prym–Tyurin variety of exponent  $e$ . First consider the map  $\beta: C \xrightarrow{\alpha^{(1)}} \text{Pic}_{C/K}^{(1)} \rightarrow T$ . Again using  $\beta(P)$  to identify  $T$  with  $Z$ , unwinding the definitions, and using the short exact sequence (2.10), we have  $\beta = -\zeta^{-1} \widehat{\iota}_Z \lambda_C \alpha_P: C \rightarrow Z$ . This implies that  $\beta^* = -\alpha_P^* \lambda_C \iota_Z \zeta^{-1} = \iota_Z \zeta^{-1}$ , where here we are using Lemma 3.1. Consequently,  $\beta^*: \widehat{Z} \rightarrow \text{Pic}_{C/K}^0$  is an injective  $K$ -homomorphism giving (i), where we are using that  $\zeta_Z$  is the identity after we have identified  $T = Z$ . It also follows from the identity  $\beta^* = \iota_Z \zeta^{-1}$  that  $\iota_Z = \beta^* \zeta$ , which equals  $\beta^* \zeta_Z \zeta$ , again since  $\zeta_Z$  is the identity after we have identified  $T = Z$ . Therefore,  $(\beta^* \zeta_Z \zeta)^* \lambda_C = \iota_Z^* \lambda_C = e\zeta$ , so we can employ Lemma 5.1(a), and we obtain (ii).

The equality  $\iota_Z = \beta^* \zeta_Z \zeta$  also shows one direction of the assertion that the two constructions in the theorem are inverses to one another. In the other direction, given  $\beta: C \rightarrow T$  satisfying (i) and (ii), one observes that  $\beta$  factors through the Abel map  $\alpha^{(1)}: C \rightarrow \text{Pic}_{C/K}^{(1)}$  using the universal property of the Albanese torsor. This gives an isomorphism of  $T$  with  $\text{Pic}_{C/K}^{(1)}/Y$ .  $\square$

As a consequence, we get the Matsusaka Criterion:

**Corollary 5.4.** *Let  $(Z, \xi)/K$  be a principally polarized abelian variety of dimension  $g_Z > 0$  and let  $C/K$  be a smooth projective curve. Fix divisors  $\Theta_{\bar{K}}$  and  $\Xi_{\bar{K}}$  such that  $\phi_{\Theta_{\bar{K}}} = (\lambda_C)_{\bar{K}}$  and  $\phi_{\Xi_{\bar{K}}} = \xi_{\bar{K}}$ . Suppose there is a  $K$ -morphism  $\beta: C \rightarrow T$  to a torsor under  $Z$  over  $K$  such that after identifying  $T_{\bar{K}}$  and  $Z_{\bar{K}}$ , we have*

$$\beta_{\bar{K}*}[C_{\bar{K}}] \equiv \frac{[\Xi_{\bar{K}}]^{g_Z-1}}{(g_Z-1)!}. \quad (5.2)$$

*Then the composition  $\iota_Z := \beta^* \zeta_Z \xi: Z \rightarrow \text{Pic}_{C/K}^0$  is an isomorphism, and there is a unique isomorphism  $\gamma: \text{Pic}_{C/K}^{(1)} \rightarrow T$  such that  $\beta = \alpha^{(1)} \gamma$ . In other words, up to an isomorphism of torsors,  $\beta$  is the Abel map.*

*Remark 5.5.* The equality in (5.2) does not depend on the choice of identification of  $T_{\bar{K}}$  with  $Z_{\bar{K}}$ , as any two choices will differ by translation.

*Proof.* We use Lemma 5.1(d) to conclude that diagram (5.1) commutes with  $e = 1$ . Therefore, since  $\xi$  is an isomorphism, a diagram chase implies that  $\beta^* \zeta_Z$  is injective, so that  $\beta^* \zeta_Z \xi$  is injective, as well. Setting  $\iota_Z = \beta^* \zeta_Z \xi$ , Theorem 5.3 then implies that  $(Z, \xi, C, \iota_Z)$  is an embedded Prym–Tyurin variety of exponent 1. We then conclude using Corollary 2.19 that, since  $\dim Z > 0$ , and  $\lambda_C$  is indecomposable, we have  $\iota_Z(Z) = \text{Pic}_{C/K}^0$ . The final assertion follows from the universal property of the Albanese torsor. (See also [BL04, Rem. 12.2.5] for the case  $K = \mathbb{C}$ .)  $\square$

*Remark 5.6 (Matsusaka–Ran Criterion).* In fact, using [Col84, Thm. p.329], one can replace the condition (5.2) in Corollary 5.4 with the weaker condition that  $\beta_{\bar{K}*}[C_{\bar{K}}] \cdot [\Xi_{\bar{K}}] = g_Z$ . This would be the natural translation of the Matsusaka–Ran Criterion [Col84, Thm. p.329] to the case of non-closed fields. Note also that in [Col84, Thm. p.329], disjoint unions of curves are also considered; one can formulate a corresponding version of the Matsusaka–Ran Criterion over non-closed fields, as well.

**Corollary 5.7.** *Let  $(Z, \xi)/K$  be a principally polarized abelian variety of dimension  $g$  over a field. Then  $(Z, \xi)$  is a Prym–Tyurin variety of exponent  $n^{g-1}(g-1)!$  for infinitely many  $n$ , and for all  $n \geq 3$  if  $\text{char}(K) = 0$ .*

*Proof.* (See also [BL04, Cor. 12.2.4] when  $K = \mathbb{C}$ .) Let  $n \geq 3$  be an integer. (During the course of the proof we will encounter two places where, if  $K$  has positive characteristic, then  $n$  must be replaced by a suitable multiple.) According to a theorem of Lefschetz  $n\Xi$  is very ample (e.g., [Mum08, §17]), and thus defines an embedding in projective space. If  $K$  is infinite, then using Bertini’s theorem we can take the intersection of  $g-1$  hyperplanes to obtain an embedding  $\beta: C \hookrightarrow Z$  of a smooth projective curve over  $K$ . (If  $K$  is finite, it may be necessary to take a higher multiple of  $\Xi$ . That this suffices is due to Gabber and to Poonen [Gab01, Poo04]; indeed, proving that any abelian variety is a quotient of a Jacobian was the impetus for those works.) Then the image of  $C$  generates  $Z$  (e.g., [Mil08, Proof of Thm. 10.1]). By construction we obtain that

$$\beta_*[C] \equiv [n\Xi]^{g-1} = n^{g-1}[\Xi]^{g-1},$$

so that from Welters’ Criterion (Theorem 5.3), all we have to show is that  $\beta^*: \widehat{Z} \rightarrow \text{Pic}_{C/K}^0$  is injective. We may verify this injectivity over the algebraic closure of  $K$ . So we may assume that

there is a  $K$ -point  $P \in C(K)$ , and we consider the diagram

$$\begin{array}{ccc}
 C & \xrightarrow{\beta} & Z \\
 \downarrow \alpha^{(1)} & \nearrow \tilde{\beta} & \\
 \text{Pic}_{C/K}^{(1)} & & \\
 \downarrow & \nearrow \beta_P & \\
 \text{Pic}_{C/K}^0 & & 
 \end{array}$$

where  $\beta_P$  is defined as in the diagram. From Lemma 3.2 we find that  $\widehat{\beta_P} = -\phi_{\Theta}\beta^*$ . Thus it suffices to show that  $\widehat{\beta_P}$  is injective. Because  $C$  generates  $Z$ ,  $\beta_P$  is surjective; by Corollary 1.4, it now suffices to verify that  $\ker(\beta_P)$  is an abelian variety.

If  $K$  has characteristic 0, then a hyperplane section of a normal variety induces a surjection of étale fundamental groups [Gro05, Cor. XII.3.5], and thus of fundamental group schemes. In positive characteristic, the failure of Kodaira vanishing can present an obstacle to surjectivity on fundamental group schemes, which can be rectified by passing to a suitable multiple of the hyperplane section [BH07, Thm. 1.1]. Thus, possibly after replacing  $n\Xi$  with some multiple  $mn\Xi$ , if  $\text{char}(K) > 0$ , we may and do assume that  $\beta$  induces a surjection  $\beta_*: \pi_1^{\text{Nori}}(C_{\overline{K}}) \twoheadrightarrow \pi_1^{\text{Nori}}(Z_{\overline{K}})$ . Now,  $\alpha_{P,*}: \pi_1^{\text{Nori}}(C_{\overline{K}}) \rightarrow \pi_1^{\text{Nori}}(\text{Pic}_{C/\overline{K}}^0)$  is the abelianization map, and in particular surjective [Ant11, Cor. 3.8]. Therefore,  $\beta_{P,*}: \pi_1^{\text{Nori}}(\text{Pic}_{C/\overline{K}}^0) \rightarrow \pi_1^{\text{Nori}}(Z_{\overline{K}})$  is surjective, and so (Lemma 1.13)  $\ker(\beta_P)$  is an abelian variety.  $\square$

*Remark 5.8.* In positive characteristic, it is possible to specify a large supply of suitable numbers  $n$  in the statement of Corollary 5.7. Indeed, over a finite field, [BL20] provides effective bounds for the degrees of hypersurfaces which meet  $Z$  in a smooth subvariety; and over an arbitrary field of positive characteristic, [BH07, Thm. 3.5] provides effective bounds which similarly ensure that the induced map on fundamental group schemes is surjective.

In each case, the bounds are in terms of the absolute size of the degree, rather than divisibility properties. Consequently, given any  $n \geq 3$ , there exists some  $r$  such that  $(Z, \xi)$  is a Prym–Tyurin variety of exponent  $n^r(g-1)!$ .

**5.2. Prym–Tyurin Prym varieties.** Recall from §2.6.3 that to a finite morphism  $f: C \rightarrow C'$  of smooth projective curves over a field  $K$  we have associated a Prym variety,  $P(C/C')$ , defined to be the complement of  $Y := f^* \text{Pic}_{C'/K}^0$ , and that  $P(C/C')$  comes with an inclusion  $\iota: P(C/C') \hookrightarrow \text{Pic}_{C/K}^0$ . We say that  $P(C/C')$  is a Prym–Tyurin Prym variety of exponent  $e$  if  $\iota^* \lambda_C = e\xi$  for the positive integer  $e$  and some principal polarization  $\xi$  on  $P(C/C')$ . Recall that (2.14) and Corollary 3.10 give alternative descriptions of  $P(C/C')$ , and that in the case that  $C/C'$  is a cyclic Galois cover, Remark 2.30 gives yet another description.

As it turns out, there are a number of restrictions on the exponent and the type of cover that can give rise to a Prym–Tyurin Prym variety. Over an algebraically closed field of characteristic not equal to 2, Mumford [Mum74] classified those degree 2 covers  $f$  which give rise to a Prym–Tyurin Prym variety. We follow the exposition of [BL04, Thm. 12.3.3], and extend it to non-closed fields of all characteristics. Note that there is a slight oversight in [BL04], which is subsequently corrected in [LO11] (the case missing in [BL04] corresponds to (c) in Theorem 5.12, below).

**5.2.1. Inseparable covers.** We want to collect some facts about inseparable covers and Frobenius morphisms. To this end, in this section assume that  $\text{char}(K) = p > 0$ .

Let  $T$  be any scheme of characteristic  $p$ . The  $p^{\text{th}}$  power map on  $\mathcal{O}_T$  defines the absolute Frobenius morphism  $\text{fr}_T : T \rightarrow T$ . If  $X/K$  is any scheme, set  $X^{(p/K)} = X^{(p)} = X \times_{\text{Spec } K, \text{fr}_{\text{Spec } K}} \text{Spec } K$ . The absolute Frobenius morphism  $\text{fr}_X$  of  $X$  is a morphism over  $\text{fr}_K$ , and so, by the definition of the fiber product, factors through a canonically-defined relative Frobenius  $K$ -morphism  $F_{X/K}$ :

$$\begin{array}{ccccc}
 & & \text{fr}_X & & \\
 & & \curvearrowright & & \\
 X & \xrightarrow{F_{X/K}} & X^{(p)} & \xrightarrow{\quad} & X \\
 & \searrow & \downarrow & & \downarrow \\
 & & \text{Spec } K & \xrightarrow{\text{fr}_K} & \text{Spec } K
 \end{array}$$

The relative Frobenius is functorial. In particular, a morphism  $\alpha : X \rightarrow Y$  of  $K$ -schemes induces a morphism  $\alpha^{(p)}$  such that the following diagram commutes:

$$\begin{array}{ccc}
 X & \xrightarrow{\alpha} & Y \\
 \downarrow F_X & & \downarrow F_Y \\
 X^{(p)} & \xrightarrow{\alpha^{(p)}} & Y^{(p)}
 \end{array}$$

If  $r \geq 2$ , we inductively define  $X^{(p^r)} = (X^{(p^{r-1})})^{(p)}$ , and with a slight abuse of notation write  $F_X^{or}$  for  $F \circ_{X^{(p^{r-1})}} \circ \cdots \circ F_X : X \rightarrow X^{(p^r)}$ .

In the special case where  $X$  is actually a commutative group scheme, one can use this group structure to canonically define a so-called Verschiebung morphism  $V_{X/K} : X^{(p)} \rightarrow X$ , which has the property that  $V_{X/K} \circ F_{X/K} = [p]_{X/K}$ , the multiplication-by- $p$  map. This is worked out in generality in [GP11, VIIA.4.3], and in the special case we need in [EvdGM, §5.2]; see also [Oda69]. If  $X$  is an abelian variety, then there is a canonical isomorphism  $\widehat{X^{(p)}} \cong (\widehat{X})^{(p)}$  [Oda69, Prop. 2.1], and Frobenius and Verschiebung are dual in the sense that  $\widehat{V_{X/K}} = F_{\widehat{X}/K}$  and  $\widehat{F_{X/K}} = V_{\widehat{X}/K}$  [Oda69, Cor. 2.2].

Now consider a cover  $f : C \rightarrow C'$  of curves over  $K$ . Then  $f$  factors uniquely as [Sta16, Prop. 0CD2]

$$\begin{array}{ccccc}
 & & f & & \\
 & & \curvearrowright & & \\
 C & \xrightarrow{i=F_{C/K}^r} & C^{(p^r)} = C'' & \xrightarrow{j} & C'
 \end{array} \tag{5.3}$$

where  $i$  is purely inseparable and  $j$  is separable.

As the genus of  $C$  is equal to that of  $C^{(p^r)}$  (e.g, [Sta16, Lem. 0CD0]), from Lemma 3.5 we have that  $(F_{C/K}^r)^* : \text{Pic}_{C^{(p^r)}/K}^0 \rightarrow \text{Pic}_{C/K}^0$  is an isogeny, so that  $i^* : \text{Pic}_{C''/K}^0 \rightarrow \text{Pic}_{C/K}^0$  is an isogeny, and thus  $P(C/C'')$  is trivial.

For later use, we will also want to recall from [Oda69, Prop. 2.1] the effect of the relative Frobenius  $F_{C/K}$  on Picard schemes. There is a canonical identification  $\text{Pic}_{C^{(p)}/K}^0 \cong (\text{Pic}_{C/K}^0)^{(p)}$ ;  $(F_{C/K})_* : \text{Pic}_{C/K}^0 \rightarrow \text{Pic}_{C^{(p)}/K}^0$  and  $(F_{C/K})^* : \text{Pic}_{C^{(p)}/K}^0 \rightarrow \text{Pic}_{C/K}^0$  are then canonically identified with, respectively, the relative Frobenius  $F$  and Verschiebung  $V$  maps of  $\text{Pic}_{C/K}^0$ ; this together with

Lemma 3.5 becomes the factorization  $[p] = F \circ V$ :

$$\begin{array}{ccccc}
& & & & [p] \\
& & \searrow & & \nearrow \\
\text{Pic}_{C^{(p)}/K}^0 & \xrightarrow{F_{C/K}^*} & \text{Pic}_{C/K}^0 & \xrightarrow{F_{C/K,*}} & \text{Pic}_{C^{(p)}/K}^0 \\
\parallel & & \parallel & & \parallel \\
(\text{Pic}_{C/K}^0)^{(p)} & \xrightarrow{V} & \text{Pic}_{C/K}^0 & \xrightarrow{F} & (\text{Pic}_{C/K}^0)^{(p)} \\
& & \nearrow & & \searrow \\
& & & & [p]
\end{array}$$

**Proposition 5.9** (Inseparable covers). *Let  $f: C \rightarrow C'$  be a finite morphism of smooth projective curves over a field  $K$ , factored as in (5.3). Then we have an isomorphism of Prym varieties  $P(C/C')^{(p^r)} = P(C^{(p^r)}/C')$ . Moreover,  $P(C/C')$  is a Prym–Tyurin Prym variety of exponent  $e$ , i.e.,  $\lambda_C|_{P(C/C')} = e\xi$  for a principal polarization  $\xi$  on  $P(C/C')$ , if and only if the Prym variety  $P(C'/C')$  associated to the separable cover  $C'/C'$  is a Prym–Tyurin Prym variety of exponent  $e$ .*

*Proof.* By considering the factorization (5.3), and by induction on the degree of inseparability of the morphism  $f$ , it suffices to prove that given morphisms of curves

$$C \xrightarrow{F_C} C^{(p)} \xrightarrow{h} C'$$

with  $h$  finite, we have  $P(C/C')$  is a Prym–Tyurin Prym variety of exponent  $e$  if and only if  $P(C^{(p)}/C')$  is a Prym–Tyurin Prym variety of exponent  $e$ .

Let  $X = \text{Pic}_C^0$ ; then  $\text{Pic}_{C^{(p)}}^0$  is canonically isomorphic to  $X^{(p)}$  [Oda69, Prop. 2.1]. Note that  $P(C^{(p)}/C')$  is the complement (with respect to the polarization  $\lambda_{C^{(p)}}$ ) of  $h^* \text{Pic}_{C'}^0$  in  $X^{(p)}$ . The equality of morphisms  $V_X = (F_C)^*$  implies that  $P(C/C')$  is the complement in  $X$  (with respect to  $\lambda_C$ ) of  $(hF_C)^* \text{Pic}_{C'}^0 = V_X(h^* \text{Pic}_{C'}^0)$ .

Similarly, the equality of morphisms  $V_X = (F_C)^*$  and  $F_X = (F_C)_*$ , together with the duality  $\widehat{F}_X = V_{\widehat{X}}$  and the calculation of Lemma 3.4, shows that the upper-right hand corner in the following diagram commutes. (The bottom row is obtained from the middle row by the functoriality of Frobenius.)

$$\begin{array}{ccccc}
h^* \text{Pic}_{C'}^0 & \hookrightarrow & X^{(p)} & \xrightarrow{\lambda_{C^{(p)}}} & \widehat{X}^{(p)} \\
\downarrow V_X & & \downarrow V_X & & \downarrow V_{\widehat{X}} \\
V_X(h^* \text{Pic}_{C'}^0) & \hookrightarrow & X & \xrightarrow{\lambda_C} & \widehat{X} \\
\downarrow F & & \downarrow F & & \downarrow F \\
(V_X(h^* \text{Pic}_{C'}^0))^{(p)} & \longrightarrow & X^{(p)} & \xrightarrow{(\lambda_C)^{(p)}} & \widehat{X}^{(p)}
\end{array}$$

The composition of arrows in each column is the multiplication-by- $p$  map. Consequently, considering the right two columns, we have  $[p] \circ \lambda_{C^{(p)}} = (\lambda_C)^{(p)} \circ [p]$ , so that by the freeness of the endomorphism ring as a  $\mathbb{Z}$ -module, we have  $\lambda_{C^{(p)}} = (\lambda_C)^{(p)}$ ; and if we let  $Y = V_X(h^* \text{Pic}_{C'}^0)$ , then  $h^* \text{Pic}_{C'}^0 = Y^{(p)}$ . We therefore have a commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & P(\widehat{C/C'}) & \longrightarrow & 0 \\
& & \downarrow F & & \downarrow F & & \downarrow F & & \\
0 & \longrightarrow & Y^{(p)} & \longrightarrow & X^{(p)} & \longrightarrow & P(\widehat{C/C'})^{(p)} & \longrightarrow & 0
\end{array}$$

and in particular, if we set  $Z = P(C/C')$ , then

$$P(C^{(p)}/C') = P(C/C')^{(p)} = Z^{(p)}.$$

Moreover,

$$Y^{(p)} \times_{X^{(p)}} Z^{(p)} = (Y \times_X Z)^{(p)}.$$

For any integer  $e$ , an easy calculation shows  $([e]_Z)^{(p)} = [e]_{Z^{(p)}}$ , and thus  $(Z[e])^{(p)} = Z^{(p)}[e]$ . Consequently, by Corollary 2.17 and Remark 2.25, we may conclude that  $P(C^{(p)}/C')$  is Prym–Tyurin Prym variety of exponent  $e$  if and only if  $Y^{(p)} \times_{X^{(p)}} Z^{(p)} = Z^{(p)}[e]$ , if and only if  $Y \times_X Z = Z[e]$ , if and only if  $P(C/C')$  is a Prym–Tyurin Prym variety of exponent  $e$ .  $\square$

5.2.2. *Kernel of the pull-back morphism for line bundles.* Next, we recall the relation between covers of curves and the kernel of the pull-back morphism on line bundles.

Say that a finite morphism of curves  $C \rightarrow C'$  is an abelian cover if  $C$  is a torsor over  $C'$  under some finite commutative group scheme.

**Lemma 5.10.** *Let  $f : C \rightarrow C'$  be a finite morphism of curves over  $K$ , and let  $f^*$  be the induced morphism  $f^* : \text{Pic}_{C'/K}^0 \rightarrow \text{Pic}_{C/K}^0$ .*

- (a) *Suppose  $C$  is a torsor under the finite commutative group scheme  $G$ . If  $C$  admits a  $K$ -rational point, then  $\ker(f^*) \cong G^\vee$ ; in general,  $\ker(f^*)^\vee$  is isomorphic to a twist of  $G$ .*
- (b) *Suppose  $f : C \rightarrow C'$  is purely inseparable of degree  $p^r$ . After choosing an isomorphism  $C' = C^{(p^r)}$ , we have  $\ker(f^*) = \ker(V_{\text{Pic}_C^0}^{\text{or}})$ . Moreover,  $f$  is abelian if and only if  $C$  and  $C'$  have genus one.*
- (c) *There is an abelian cover  $C^{\text{abn}} \rightarrow C'$ , which is initial among all abelian covers of  $C'$  through which  $C \rightarrow C'$  factors, and sits in the following diagram:*

$$\begin{array}{ccc} C & \xrightarrow{f} & C' \\ & \searrow & \nearrow \\ & C^{\text{abn}} & \end{array} \quad (5.4)$$

$f^{\text{n-ab}}$        $f^{\text{abn}}$

*If  $C$  admits a  $K$ -rational point, then the cover  $C^{\text{abn}} \rightarrow C'$  is a torsor under  $\ker(f^*)^\vee$ ; in general,  $C^{\text{abn}}$  is a torsor under some twist of  $\ker(f^*)^\vee$ . If  $K$  is perfect or if  $f$  is separable, then formation of  $C^{\text{abn}}$  is compatible with algebraic extension of the base field.*

- (d) *If  $f^*$  is an inclusion, then  $C^{\text{abn}} = C'$ . If  $f$  is separable and if  $C^{\text{abn}} = C'$ , then  $f^*$  is an inclusion.*
- (e) *Suppose  $f$  is separable. Then  $f$  is an abelian cover if and only if  $\deg(f) = \deg(f^*)$ .*

*Proof.* Suppose that  $C$  admits a  $K$ -point,  $P$ , and let  $P' = f(P)$ . The pointed Abel map  $\alpha_P : (C, P) \rightarrow (\text{Pic}_{C/K}^0, 0)$  induces a map  $\pi_1(\alpha_P) : \pi_1^{\text{Nori}}(C) \rightarrow \pi_1^{\text{Nori}}(\text{Pic}_{C/K}^0)$  of fundamental group schemes, which gives an isomorphism on the abelianization of the fundamental group scheme of  $C$  [Ant11, Cor. 3.8]:

$$\pi_1^{\text{Nori}}(C) \twoheadrightarrow \pi_1^{\text{Nori}}(C)^{\text{abn}} \xrightarrow{\sim} \pi_1^{\text{Nori}}(\text{Pic}_{C/K}^0).$$

Now consider the finite morphism  $f : (C, P) \rightarrow (C', P')$  of pointed curves. In the following diagram, the existence of the top row is built into the theory of Nori’s fundamental group; the descent to the middle row is because, by hypothesis,  $G$  is abelian; and the commutativity of the bottom

rectangle follows from (3.5) and the chosen trivializations of  $\text{Pic}_{C/K}^{(1)}$  and  $\text{Pic}_{C'/K}^{(1)}$ :

$$\begin{array}{ccccccc}
0 & \longrightarrow & \pi_1^{\text{Nori}}(C) & \xrightarrow{\pi_1(f)} & \pi_1^{\text{Nori}}(C') & \longrightarrow & G \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \sim \\
& & \pi_1^{\text{Nori}}(C)^{\text{abn}} & \xrightarrow{\pi_1(f)^{\text{abn}}} & \pi_1^{\text{Nori}}(C')^{\text{abn}} & \longrightarrow & G \longrightarrow 0 \\
& & \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\
& & \pi_1^{\text{Nori}}(\text{Pic}_{C/K}^0) & \xrightarrow{\pi_1(f_*)} & \pi_1^{\text{Nori}}(\text{Pic}_{C'/K}^0) & \longrightarrow & G \longrightarrow 0
\end{array}$$

In particular,  $\text{coker}(\pi_1(f_*)) \cong G$ . By Lemma 1.14,  $\ker(\widehat{f}_*) \cong G^\vee$ . Now (a) follows because  $\ker(\widehat{f}_*) \cong \ker(f^*)$  (Lemma 3.4). If  $C$  admits no  $K$ -point, then  $C$  is still a torsor over  $C'$  under a group scheme which becomes isomorphic to  $\ker(f^*)^\vee$  after a finite extension of  $K$ .

For (b), suppose  $f : C \rightarrow C'$  is a purely inseparable abelian cover of degree  $p^r$ . The description of  $\ker(f^*)$  follows from our discussion of Frobenius above. For the remaining claim, on one hand, since  $f$  is purely inseparable,  $C$  and  $C'$  have the same genus. On the other hand, because  $C$  is a torsor over  $C'$ , its Euler characteristic  $\chi(C)$  satisfies  $\chi(C) = \deg(f)\chi(C')$ . Therefore,  $\chi(C) = \chi(C') = 0$ , and the common genus of  $C$  and  $C'$  is one.

For (c), suppose  $C_1$  and  $C_2$  are torsors over  $C'$  under respective finite commutative group schemes  $G_1$  and  $G_2$  through which  $f$  factors. Then  $C_{12} := C_1 \times_{C'} C_2$  is a torsor under the finite commutative group scheme  $G_1 \times G_2$ , and  $f$  factors through  $C_{12}$ . Since  $C \rightarrow C'$  is finite, we may find an initial such cover, and produce the factorization (5.4). The description of the covering group of  $C^{\text{abn}} \rightarrow C'$  is given in (a).

We investigate the compatibility of the formation of  $C^{\text{abn}}$  with algebraic field extensions  $L/K$ . If  $f$  is separable, the claimed compatibility is classical; indeed,  $C^{\text{abn}}$  is the curve whose function field  $K(C^{\text{abn}})$  is the maximal unramified abelian extension of  $K(C')$  inside  $K(C)$ . Otherwise, suppose that  $L/K$  is finite and separable; we may assume  $L/K$  is actually Galois. Then the collection of abelian covers of  $C'_L$  through which  $f_L$  factors is stable under  $\text{Gal}(L/K)$ , and thus  $(C_L)^{\text{abn}}$  descends to  $K$ .

For (d), suppose  $f^*$  is an inclusion. Let  $C'' \rightarrow C'$  be a torsor over  $C'$  under the finite commutative group scheme  $G$ . If  $C \rightarrow C'$  factors through  $C''$ , then  $f^*$  factors through  $g^*$ . Since  $\ker(g^*) \subseteq \ker(f^*) = \{1\}$ , the trivial group scheme, we find that  $C''$  is a torsor under  $\ker(g^*)^\vee \cong \{1\}$ .

For the converse, suppose  $f$  is separable. Then the usual theory of the étale fundamental group shows that étale (Galois) subcovers of  $C'$  correspond to (normal) subgroups of  $\pi_1^{\text{ét}}(C')$  which contain the image of  $\pi_1^{\text{ét}}(C)$ , and étale abelian subcovers of  $C'$  correspond to subgroups of  $\pi_1^{\text{ab}}(C')$  which contain the image of  $\pi_1^{\text{ab}}(C)$ . Consequently,  $C^{\text{abn}}$  is a torsor over  $C'$  under  $\ker(f^*)^\vee$ . In particular,  $C^{\text{abn}} = C'$  if and only if  $f^*$  is an inclusion.

Finally, we address (e); suppose  $f$  is separable. Part (a) implies that if  $f$  is abelian with covering group the étale group scheme  $G$ , then  $\deg(f) = \text{ord}(G) = \text{ord}(G^\vee) = \deg(f^*)$ . Conversely, if  $\deg(f) = \deg(f^*)$ , then  $\deg(f^{n-\text{ab}}) = 1$ , and thus  $C \rightarrow C'$  is an abelian cover.  $\square$

*Remark 5.11.* Lemma 5.10, via its reliance on [Ant11], encodes (and is built upon) the starting point of geometric class field theory: for a fixed curve  $C'$ , there is a bijection between cyclic étale covers  $C \rightarrow C'$  of degree  $N$  and multiplicative cyclic subgroups of  $\text{Pic}_{C'/K}^0$  of degree  $N$ , i.e., sub-group schemes of  $\text{Pic}_{C'/K}^0$  which are isomorphic, over  $\overline{K}$ , to  $(\mathbb{Z}/N)^\vee = \mu_N$ . (In fact, this is a special case of the general fact that, for a pointed smooth proper scheme  $X$  over  $S$  and a finite commutative group scheme  $G/S$ , there is a canonical bijection  $H_\bullet^1(X, G) \cong \text{Hom}(G^\vee, \text{Pic}_{X/S}^\tau)$ , where the left-hand side

denotes the subgroup of torsors which are pointed over the given section of  $X$ , and  $\text{Pic}_{X/S}^\tau$  denote the torsion component of the Picard scheme [Ant11, Prop. 3.2].) In particular, if  $C'_{\bar{K}}$  admits such a cover, then there exists an inclusion  $\mu_{N,\bar{K}} \hookrightarrow \text{Pic}_{C'/\bar{K}}^0$ .

If  $N$  is invertible in  $K$ , this is always possible; but if  $p = \text{char}(K) > 0$ , then the existence of such an inclusion is a nontrivial constraint on  $C'$ .

Indeed, recall that if  $A/K$  is an abelian variety over a field, then there exists an integer  $f$ ,  $0 \leq f \leq \dim A$ , such that the étale quotient  $A[p]^{\text{ét}}$  has order  $p^f$ ; this is the  $p$ -rank of  $A$ . Equivalently,  $A[p](\bar{K}) \cong (\mathbb{Z}/p)^f$ . Since  $A[p]^\vee \cong A[p]$ , the  $p$ -rank may also be computed as  $\dim_{\mathbb{F}_p} \text{Hom}(\mu_{p,\bar{K}}, A[p]_{\bar{K}})$ . The abelian variety is called ordinary if  $f = \dim A$ , i.e., its  $p$ -rank is as large as possible. The  $p$ -rank of a curve is by definition the  $p$ -rank of its Jacobian, and a curve is called ordinary if its Jacobian is.

Thus,  $C'$  admits an étale cyclic  $p$ -cover (over  $\bar{K}$ ) if and only if its  $p$ -rank is positive. We return to this point in Remark 5.15.

5.2.3. *Classification of Prym–Tyurin Prym varieties.* We now classify all Prym–Tyurin Prym varieties over a field. The cases where the genus of the base curve is 0, or the genus of the base curve is positive and equal to the genus of the cover (i.e., the case  $g(C) = g(C') = 1$  and  $f$  is étale), or the case where the cover has degree 1, are trivial, giving rise to Prym varieties of exponent 1 (see Corollary 2.19), and so we exclude those cases below.

**Theorem 5.12** (Classification of Prym–Tyurin Prym varieties [BL04, Thm. 12.3.3]). *Let  $f: C \rightarrow C'$  be a finite morphism of smooth projective curves over a field  $K$  with respective genera  $g > g' \geq 1$ , let*

$$C \xrightarrow{i} C'' \xrightarrow{j} C'$$

*be the unique, up to isomorphism, factorization of  $f$  with  $j$  separable and  $i$  purely inseparable, let  $d$  be the degree of  $j$ , and let  $e$  be a positive integer.*

*The Prym variety  $P(C/C')$  is a Prym–Tyurin variety of exponent  $e$ ; i.e.,  $\lambda_C|_{P(C/C')} = e\xi$  for a principal polarization  $\xi$  on  $P(C/C')$ , if and only if  $C''/C'$  is one of the following types:*

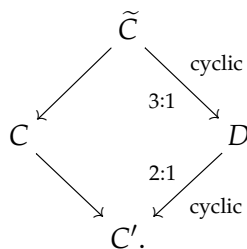
- (a)  $d = 2$  and  $j$  is étale, in which case  $e = 2$  and  $\dim P(C/C') = g' - 1$ ;
- (b)  $d = 2$  and the ramification divisor of  $j$  has degree 2, in which case  $e = 2$  and  $\dim P(C/C') = g'$ ;
- (c)  $d = 3$ ,  $j$  is étale and noncyclic, and  $g' = 2$ , in which case  $e = 3$  and  $\dim P(C/C') = g' = 2$ ;
- (d)  $g = 2$  and  $g' = 1$ , in which case  $e = \deg j^{n\text{-ab}}$  and  $\dim P(C/C') = g' = 1$ , where  $j^{n\text{-ab}}$  is the morphism from the factorization of  $j$  in (5.4) via the maximal abelian cover defined in Lemma 5.10.

*Remark 5.13* (Degree 2 covers). In cases (a) and (b) the cover  $j$  is cyclic of degree 2, and so is Galois. Consequently, there is an involution  $\sigma: C'' \rightarrow C''$  with  $C' = C''/\langle\sigma\rangle$ , and  $P(C''/C') = \text{Im}(1 - \sigma^*)$  (Remark 2.30).

*Remark 5.14* (Degree 2 ramification). In case (b) there are two possibilities. Since  $\deg(j) = 2$ , it is Galois; and if  $P$  is a ramification point of  $j$ , then its ramification index is therefore exactly two. Either  $\text{char}(K) \neq 2$ , and  $j$  is (necessarily tamely) ramified at exactly two geometric points; or  $\text{char}(K) = 2$ ,  $j$  is (necessarily wildly) ramified at a single point, and the lower ramification filtration is trivial at step 2. We call the latter case weakly wildly ramified; any such cover looks formally locally like  $K[[x]] \hookrightarrow K[[x]][y]/(y^2 - y - x)$ . We will never need this explicit description, and refer the reader to [Ser79, Ch. IV] for a reminder on ramification filtrations.

*Remark 5.15* ( $p$ -ranks and Pryms). Recall (Remark 5.11) that if  $C'/K$  is a smooth projective curve over a field of characteristic  $p > 0$ , then  $C'$  geometrically admits a cyclic étale cover of degree  $p$  if and only if its  $p$ -rank is positive. We briefly explore the interaction between the  $p$ -rank of  $C'$  and the classification in Theorem 5.12.

- (a) Suppose that  $K$  has characteristic 2.
- (i) If  $f: C \rightarrow C'$  satisfies the hypotheses of Theorem 5.12 case (a), we have seen that  $C'$  has positive 2-rank (Remark 5.11).
  - (ii) If  $f$  satisfies the hypotheses of Theorem 5.12 case (c), this also forces  $C'$  to have positive 2-rank. Indeed, we assume that  $C = C''$ , i.e., that  $C \rightarrow C'$  is separable. Let  $\tilde{C} \rightarrow C'$  be the Galois closure of  $C$  over  $C'$ . It is étale over  $C'$  and its Galois group, being a non-cyclic transitive subgroup of  $S_3$ , is  $\text{Aut}(\tilde{C}/C') \cong S_3$ . Let  $D$  be the quotient of  $\tilde{C}$  by the unique, thus normal, subgroup of order 3. We have a diagram of étale covers of curves



In characteristic  $p = 2$ , the existence of the étale double cover  $D \rightarrow C'$  once again forces  $C'$  to have positive 2-rank.

- (b) Note that in characteristic  $p = 3$ , if  $f$  satisfies the hypotheses of Theorem 5.12 case (c), then one can apply the same analysis as in the previous case, and the existence of a cyclic étale triple cover  $\tilde{C} \rightarrow D$  forces  $D$  to have positive  $p$ -rank. However, in characteristic 3 this is automatic [FvdG04, (7.1)]: an étale double cover of a curve of genus 2 has positive 3-rank.
- (c) In characteristic  $p \geq 5$ , we make a side remark regarding Theorem 5.12 case (a) that there exist étale double covers of curves of genus 2 which have  $p$ -rank zero [OP19, Prop. 6.1].

*Proof of Theorem 5.12.* As promised, we follow the argument of [BL04, Thm. 12.3.3]. First, from Proposition 5.9, it suffices to prove the case that  $f = j$ . We set  $Z = P(C/C')$  for brevity.

We start by showing that if  $Z$  is a Prym–Tyurin Prym variety of exponent  $e$ , then  $f$  is in one of the cases delineated in the statement of the theorem. First, since  $\dim \text{Pic}_{C/K}^0 = g > g - g' = \dim Z > 0$  we may assume that  $e \geq 2$  (Corollary 2.19). By hypothesis,  $i_Z^* \lambda_C = e \xi$  for some principal polarization  $\xi$  on  $Z$ , which, as we noted at the beginning of §5.1 (equivalently, Remark 2.25) is equivalent to  $K(i_Z^* \Theta) = Z[e]$ ; by Corollary 2.17 (see also Remark 2.18), we then have  $K(i_Y^* \lambda_C) \cong Z[e]$ . In particular,  $\dim Y \geq \dim Z$  (since  $K(i_Y^* \lambda_C) \subseteq Y[e]$ , and so  $e^{2dz} = \deg[e]_Z = \deg K(i_Y^* \lambda_C) \leq \deg[e]_Y = e^{2d_Y}$ ), i.e.,  $g' \geq g - g'$ , and so

$$2g' \geq g. \quad (5.5)$$

In our setting the Riemann–Hurwitz formula reads  $2g - 2 = d(2g' - 2) + \delta$ , where  $\delta$  is the degree of the ramification divisor, which combined with (5.5) gives

$$2g' - 1 \geq g - 1 = d(g' - 1) + \frac{\delta}{2} \geq d(g' - 1) \quad (5.6)$$

and we find that

$$d - 1 \geq (d - 2)g'. \quad (5.7)$$

- Suppose  $d \geq 3$  and  $g' \geq 3$ . Then one easily derives a numerical contradiction in (5.7).
- Suppose  $d \geq 3$  and  $g' = 2$ . Then from (5.7) we must have  $d = 3$ , giving equality in (5.6) so that  $\delta = 0$  and  $g = 4$ . In particular,  $f$  is étale, and  $\dim Y = \dim Z = 2$ . By Lemma 3.9,  $e|d$ ; since  $e > 1$ , we have  $e = 3$ . By hypothesis  $Z$  is a Prym–Tyurin variety, and thus  $K(i_Z^* \lambda_C) = Z[3] \cong K(i_Y^* \lambda_C)$ . Therefore, since  $\dim Y = \dim Z$ , we have  $K(i_Y^* \lambda_C) = Y[3]$ .

- Suppose  $f$  were cyclic; we will derive a contradiction. Then  $f^*$  is not injective (Lemma 5.10), so  $f^*$  factors as  $\iota_Y h$  with  $h$  a nontrivial isogeny. On one hand, this implies that  $\deg((f^*)^* \lambda_C) = \deg((\iota_Y h)^* \lambda_C) = \deg(\widehat{h}) \deg(\iota_Y^* \lambda_C) \deg(h) > \deg \iota_Y^* \lambda_C$ , and therefore the orders of their kernels satisfy  $|\mathbf{K}((f^*)^* \lambda_C)| = |\mathbf{K}((\iota_Y h)^* \lambda_C)| > |\mathbf{K}(\iota_Y^* \lambda_C)| = |Y[3]| = |\mathrm{Pic}_{C'/K}^0[3]|$ . On the other hand, Lemma 3.6 implies that  $\mathbf{K}((f^*)^* \lambda_C) = \mathrm{Pic}_{C'/K}^0[3]$ , contradicting the previous degree estimate.
- If  $f$  is *not* cyclic, then we are in case (c) of the theorem.
- Suppose  $d \geq 2$  and  $g' = 1$ . Then by (5.5) we have  $g = 2$ . So we are in case (d); the only thing left to show is the statement about the exponent. As we have seen, the exponents of  $Z$  and  $Y$  are the same (Remark 2.18), so it suffices to find the exponent of  $Y$ . For this, consider the factorization of (5.4):

$$\begin{array}{ccc} C & \xrightarrow{f} & C' \\ & \searrow f^{\mathrm{n-ab}} & \nearrow f^{\mathrm{abn}} \\ & & C^{\mathrm{abn}} \end{array} \quad (5.8)$$

By assumption  $f$  is separable, and consequently so is  $f^{\mathrm{abn}}$ . Now,  $C^{\mathrm{abn}}$  is a torsor over  $C'$  under a finite flat commutative group scheme; since  $f^{\mathrm{abn}}$  is separable, this group scheme is reduced, and so  $f^{\mathrm{abn}}$  is étale. From Riemann–Hurwitz, we conclude that  $g(C^{\mathrm{abn}}) = 1$ , so that  $(f^{\mathrm{abn}})^*$  is an étale isogeny of elliptic curves, and the image of  $f^*$  is equal to the image of  $(f^{\mathrm{n-ab}})^*$ . We then use Lemma 3.6 which says  $((f^{\mathrm{n-ab}})^*)^* \lambda_C = \deg(f^{\mathrm{n-ab}}) \lambda_{C^{\mathrm{abn}}}$ .

- Suppose  $d = 2$  and  $g' \geq 2$ , then one shows that  $\delta \leq 2$ . Parity shows that  $\delta \in \{0, 2\}$ , and thus we are in case (a) or (b); recall that it follows immediately from (3.9) that  $e = 2$ .

Now, conversely, we show that in any of the cases (a)–(d) of the theorem,  $\mathbf{K}(\iota_Z^* \lambda_C) = Z[e]$ , and thus that  $P(C/C')$  is a Prym–Tryurin Prym. Note that once we have established this, then the discussion above about the exponent applies in all of the cases.

In case (d),  $Z$  is an elliptic curve, and so any polarization is a multiple of the principal polarization.

In case (b), since  $\delta = 2$  we have by Riemann–Hurwitz that  $g = 2g'$ . Moreover,  $f$  is totally ramified at its ramification points, and thus the only (abelian) torsor through which  $C \rightarrow C'$  factors is the trivial torsor  $C'$  itself. Therefore,  $f^*$  is injective (Lemma 5.10),  $Y = \mathrm{Pic}_{C'/K}^0$ , and  $\iota_Y = f^*$ . This means  $\iota_Y^* \lambda_C = (f^*)^* \lambda_C$ , and by Lemma 3.6 we find that this is  $2\lambda_{C'}$  so that  $\mathbf{K}(\iota_Y^* \lambda_C) = \mathrm{Pic}_{C'/K}^0[2]$ . Since  $\dim Y = \dim Z$ , by Corollary 2.17 (see also Remark 2.18) we have  $\mathbf{K}(\iota_Z^* \lambda_C) = Z[2]$ , and  $Z$  is a Prym–Tyurin variety.

Case (c) is similar; we have  $g' = 2$ , so that, by Riemann–Hurwitz,  $g = 4$ , which implies that  $\dim Y = \dim Z = 2$ . Since  $\deg f = 3$  is prime,  $f: C \rightarrow C'$  does not factor through a nontrivial cover of curves. Because  $f$  itself is not cyclic,  $f^*$  is thus injective (Lemma 5.10),  $Y = \mathrm{Pic}_{C'/K}^0$ , and  $\iota_Y = f^*$ , and so we have, again by Lemma 3.6, that  $\mathbf{K}(\iota_Y^* \Theta) = \mathrm{Pic}_{C'/K}^0[3]$ . Consequently, again by Corollary 2.17 (see also Remark 2.18),  $\mathbf{K}(\iota_Z^* \lambda_C) = Z[3]$ , and  $Z$  is a Prym–Tyurin variety.

Finally, in case (a), we have that  $f$  is an étale double cover. Note that from Riemann–Hurwitz we have  $g = 2g' - 1$ . From Lemma 5.10 there is a factorization  $f^* = \iota_Y \circ h$ , where  $h: \mathrm{Pic}_{C'/K}^0 \rightarrow Y$  is an isogeny of degree 2. Since  $\mathbf{K}(h^* \iota_Y^* \lambda_C) = \mathrm{Pic}_{C'/K}^0[2]$  (Lemma 3.6), we have that  $\mathbf{K}(\iota_Y^* \lambda_C)$  is a sub-group scheme of  $Y[2]$  of index 4. From Remark 2.18 we have  $\mathbf{K}(\iota_Y^* \lambda_C) \cong \mathbf{K}(\iota_Z^* \lambda_C)$ , and the containment  $\mathbf{K}(\iota_Y^* \lambda_C) \subseteq Y[2]$  implies  $\mathbf{K}(\iota_Z^* \lambda_C) \subseteq Z[2]$ . Since  $\dim Z = g - g' = g' - 1$ , after considering the degrees of these group schemes, we see that the only possible sub-group scheme of  $Z[2]$  isomorphic to this group scheme is  $Z[2]$  itself. Therefore,  $\mathbf{K}(\iota_Z^* \lambda_C) = Z[2]$ , and  $Z$  is a Prym–Tyurin variety of exponent 2.  $\square$

As explained in Remark 2.30, Prym varieties often arise in nature from a curve with automorphisms. Using Theorem 5.12, it is not hard to classify those group actions giving rise to Prym–Tyurin Prym varieties. In particular, we extract the following converse to Remark 5.13, which is entirely classical away from characteristic two, but seems less well-known in the case of even characteristic.

**Corollary 5.16** (Prym–Tyurin Prym varieties associated to involutions). *Let  $C$  be a smooth projective curve of genus  $g$  over a field  $K$ , and let  $\sigma: C \rightarrow C$  be a nontrivial separable involution. Let  $C' = C / \langle \sigma \rangle$  be the quotient curve. Then the Prym variety  $P(C/C')$  is a Prym–Tyurin variety if and only if either:*

- (a)  $\iota$  acts without fixed points, or
- (b) (i)  $\text{char}(K) \neq 2$  and  $\sigma$  has exactly two fixed points; or
  - (ii)  $\text{char}(K) = 2$ ,  $\sigma$  has exactly one fixed point, and the action of  $\sigma$  is weakly wildly ramified there, in the sense of Remark 5.14, that at the unique fixed point,  $C$  looks formally locally like  $K[[x]][y]/(y^2 - y - x)$ , with local involution  $y \mapsto y + 1$ .

In both cases the exponent is  $e = 2$ . In case (a) one has  $\dim P(C/C') = g' - 1 = (g - 1)/2$ , and in case (b) one has  $\dim P(C/C') = g' = g/2$ .

*Proof.* This is an immediate consequence of Theorem 5.12 and Remark 5.14.  $\square$

Consider a cyclic  $p$ -cover  $C \rightarrow C'$  of curves in characteristic  $p$ . The Deuring–Shafarevich formula allows one to compute the  $p$ -rank  $r(C)$  in terms of the  $p$ -rank  $r(C')$ , and thus that of  $P(C/C')$ . (See [Shi11] for a precise statement and the history of this problem, and [CU23] for a recent and far-reaching generalization.) The  $p$ -rank is a partial invariant of the  $p$ -torsion group scheme of an abelian variety. In the special case of a Prym–Tyurin Prym variety, we can actually compute the isomorphism class of  $P(C/C')[p]$ :

**Lemma 5.17.** *Let  $f: C \rightarrow C'$  be a finite separable morphism of degree  $d$  of smooth curves over a field  $K$  of characteristic  $p$ . Let  $Y = f^* \text{Pic}_{C'/K}^0$ , and let  $Z = P(C/C')$  be the corresponding Prym variety.*

- (a) *Suppose  $d = p = 2$  and  $f$  is étale. Then  $Y[2]_{\bar{K}} \cong \text{Pic}_{C'/K}^0[2]_{\bar{K}}$ , and there is an exact sequence of group schemes*

$$0 \longrightarrow Z[2] \longrightarrow Y[2] \longrightarrow H \longrightarrow 0$$

where  $H_{K^{\text{perf}}} \cong (\mu_2 \oplus \mathbb{Z}/2)$ .

- (b) *Suppose  $d = p = 2$  and the ramification divisor of  $f$  has degree 2. Then  $Z[2] \cong \text{Pic}_{C'/K}^0[2]$ .*
- (c) *Suppose  $d = p = 3$ ,  $f$  is étale and noncyclic, and  $C'$  has genus 2. Then  $Z[3] \cong \text{Pic}_{C'/K}^0[3]$ .*

*Proof.* In cases (b) and (c), we have already seen in the proofs of the corresponding cases of Theorem 5.12 that  $Y \cong \text{Pic}_{C'}^0$ , and that  $Z[p] = K(\iota_Z^* \lambda_C) = K(\iota_Y^* \lambda_C) = Y[p]$ .

Thus, we now consider case (a). Let  $r(Y)$  and  $r(Z)$  denote the  $p$ -ranks of  $Y$  and  $Z$ , respectively, and let  $r(C)$  and  $r(C')$  be the respective  $p$ -ranks of  $\text{Pic}_{C/K}^0$  and  $\text{Pic}_{C'/K}^0$ . For later use, we recall that the Deuring–Shafarevich formula (e.g., [Shi11, Thm. 1.1]) implies that  $r(C) = 2r(C') - 1$ . Since the  $p$ -rank is an isogeny invariant, and additive for direct sums of abelian varieties, we find that  $r(Z) = r(Y) - 1 = r(C') - 1$ .

We have  $N := \ker(f^*) \cong \mu_2$ , a group scheme of exponent two; then

$$Y[2] \cong \left( \frac{\text{Pic}_{C'}^0[4]}{N} \right) [2].$$

Over  $\bar{K}$ , we have  $\text{Pic}_{C'}^0[4]_{\bar{K}} \cong T \oplus G$  where  $T \cong (\mu_4)^{\oplus r(C')}$  and  $\text{Hom}(\mu_2, G) = (0)$ . Therefore,  $N \subset T$  and we have

$$(\text{Pic}_{C'}^0[4]_{\bar{K}}/N)[2] \cong (T/N)[2] \oplus G[2] \cong \text{Pic}_{C'}^0[2]_{\bar{K}},$$

where the last isomorphism follows from the fact that  $T/N$  is again a finite multiplicative group scheme of rank  $r(C')$ .

We have seen above that  $K(\iota_Z^* \lambda_{C'}) = Z[2] = Z \times_{\text{Pic}_{C/K}^0} Y \subset Y[2]$ , and so there is an exact sequence of 2-torsion group schemes

$$0 \longrightarrow Z[2] \longrightarrow Y[2] \longrightarrow H \longrightarrow 0.$$

Rank conditions show that  $H$  is a self-dual 2-torsion group scheme of order  $2^{2 \cdot (\dim Y - \dim Z)} = 4$ . Because  $r(Y) = r(Z) - 1$ , we have  $\dim_{\mathbb{F}_2} H^{\text{et}} = 1$ . Since  $H$  is self-dual,  $H$  is an extension of  $\mathbb{Z}/2$  by  $\mu_2$ ; in particular,  $H_{K^{\text{perf}}} \cong \mu_2 \oplus \mathbb{Z}/2$ .  $\square$

*Remark 5.18.* The analogous claim of 5.17 for situation Theorem 5.12 is false in every positive characteristic. Indeed, given a field  $K$  of characteristic  $p > 0$ , choose elliptic curves  $E_0$  and  $E_1$  over  $K$  with  $E_0$  supersingular and  $E_1$  ordinary, i.e., with  $r(E_i) = i$ ; note that if  $E'_i$  is isogenous to  $E_i$  for  $i = 0, 1$ , then  $E'_0[p]_{\bar{K}} \not\cong E'_1[p]_{\bar{K}}$ . Let  $A/K$  be a principally polarized abelian surface over  $K$  which is isogenous to  $E_0 \times E_1$ , but is not isomorphic to a product of elliptic curves. Then  $A$  is the Jacobian of a smooth projective curve,  $C$ . Moreover,  $C$  admits a nontrivial map to  $E_0$ , but  $P(C/E_0)$  is isogenous to  $E_1$ .

## 6. PRYM SCHEMES

While the first part of this paper treated abelian schemes over arbitrary bases, the reader will have noticed our retreat in §4 and §5 to objects over fields. We now return to working over a connected locally Noetherian scheme  $S$ . For ease of notation, given a polarized abelian variety  $(Z, \xi)$  over a field  $K$ , we will write  $[\xi_{\bar{K}}]$  for the class  $[\Xi_{\bar{K}}]$  for any ample divisor  $\Xi_{\bar{K}}$  on  $Z_{\bar{K}}$  such that  $\xi_{\bar{K}} = \phi_{\Xi_{\bar{K}}}$ .

**6.1. Prym–Tyurin schemes and a relative Welters’ criterion.** The starting point is a relative version of Welters’ Criterion:

**Theorem 6.1** (Relative Welters’ Criterion). *Let  $(Z, \xi)$  be a principally polarized abelian scheme of dimension  $g_Z$  over  $S$  and let  $C/S$  be a smooth proper curve.*

(a) *Suppose there is an  $S$ -morphism  $\beta: C \rightarrow T$  to a torsor under  $Z$  over  $S$  such that*

(i) *the composition  $\widehat{Z} \xrightarrow[\sim]{\zeta_Z} \text{Pic}_{T/S}^0 \xrightarrow{\beta^*} \text{Pic}_{C/S}^0$  is an injective  $S$ -homomorphism of abelian varieties, where  $\zeta_Z$  is the canonical isomorphism (1.6), and*

(ii) *for every geometric point  $\bar{s}$  of  $S$  (equivalently, for a single geometric point  $\bar{s}$  of  $S$ ), after identifying  $T_{\bar{s}}$  and  $Z_{\bar{s}}$ , we have  $\beta_{\bar{s}*}[C_{\bar{s}}] \equiv e \frac{[\xi_{\bar{s}}]^{g_Z - 1}}{(g_Z - 1)!}$ .*

*Then the inclusion  $\iota_Z := \beta^* \zeta_Z \xi: Z \hookrightarrow \text{Pic}_{C/S}^0$  makes  $(Z, \xi)$  a Prym–Tyurin scheme of exponent  $e$ ; i.e.,  $\iota_Z^* \lambda = e\xi$ , so that  $(Z, \xi, C, \iota_Z)$  is an embedded Prym–Tyurin scheme of exponent  $e$ .*

(b) *Conversely, suppose there is an inclusion  $\iota_Z: Z \hookrightarrow \text{Pic}_{C/S}^0$  making  $(Z, \xi)$  a Prym–Tyurin scheme of exponent  $e$ ; i.e.,  $(Z, \xi, C, \iota_Z)$  is an embedded Prym–Tyurin scheme of exponent  $e$ . Let  $Y$  be the complement of  $Z$ . Then, under the isomorphism  $Z \xrightarrow[\sim]{-\xi} \widehat{Z} \xrightarrow[\sim]{} \text{Pic}_{C/S}^0 / Y$ , (see (2.10))  $T = \text{Pic}_{C/S}^{(1)} / Y$  is a torsor under  $Z$ , and the composition  $\beta: C \xrightarrow{\alpha^{(1)}} \text{Pic}_{C/S}^{(1)} \rightarrow T$  of the Abel map with the quotient map satisfies conditions (i) and (ii) above.*

*Moreover, these constructions are inverse to one another, up to canonical isomorphisms.*

*Proof.* It suffices to establish the claim on geometric fibers, and so this reduces to Theorem 5.3.  $\square$

As a consequence, we get the Matsusaka Criterion:

**Corollary 6.2** (Relative Matsusaka Criterion). *Let  $(Z, \xi)$  be a principally polarized abelian scheme of dimension  $g_Z > 0$  over  $S$  and let  $C/S$  be a smooth proper curve. Suppose there is an  $S$ -morphism  $\beta: C \rightarrow T$  to a torsor under  $Z$  over  $S$  such that for every geometric point  $\bar{s}$  of  $S$  (equivalently, for a single geometric point  $\bar{s}$  of  $S$ ), after identifying  $T_{\bar{s}}$  and  $Z_{\bar{s}}$ , we have*

$$\beta_{\bar{s}*}[C_{\bar{s}}] \equiv \frac{[\xi_{\bar{s}}]^{g_Z-1}}{(g_Z-1)!}. \quad (6.1)$$

*Then the composition  $\iota_Z := \beta^* \zeta_Z \xi: Z \rightarrow \text{Pic}_{C/S}^0$  is an isomorphism, and there is a unique isomorphism  $\gamma: \text{Pic}_{C/S}^{(1)} \rightarrow T$  such that  $\beta = \alpha^{(1)} \gamma$ . In other words, up to an isomorphism of torsors,  $\beta$  is the Abel map.*

*Proof.* It suffices to check the assertions on geometric fibers, and so this reduces to Corollary 5.4.  $\square$

**Remark 6.3** (Relative Matsusaka–Ran Criterion). In fact, using [Col84, Thm. p.329], one can replace the condition (6.1) in Corollary 6.2 with the weaker condition that  $\beta_{\bar{s}*}[C_{\bar{s}}] \cdot [\Xi_{\bar{s}}] = g_Z$ .

**6.2. Prym–Tyurin Prym schemes.** We now prove our main theorem.

**Theorem 6.4** (Classification of Prym–Tyurin Prym schemes). *Let  $f: C \rightarrow C'$  be a finite morphism of smooth proper curves over  $S$  with geometrically connected fibers of respective genera  $g > g' \geq 1$ , and suppose that  $f$  is fiberwise separable. Let  $e$  be a positive integer. The following are equivalent.*

- (a)  $P(C/C')$  is a Prym–Tyurin Prym scheme of exponent  $e$ ; i.e.,  $\lambda_C|_{P(C/C')} = e\xi$  for a principal polarization  $\xi$  on  $P(C/C')$ .
- (b) For every point  $s$  in  $S$ , we have that  $P(C_s/C'_s)$  is a Prym–Tyurin Prym variety of exponent  $e$ .
- (c) For every geometric point  $\bar{s}$  in  $S$ , we have that  $P(C_{\bar{s}}/C'_{\bar{s}})$  is a Prym–Tyurin Prym variety of exponent  $e$ .
- (d) For every point  $s$  of  $S$ ,  $f_s$  is one of the four types delineated in Theorem 5.12 with corresponding exponent  $e$ .
- (e) For every geometric point  $\bar{s}$  of  $S$ ,  $f_{\bar{s}}$  is one of the four types delineated in Theorem 5.12 with corresponding exponent  $e$ .
- (f) For one point  $s$  of  $S$ , we have that  $P(C_s/C'_s)$  is a Prym–Tyurin Prym variety of exponent  $e$ .
- (g) For one geometric point  $\bar{s}$  of  $S$ , we have that  $P(C_{\bar{s}}/C'_{\bar{s}})$  is a Prym–Tyurin Prym variety of exponent  $e$ .
- (h) For one point  $s$  of  $S$ , we have that  $f_s$  is one of the four types delineated in Theorem 5.12 with corresponding exponent  $e$ .
- (i) For one geometric point  $\bar{s}$  of  $S$ , we have that  $f_{\bar{s}}$  is one of the four types delineated in Theorem 5.12 with corresponding exponent  $e$ .

*Proof.* The equivalence of (a), (b), and (c) (resp. (f) and (g)) follows from Lemma 1.11. The equivalence of (b) and (d) (resp. (f) and (h)) is Theorem 5.12, as is the equivalence of (c) and (e) (resp. (g) and (i)). In other words, we have (a)  $\iff$  (b)  $\iff$  (c)  $\iff$  (d)  $\iff$  (e) and (f)  $\iff$  (g)  $\iff$  (h)  $\iff$  (i).

We complete the proof by showing the equivalence of (e) and (i). The degree of the finite flat morphism  $f$  is constant on geometric fibers. As a consequence, Riemann–Hurwitz gives that the degrees of the ramification divisors are also constant. All of the conditions in Theorem 5.12, with the exception of the condition for a noncyclic cover in Theorem 5.12(c), are stated in terms of the genera of the curves, the degrees of the covers, and the degrees of the ramification divisors. The equivalence of (e) and (i) then follows in all cases of Theorem 5.12 once we recall that the degree (over  $C'$ ) of the maximal étale abelian subcover of  $C'' \rightarrow C'$  is constant on  $S$  (Lemma 3.8).  $\square$

**Remark 6.5.** In fact, the proof of Theorem 6.4 shows that if, for some point  $s \in S$ ,  $f_s$  is of type (t) for one of the four types delineated in Theorem 5.12, then the same is true for every point  $s' \in S$ . Note

that the good reduction of an étale cover stays étale, while in mixed characteristic, it is possible for a degree 2 cover ramified in exactly two geometric points to specialize to a degree 2 cover weakly wildly ramified at a single geometric point.

We now turn to Prym–Tyurin Prym schemes associated to involutions. Note that we restrict the genus of  $C$  to be  $g \geq 3$  to ensure that the genus of the associated quotient curve is  $g' \geq 2$ ; in this way we avoid the issue of whether the associated quotient  $S$ -curve exists as a scheme, and not only as an algebraic space (Remark 2.30). This excludes case (d) from Theorem 5.12 (which is the same as case (b) if  $g = 2$ ), and consequently, we are excluding case (b) from Corollary 5.16 when  $g = 2$ .

**Corollary 6.6** (Prym–Tyurin Prym schemes associated to involutions). *Let  $C$  be a smooth projective curve of genus  $g \geq 3$  over  $S$ , and let  $\sigma: C \rightarrow C$  be a nontrivial involution such that for every geometric point  $\bar{s}$  of  $S$  we have that  $\sigma_{\bar{s}}$  is separable. Let  $C' = C / \langle \sigma \rangle$  be the quotient curve. Then the Prym scheme  $P(C/C')$  is a Prym–Tyurin scheme if and only if for every geometric point  $\bar{s}$  of  $S$  we have that  $\sigma_{\bar{s}}$  is one of the two types delineated in Corollary 5.16. In both cases the exponent is  $e = 2$ ; in case (a) one has  $\dim_S P(C/C') = g' - 1 = (g - 1)/2$ , and in case (b) one has  $\dim_S P(C/C') = g' = g/2$ .*

*Proof.* This is an immediate consequence of Corollary 5.16. □

## 7. APPLICATIONS

In this section, we would like to briefly indicate how the tools developed here allow for easy extensions of some results about abelian varieties. We restrict ourselves here to giving an impressionistic summary of the available results, and hope that the interested reader will follow the references for more details. (Better yet, we hope that the interested reader will develop entirely new applications we have not anticipated!)

**7.1. Prym–Tyurin varieties via Hecke algebras.** We have seen that a principally polarized abelian variety is a Prym–Tyurin variety of arbitrarily large exponent; consequently, it is of some interest to produce Prym–Tyurin varieties of small exponent.

One method, developed by Carocca et. al., is as follows. Consider a  $G$ -Galois cover  $f: D \rightarrow C$  of smooth projective curves over  $\mathbb{C}$ . If  $H \subseteq G$  is a subgroup, then the Hecke algebra  $\mathbb{Q}[H \backslash G / H]$  acts on the Jacobian of the intermediate curve  $E = D/H$ . To a rational irreducible representation of  $G$  one associates an idempotent of the Hecke algebra. The main result of [CLRR09] is a sufficient condition for the image of this idempotent to be a Prym–Tyurin variety. The criterion is in terms of the restriction of this data to  $H$  and the ramification data of the covering map  $f$ ; and part of the formula there is an explicit calculation of the Prym–Tyurin exponent which arises.

It turns out that their main theorem is valid for a cover of curves over an arbitrary field  $K$ , provided the action of  $G$  is defined over  $K$  and the cover is tamely ramified. The tameness condition is necessary for the genus calculation in [CLRR09, Prop. 4.8], and for the description of intermediate covers in [Roj07].

More generally, it seems likely to us that many constructions involving decompositions of abelian varieties pass from the complex numbers to arbitrary fields without serious difficulties.

**7.2. Point counts on Pryms.** Consider an abelian variety  $X$  over a finite field  $\mathbb{F}_q$ ; then Weil gives bounds on  $\#X(\mathbb{F}_q)$ . If  $X$  is actually the Jacobian of a curve  $C$ , then information about the curve (such as gonality) can be exploited to give bounds on  $\#X(\mathbb{F}_q)$  which are often strictly stronger than the usual Weil bounds [LMD90]. In the case where  $q$  is odd and  $X$  is a classical Prym, i.e., a Prym–Tyurin variety attached to an étale double cover of curves, Perret [Per06] and then Aubry and Haloui [AH16] used a similar strategy to bound the number of points. The only reason we can surmise for restricting to odd characteristic is perhaps some uncertainty concerning the status

of the Prym construction in characteristic two, which might be traced to the original restrictions in Mumford’s paper [Mum74]. We are happy to report that the main results of [AH16, Per06] are valid over fields of even characteristic as well.

More generally, constructions involving the Prym variety of an étale double cover of curves can be used without fear in even characteristic.

**7.3. Moduli spaces.** In [FvdG04], Faber and van der Geer use the classical Prym construction to study complete subvarieties of  $\mathcal{M}_{g',k}$ , the moduli space of curves of genus  $g' \geq 2$  over a field  $k$  of characteristic  $p > 2$ . To this end, they work with  $\mathcal{R}_{g'}$ , the moduli space of étale double covers  $C \rightarrow C'$ , where  $C'$  has genus  $g'$ ; this stack is naturally a smooth Deligne–Mumford stack over  $\text{Spec } \mathbb{Z}[\frac{1}{2}]$  (e.g., [Bea77a, §(6.5.1)], [DM69, §5]). Faber and van der Geer work extensively with the Prym map  $\mathcal{R}_{g'} \rightarrow \mathcal{A}_{g'-1}$ , which sends such a cover to the principally polarized Prym variety  $(P(C/C'), \xi)$  [Bea77a, §(6.2)]. The construction of the Prym map is made via [GP11, Cor VI<sub>B</sub>4.4] using the universal family, and (implicitly) the fact that away from characteristic two, the norm map has reduced, thus smooth, kernel. The Prym map for a family of covers  $C/C'$  over a general base  $S$  is then defined via the composition  $S \rightarrow \mathcal{R}_{g'} \rightarrow \mathcal{A}_{g'-1}$ . A result such as ours, involving complements in arbitrary families, provides perhaps a more direct construction of the Prym map. Moreover, we can actually define the moduli space  $\mathcal{R}_{g'}$ , and the morphism  $\mathcal{R}_{g'} \rightarrow \mathcal{A}_{g'-1}$ , over  $\text{Spec } \mathbb{Z}$ .

More precisely, define a category fibered in groupoids (CFG)  $\mathcal{R}_{g'}$  over the étale site of schemes (over  $\mathbb{Z}$ ) in the following way. Over a scheme  $S$ , we define  $\mathcal{R}_{g'}(S)$  to be the category of pairs  $(C/S, \sigma)$  where  $C/S$  is a smooth projective curve of genus  $2g' + 1$ , and  $\sigma : C \rightarrow C$  is a nontrivial  $S$ -involution of  $C$  such that every geometric fiber of the quotient map  $C \rightarrow C' := C/\langle \sigma \rangle$  is an étale double cover. Morphisms in  $\mathcal{R}_{g'}$  are defined via pull-back in the obvious way. There are natural forgetful functors of CFGs  $F_{2g'+1} : \mathcal{R}_{g'} \rightarrow \mathcal{M}_{2g'+1}$  and  $F_{g'} : \mathcal{R}_{g'} \rightarrow \mathcal{M}_{g'}$ .

**Proposition 7.1.** *The CFG  $\mathcal{R}_{g'}$  is a Deligne–Mumford stack over the étale site of schemes (over  $\mathbb{Z}$ ), and each fiber of  $\mathcal{R}_{g'} \rightarrow \text{Spec}(\mathbb{Z})$  is geometrically irreducible.*

*Proof.* Beauville’s proof ([Bea77a, (6.5.1)]) (where one allows stable curves, but restricts to schemes over  $\mathbb{Z}[1/2]$ ) is also valid in this context, too. The key point is to show that the forgetful functor  $F_{2g'+1} : \mathcal{R}_{g'} \rightarrow \mathcal{M}_{2g'+1}$  is representable, as it is a standard result that this implies that  $\mathcal{R}_{g'}$  is a DM stack (see, e.g., [CMW18] for a detailed exposition). Concretely, let  $S$  be a scheme and  $C/S$  a relative curve of genus  $2g' + 1$ , determined by a morphism  $S \rightarrow \mathcal{M}_{2g'+1}$ . We must show that the fibered product  $\mathcal{R}_{g'} \times_{\mathcal{M}_{2g'+1}} S$  is representable by a scheme. Insofar as this fibered product represents the functor of a special kind of automorphism of  $C$ , we may proceed as follows. There is a chain of functors  $\text{Aut}_S(C) \supset \text{Inv}_S(C) \supset \text{Inv}_S^\circ(C) \supset \text{Inv}_S^{\text{ét}}(C) \cong \mathcal{R}_{g'} \times_{\mathcal{M}_{2g'+1}} S$  defined as, respectively, the automorphisms of  $C$ ; the involutions of  $C$ ; the fiberwise nontrivial involutions of  $C$ ; and the fiberwise étale nontrivial involutions of  $C$ . We now show successively that each of these functors is representable (by a scheme).

Since  $2g' + 1 \geq 5 > 1$ ,  $C$  is projective over  $S$  and so the functor  $\text{Aut}_S(C)$  is representable. The composition  $\text{Aut}_S(C) \rightarrow \text{Aut}_S(C) \times_S \text{Aut}_S(C) \rightarrow \text{Aut}_S(C)$  of the diagonal with composition of automorphisms shows, via pull-back of the identity, that  $\text{Inv}_S(C)$  is representable by a closed subscheme of  $\text{Aut}_S(C)$ . A similar argument shows that the sub-functor  $\text{Inv}_S^\circ(C)$  of non-trivial involutions is representable by an open subscheme of  $\text{Inv}_S(C)$ . Finally, since the étale locus is open, the sub-functor  $\text{Inv}_S^{\text{ét}}(C)$  of non-trivial involutions where the quotient map gives an étale double cover is representable by an open subscheme of  $\text{Inv}_S^\circ(C)$ . Therefore,  $\mathcal{R}_{g'}$  is a DM stack.

For an alternative approach to showing that  $\mathcal{R}_{g'}$  is a DM stack, one can directly show that the forgetful morphism  $F_{g'} : \mathcal{R}_{g'} \rightarrow \mathcal{M}_{g'}$  is representable. For this, one can appeal to [Ant11, Prop. 3.2], which is the relative version of a special case of Lemma 5.10: if  $\omega : C' \rightarrow S$  is a

smooth proper curve, then there is a bijection between nontrivial étale double covers of  $C'$  and nonzero sections of  $R^1\omega_*(\mu_{2,C'})$ ; and these sections are canonically identified with nonzero sections of  $\text{Pic}_{C'/S}^0[2]$ .

This perspective allows us to quickly prove geometric irreducibility of  $\mathcal{R}_{g'}$  in characteristic two. All other characteristics are settled in [Bea77a, p.181]; see also [DM69, Thm. 5.13]. Let  $\mathcal{M}_{g',\mathbb{F}_2}^{\text{ord}}$  denote the ordinary locus in  $\mathcal{M}_{g',\mathbb{F}_2}$  (i.e., the locus where the 2-torsion is of full rank); it is open and dense (e.g., [FvdG04, Thm. 2.3]). Let  $\mathcal{R}_{g',\mathbb{F}_2}^{\text{ord}} = \mathcal{R}_{g',\mathbb{F}_2} \times_{\mathcal{M}_{g',\mathbb{F}_2}} \mathcal{M}_{g',\mathbb{F}_2}^{\text{ord}}$  be the open substack parametrizing étale double covers  $C \rightarrow C'$  where  $C'$  is ordinary. Then  $\mathcal{R}_{g',\mathbb{F}_2}^{\text{ord}}$  is again dense in  $\mathcal{R}_{g',\mathbb{F}_2}$ , and so it suffices to show that  $\mathcal{R}_{g',\mathbb{F}_2}^{\text{ord}}$  is geometrically irreducible. To see this density, it suffices to show that if  $K$  is any algebraically closed field of characteristic two, and if  $C'/K$  is a curve of genus  $2g' + 1$  and  $\chi$  is a section of  $\text{Pic}_{C'/K}^0[2]^{\text{ét}}$ , then the data  $(C', \beta)$  deforms to the data of an ordinary curve and a point of order two on its Picard scheme. Using the density of  $\mathcal{M}_{2g'+1}^{\text{ord}}$ , we may find an equicharacteristic discrete valuation ring  $S$  and a curve  $\tilde{C}'/S$  with ordinary generic fiber and with special fiber isomorphic to  $C'$ . Let  $G$  be the finite flat group scheme  $\text{Pic}_{\tilde{C}'/S}^0[2]$ . After a suitable fppf base extension  $T \rightarrow S$  (in fact,  $G \rightarrow S$  itself will do) there is a section of  $G_T \rightarrow T$  which specializes to  $\chi$ .

Now let  $\mathcal{I}_{g',\mathbb{F}_2}$  be the moduli stack of data  $(C' \rightarrow S, \alpha: \text{Pic}_{C'/S}^0[2]^{\text{ét}} \xrightarrow{\sim} (\mathbb{Z}/2)_S^{g'})$ . On one hand, a monodromy calculation [Eke91, Thm. 2.1] shows that  $\mathcal{I}_{g',\mathbb{F}_2}$  is geometrically irreducible. On the other hand, the forgetful functor  $\mathcal{I}_{g',\mathbb{F}_2} \rightarrow \mathcal{M}_{g'}^{\text{ord}}$  admits a factorization through  $\mathcal{R}_{g',\mathbb{F}_2}^{\text{ord}}$ , say by letting the isomorphism  $\alpha$  determine the section  $\alpha^{-1}([1, 0, \dots, 0])$ ; and since any section of  $\text{Pic}_{C'/S}^0[2]^{\text{ét}}$  can, after fppf base extension  $T \rightarrow S$ , be completed to a basis of sections, this morphism is surjective. Therefore,  $\mathcal{R}_{g',\mathbb{F}_2}^{\text{ord}}$  is geometrically irreducible; by density, so is  $\mathcal{R}_{g',\mathbb{F}_2}$ .  $\square$

For clarity, we summarize our discussion with the following statement. (Note that outside of characteristic 2, this is proven in [Bea77a]; there the Prym map is defined *a priori* over reduced versal spaces, and then *a posteriori* over arbitrary bases via the induced morphisms to versal spaces.)

**Proposition 7.2.** *There is a Prym morphism of stacks over  $\text{Spec } \mathbb{Z}$*

$$\begin{aligned} P : \mathcal{R}_{g'} &\longrightarrow \mathcal{A}_{g'-1} \\ (C \rightarrow C') &\mapsto P(C/C') \end{aligned}$$

*Proof.* The construction of  $\mathcal{R}_{g'}$  is Proposition 7.1; the existence of the Prym morphism is extracted from our main theorem in Corollary 6.6.  $\square$

**7.4. Group schemes are summands of the torsion of Jacobians.** The titular claim of [PU21] is that every  $BT_1$  group scheme appears in a Jacobian. We can quickly recover much of that abstract statement, and extend it, as follows.

**Lemma 7.3.** *Let  $K$  be a field of characteristic  $p > 0$ , and let  $Z$  be a principally polarized abelian variety with  $g := \dim Z < p$ . Then there exists a smooth projective curve  $C/K$  such that, for each  $m \geq 1$ ,  $Z[p^m]$  is a summand of  $\text{Pic}_{C/K}^0[p^m]$  as a principally-quasipolarized finite group scheme.*

(In fact, when  $K = \bar{K}$  and  $m = 1$ , [PU21, Thm. 1.1] is much more precise, in that the authors show that for  $C$  one may take a quotient of a Fermat curve whose degree is explicitly bounded in terms of  $g$ .)

*Proof.* Fix some  $n \geq 3$  with  $p \nmid n$ . By Corollary 5.7 and Remark 5.8, there exists a smooth projective curve  $C/K$  such that  $Z$  is a sub-abelian variety of  $\text{Pic}_{C/K}^0$  of exponent  $e = n^r(g-1)!$  for some

$r$ , and  $\Theta_C|_Z = e\Xi$ . Let  $Y$  be the complement of  $Z$ . Since  $\gcd(p, n^r(g-1)!) = 1$ , the isogeny  $\mu: Y \oplus Z \rightarrow \text{Pic}_{C/K}^0$  induces, for each  $m$ , an isomorphism  $Y[p^m] \oplus Z[p^m] \xrightarrow{\sim} \text{Pic}_{C/K}^0[p^m]$ .  $\square$

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